J. Math. Study
doi: 10.4208/jms.v48n3.15.06

Vol. **48**, No. 3, pp. 275-289
September 2015

Second Order Estimates for Non-concave Hessian Type Elliptic Equations on Riemannian Manifolds

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Received 18 March, 2015; Accepted 16 July, 2015

Abstract. In this paper, we derive second order estimates for a class of non-concave Hessian type elliptic equations on Riemannian manifolds. By applying a new method for C^2 estimates, we can weaken some conditions, which works for some non-concave equations. Gradient estimates are also obtained.

AMS subject classifications: 35J15, 58J05, 35B45.

Key words: Non-concave fully nonlinear elliptic equations, Riemannian manifolds, a priori estimates, Dirichlet problem, subsolutions.

1 Introduction

In this paper we continue our previous work in [5, 6] to study the Dirichlet problem for Hessian type equations

$$\begin{cases} f(\lambda(\nabla^2 u + A[u])) = \psi(x,u,\nabla u) \text{ in } M, \\ u = \varphi \text{ on } \partial M \end{cases} \tag{1.1}$$

on a compact Riemannian manifold (M^n,g) of dimension $n \geq 2$ with smooth boundary ∂M, where f is a symmetric smooth function of n variables, $\nabla u = du$ which is often identified with the gradient of u, $\nabla^2 u$ denotes the Hessian of u, $A[u] = A(x,u,\nabla u)$ is a $(0,2)$ tensor which may depend on u and ∇u, and

$$\lambda(\nabla^2 u + A[u]) = (\lambda_1,\ldots,\lambda_n)$$

denotes the eigenvalues of $\nabla^2 u + A[u]$ with respect to the metric g.

Our motivation to study equations (1.1) is mainly from their significant applications in differential geometry (see [5]).

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As in [1], the function $f\in C^2(\Gamma)$ is assumed to be defined in an open, convex, symmetric cone $\Gamma\subset\mathbb{R}^n$ with vertex at the origin,

$$\Gamma_n\equiv\{\lambda\in\mathbb{R}^n:\text{each component }\lambda_i>0\}\subseteq\Gamma\neq\mathbb{R}^n,$$

and to satisfy

$$f_i\equiv\frac{\partial f}{\partial\lambda_i}>0\text{ in }\Gamma,\ 1\leq i\leq n, \tag{1.2}$$

which implies (1.1) is elliptic for the solution $u\in C^2(\bar{M})$ satisfying $\lambda(\nabla^2u+A[u])\in\Gamma$; we shall call such functions *admissible* (see [1]).

Another fundamental assumption is

$$\sup_{\partial\Gamma}f<\psi<\sup_{\Gamma}f,\text{ where }\sup_{\partial\Gamma}f\equiv\sup_{\lambda_0\in\partial\Gamma}\limsup_{\lambda\to\lambda_0}f(\lambda) \tag{1.3}$$

which prevents the degeneracy of (1.1).

In the literature, the function f is often assumed to be concave in Γ, which is crucial to the second order estimates and Evans-Krylov theorem. We wish to establish the *a priori* estimates and existence of solutions for (1.1) with respect to some non-concave f. The contribution of this paper is to weaken the concavity condition of f in establishing the second order estimates. We first consider those f which are concave when $|\lambda|$ is sufficiently large. Our main idea is to treat another function $\widetilde{f}$ instead of f, where $\widetilde{f}$ is a concave function on Γ satisfying that $\widetilde{f}\geq f$ in Γ, $\widetilde{f}(\lambda)=f(\lambda)$ when $|\lambda|$ is sufficiently large and that

$$\widetilde{f}_i\equiv\frac{\partial\widetilde{f}}{\partial\lambda_i}>0\text{ in }\Gamma. \tag{1.4}$$

We assume in this paper that $A[u]$ and $\psi[u]$ are smooth on $\bar{M}$ for $u\in C^\infty(\bar{M})$ and $\varphi\in C^\infty(\partial M)$. As in [5] and [6], we shall use the notations

$$A^{\xi\eta}(x,\cdot,\cdot):=A(x,\cdot,\cdot)(\xi,\eta),\ \xi,\eta\in T_x^*M$$

and therefore, $A^{\xi\eta}[v]:=A^{\xi\eta}(x,v,\nabla v)$ for a function $v\in C^2(M)$.

We assume that

$$-\psi(x,z,p)\text{ and }A^{\xi\xi}(x,z,p)\text{ are concave in }p, \tag{1.5}$$

$$-\psi_z,A_z^{\xi\xi}\geq0,\quad\forall\xi\in T_xM. \tag{1.6}$$

and moreover, there exists an admissible subsolution $\underline{u}\in C^2(\bar{M})$ satisfying

$$\begin{cases}f(\lambda(\nabla^2\underline{u}+A[\underline{u}]))\geq\psi(x,\underline{u},\nabla\underline{u})\text{ in }\bar{M},\\ \underline{u}=\varphi\text{ on }\partial M.\end{cases} \tag{1.7}$$

Now we can state our main theorem as follows.