J. Math. Study
doi: 10.4208/jms.v48n4.15.04

Vol. **48**, No. 4, pp. 362-374
December 2015

Ball Convergence for Higher Order Methods under Weak Conditions

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Received 6 May, 2015; Accepted 29 September, 2015

Abstract. We present a local convergence analysis for higher order methods in order to approximate a locally unique solution of an equation in a Banach space setting. In earlier studies, Taylor expansions and hypotheses on higher order Fréchet-derivatives are used. We expand the applicability of these methods using only hypotheses on the first Fréchet derivative. Moreover, we obtain a radius of convergence and computable error bounds using Lipschitz constants not given before. Numerical examples are also presented in this study.

AMS subject classifications: 65G99, 65D99, 65G99, 47J25, 45J05

Key words: Higher order method, Banach space, Fréchet derivative, local convergence.

1 Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the nonlinear equation

$$F(x)=0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Denote by $L(X,Y)$ the space of bounded linear operators from X into Y.

A lot of problems from Computational Sciences and other disciplines can be brought in the form of Eq. (1.1) using Mathematical Modeling [2, 5, 10, 17, 22]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [1–22].

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The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semi-local convergence analysis of Newton-like methods such as [1–22]. In order to obtain a higher order of convergence, Newton-like methods have been studied such as Potra-Ptak, Chebyshev, Cauchy, Halley and Ostrowski, methods [2,5,16,22]. The number of function evaluations per step increases with the order of convergence. In the scalar case the efficiency index [16,22] $EI=p^{\frac{1}{m}}$ provides a measure of balance where p is the order of the methods and m is the number of function evaluations.

We study the local convergence of the two-step methods defined for each $n=0,1,2...$ by

$$\begin{aligned} y_n &= x_n - \Theta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - \frac{1}{2}F'(x_n)^{-1}F(x_n) + (F'(x_n) - 3F'(y_n))^{-1}F(x_n) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} y_n &= x_n - \Theta F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= x_n - H(x_n,y_n)F'(x_n)^{-1}F(x_n) \end{aligned} \tag{1.3}$$

where x_0 is an initial point, $\Theta \in \mathbb{R}$ a parameter and $H: X^2 \to L(X,Y)$ a given continuous operator. Method (1.2) was studied by Basu in [7], when $X=Y=\mathbb{R}^m$ and $\Theta=\frac{2}{3}$. The method (1.2) was shown to be of order four using Taylor expansions and hypotheses reaching up to the sixth derivative of F. Notice that method (1.2) is really a particular case of Jarratt's method [2,5,16,22]. Moreover, method (1.3) was studied by Chun *et al.* in [8] the same way. This method is also of order four assuming that function H satisfies certain initial conditions [8]. The case $H(x_n,y_n)=F'(x_n)^{-1}F(y_n)$ was also studied in [8] (see also our Remark 2.3 (6)). Method (1.2) uses two inverses and one function evaluation. Two-step Newton methods comparable to method (1.2) are given by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(x_n) \end{aligned} \tag{1.4}$$

or

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= y_n - F'(y_n)^{-1}F(y_n). \end{aligned} \tag{1.5}$$

Method (1.4) uses two inverses and one function evaluation (so does method (1.2)) but it is of order three. Moreover, method (1.5) uses two inverses and two function evaluations