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Existence of Renormalized Solutions of Nonlinear Elliptic Problems in Weighted Variable-Exponent Space

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Abstract. In this article, we study a general class of nonlinear degenerated elliptic problems associated with the differential inclusion $\beta(u)-div(a(x,Du)+F(u))\ni f$ in Ω where $f\in L^1(\Omega)$. A vector field $a(.,.)$ is a Carathéodory function. Using truncation techniques and the generalized monotonicity method in the framework of weighted variable exponent Sobolev spaces, we prove existence of renormalized solutions for general L^1-data.

AMS subject classifications: 35J15, 35J70, 35J85

Key words: Weighted variable exponent Sobolev spaces, truncations, Young's Inequality, elliptic operators.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$ with Lipschitz boundary if $N\geq 2$, where the variable exponent $p:\overline{\Omega}\to(1,\infty)$ is a continuous function, and ω be a weight function on Ω, i.e. each ω is a measurable a.e. positive on Ω. Let $W_0^{1,p(\cdot)}(\Omega,\omega)$ be the weighted variable exponent Sobolev space associated with the vector ω. We are interested in existence of renormalized solutions to the following nonlinear elliptic equation

$$(E,f)\begin{cases}\beta(u)-div(a(x,Du)+F(u))\ni f & \text{in} \quad \Omega\\ u=0 & \text{on} \quad \partial\Omega,\end{cases}$$

with a right-hand side f which is assumed to belong either to $L^\infty(\Omega)$ or to $L^1(\Omega)$ for Eq. (E,f). Furthermore, F and β are two functions satisfying the following assumptions:

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($\mathbf{A}_0$) $F:\ \mathbb{R}\ \to\ \mathbb{R}^N$ is locally lipschitz continuous and $\beta:\mathbb{R}\ \to\ 2^{\mathbb{R}}$ a set valued, maximal monotone mapping such that $0\in\beta(0)$, Moreover, we assume that

$$\beta^0(l)\in L^1(\Omega), \tag{1.1}$$

for each $l\in\mathbb{R}$, where β^0 denotes the minimal selection of the graph of β. Namely $\beta_0(l)$ is the minimal in the norm element of $\beta(l)$

$$\beta_0(l)=\inf\{|r|\ /r\in\mathbb{R}\quad\text{and}\quad r\in\beta(l)\}.$$

Moreover, $a:\ \Omega\times\mathbb{R}^N\ \to\ \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions :

($\mathbf{A}_1$) There exists a positive constant λ such that $a(x,\xi)\cdot\xi\geq\lambda\omega(x)|\xi|^{p(x)}$ holds for all $\xi\in\mathbb{R}^N$ and almost every $x\in\Omega$.

($\mathbf{A}_2$) $|a_i(x,\xi)|\leq\alpha\omega^{1/p(x)}(x)[k(x)+\omega^{1/p'(x)}(x)|\xi|^{p(x)-1}]$ for almost every $x\in\Omega$, all $i=1,....,N$, every $\xi\in\mathbb{R}^N$, where $k(x)$ is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, $p'(x):=p(x)/(p(x)-1)$, and $\alpha>0$.

($\mathbf{A}_3$) $(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)\geq 0$ for almost every $x\in\Omega$ and every $\xi,\eta\in\mathbb{R}^N$.

We use in this paper the framework of renormalized solutions. This notion was introduced by Diperna and P.-L. Lions [7] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (E,f) by L. Boccardo *et al.* [5] when the right hand side is in $W^{-1,p'}(\Omega)$, by J.-M. Rakotoson [17] when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [10] for the case of right hand side is general measure data. The equivalent notion of entropy solution has been introduced by Bénilan *et al.* in [4]. For results on existence of renormalized solutions of elliptic problems of type (E,f) with $a(,)$ satisfying a variable growth condition, we refer to [19], [12], [2] and [1]. One of the motivations for studying (E,f) comes from applications to electrorheological fluids (see [18] for more details) as an important class of non-Newtonian fluids.

For the convenience of the readers, we recall some definitions and basic properties of the weighted variable exponent Lebesgue spaces $L^{p(x)}(\Omega,\omega)$ and the weighted variable exponent Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$. Set

$$C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}p(x)>1\}.$$

For any $p\in C_+(\overline{\Omega})$, we define

$$p^+=\max_{x\in\overline{\Omega}}p(x),\quad p^-=\min_{x\in\overline{\Omega}}p(x).$$

For any $p\in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega)=\left\{u:\Omega\to\mathbb{R},\textit{measurable},\int_\Omega|u(x)|^{p(x)}\omega(x)dx<\infty\right\}.$$