J. Math. Study
doi: 10.4208/jms.v48n4.15.01

Vol. **48**, No. 4, pp. 315-329
December 2015

Generalized Jacobi Spectral-Galerkin Method for Nonlinear Volterra Integral Equations with Weakly Singular Kernels

Jie Shen[1,2], Changtao Sheng[1,*] and Zhongqing Wang[3]

[1] *School of Mathematical Sciences and Fujian Provincial Key Laboratory on Mathematical Modeling & High Performance Scientific Computing, Xiamen University, Xiamen 361005, Fujian, P. R. China.*
[2] *Department of Mathematics, Purdue University, West Lafayette, IN 47907-1957, USA.*
[3] *School of Science, University of Shanghai for Science and Technology, Shanghai 200093, P. R. China.*

Received 10 May, 2015; Accepted 20 August, 2015

Abstract. We propose a generalized Jacobi spectral-Galerkin method for the nonlinear Volterra integral equations (VIEs) with weakly singular kernels. We establish the existence and uniqueness of the numerical solution, and characterize the convergence of the proposed method under reasonable assumptions on the nonlinearity. We also present numerical results which are consistent with the theoretical predictions.

AMS subject classifications: 45D05, 65L60, 41A30, 65L70

Key words: Generalized Jacobi spectral-Galerkin method, nonlinear Volterra integral equations with weakly singular kernels, convergence analysis.

1 Introduction

This paper is concerned with the numerical solutions of the nonlinear Volterra integral equations(VIEs) with weakly singular kernels:

$$y(t)=f(t)+\mathcal{V}y(t):=f(t)+\int_0^t (t-s)^{-\mu}K(t,s)G(s,y(s))ds,\quad t\in I:=[0,T], \tag{1.1}$$

where $0<\mu<1$, $K\in C(D)$ with $D:=\{(t,s):\ 0\le s\le t\le T\}$, $f\in C(I)$ and G is a continuous function.

*Corresponding author. *Email addresses:* shen7@purdue.edu (J. Shen), ctsheng@stu.xmu.edu.cn (C. Sheng), zqwang@usst.edu.cn (Z. Wang)

In recent years, there has been an increasing interest in studying VIEs. The main difficulties for dealing with weakly singular VIEs are: (i) the integral operator is non-local; (ii) the solutions are usually singular near $t=0$. Brunner [4] and Lubich [17] investigated the smoothness properties of the exact solutions of VIEs with weakly singular kernels. Various numerical approaches, using the piecewise polynomial collocation methods and the Runge-Kutta methods, have been proposed for approximating VIEs with weakly singular kernels [4, 5, 12, 23]. However, these numerical methods do not particularly deal with the above two difficulties.

Spectral methods are capable of providing exceedingly accurate numerical results with relatively less degree of freedoms, and have been widely used for scientific computation, see, e.g., [1, 2, 6, 13, 14, 19, 20]. Since the spectral methods are global methods, so they could be better suited for non-local problems. Recently, many kinds of spectral collocation methods are proposed for solving VIEs with smooth kernels. Li, Tang and Xu [16] introduced a time parallel method with spectral-subdomain enhancement for VIEs; Sheng, Wang and Guo [21] presented a multistep spectral collocation method for nonlinear VIEs; Wang and Sheng [22] also proposed a multistep spectral collocation method for nonlinear VIEs with delays.

To solve VIEs with weakly singular kernels, many attempts have been made to overcome the difficulties caused by the singularities of the solutions. Chen and Tang [9, 10] proposed spectral collocation methods for weakly singular VIEs; Huang, Tang and Zhang [15] studied the supergeometric convergence of spectral collocation methods for weakly singular Volterra/Fredholm integral equations. These methods usually use orthogonal polynomials as basis functions. Another approach for solving weakly singular VIEs is to use the non polynomial singular functions (which reflect the singularities of the exact solutions) as basis functions. For example, Brunner [3] employed a non polynomial spline collocation method for VIEs with weakly singular kernels; Cao, Herdman and Xu [7] presented a non polynomial singularity preserving collocation method for VIEs with weakly singular kernels.

In this paper, we develop a non polynomial spectral-Galerkin method for VIEs with weakly singular kernels. More precisely, we construct a spectral-Galerkin method for weakly singular VIEs (1.1), using the generalized Jacobi functions as basis functions. This kind of basis functions have been used by Zayernouri & Karniadakis [24] and Chen, Shen & Wang [8] for approximating fractional differential equations. The main strategies and contributions are as follows.

- We propose a generalized Jacobi spectral-Galerkin method for nonlinear VIEs with weakly singular kernels. The basis functions can be tuned to match the singularities of the underlying solutions, and lead to an efficient implementation. The existing works (cf. [9, 10])

- We approximate the problem (1.1) directly without any variable transformations, as opposed to the approach in [9,10] where a spectral-collection method is constructed for the transformed VIEs.