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# One-Side Derived Categories of Pre-Strict $P$-Semi-Abelian Categories

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**Abstract.** In this paper, we study a class of $P$-semi-abelian categories, as well as left and right cohomological functors. Then we establish the corresponding one-side derived categories.

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**Key words**: $P$-semi-abelian category, cohomological functor, one-side derived category.

## 1 Introduction

The concept of derived categories seems to have first appeared in Verdier [9]. He introduced triangulated categories and developed localization theories to established derived categories. These theories have been studied by many mathematicians during the last four decades and applied in some branches of mathematics, such as representation theories of algebra and the algebraic geometry (see [1,2]).

The notion of semi-abelian categories was invented several times by different mathematicians under different names. At the end of the 1960's, Palamodov [8] introduced the same concept under the name of "semi-abelian categories". Therefore some authors use "$P$-semi-abelian category" to denote the categories above. The properties of $P$-semi-abelian categories was optimized by Kopylov [3,5] in recent years. In the sequel, we call the "$P$-semi-abelian category" as "semi-abelian category" for short.

Milicic [7] studied the derived category of abelian categories. The aim of this article is to generalize these properties to a class of semi-abelian categories, called pre-strict semi-abelian categories. In Section 2, we first give some necessary definitions or notations and recall basic facts. Then we construct the left and right cohomological functors in semi-abelian categoreis. In Section 3, we establish the pre-strict semi-abelian category and investigate some properties of it. Then we introduce the concept of left quasi-bimorphisms and right quasi-bimorphisms in pre-strict semi-abelian categories to form

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localizing classes compatible with triangulation respectively. Consequently, we obtain the corresponding one-side derived categories.

## 2 Right cohomological functor $H^n_-$ and left cohomological functor $H^n_+$

The theory of $P$-semi-abelian categories is being optimized by mathematicians in recent years. Here the definition of $P$-semi-abelian category is in the sense of Palamodov [7]. In this paper, we call the $P$-semi-abelian category as semi-abelian category for short. Kopylov claimed in [5] that an additive category $\mathscr{E}$ is called pre-abelian category if every morphism in $\mathscr{E}$ has a kernel and a cokernel. Furthermore, in pre-abelian categories, each morphism $\alpha$ admits the canonical decomposition $\alpha$=(im$\alpha$)$\overline{\alpha}$(coim$\alpha$), where im$\alpha$=ker(coker$\alpha$), coim$\alpha$=coker(ker$\alpha$), and $\overline{\alpha}$ is unique. If the $\overline{\alpha}$ is an isomorphism, then $\alpha$ is called strict. A pre-abelian category is called semi-abelian category if for every morphism $\alpha$, $\overline{\alpha}$ is bimorphism. The equivalent definition of semi-abelian categories can be seen in [3].

A preabelian category is semi-abelian if for every morphism $\alpha$, $\overline{\alpha}$ is a bimorphism, i.e. $\overline{\alpha}$ is a monomorphism and an epimorphism simultaneously. We refer to [5] for equivalent definitions of semi-abelianity for a preabelian category.

For an additive category $\mathscr{C}$,$C(\mathscr{A})$ is the category of complexes of $\mathscr{C}$-objects with complexes of $\mathscr{C}$-objects as objects and morphisms of complexes as morphisms.

Let $\mathscr{C}$ be a semi-abelian category. Then so is $C(\mathscr{A})$. We denote a complex $A^{\cdot}=(A^n,d^n_A)_{n\in Z}$ in $C(\mathscr{A})$ to be as follows:

$$A^{\cdot}:\cdots\to A^{n-1}\xrightarrow{d^{n-1}}A^n\xrightarrow{d^n}A^{n+1}\xrightarrow{d^{n+1}}\cdots$$

For each $n\in Z$, there is a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Ker}d^n_A \arrow[dr, "\mathrm{ker}d^n_A"] & & & \mathrm{Ker}d^{n+1}_A \arrow[dr, "\mathrm{ker}d^{n+1}_A"] & \\
A^{n-1} \arrow[rrr, "d^{n-1}_A"] \arrow[dr, "\mathrm{coker}d^{n-2}_A"'] & & & A^n \arrow[rrr, "d^n_A"] \arrow[dr, "\mathrm{coker}d^{n-1}_A"'] & & & A^{n+1} \\
 & \mathrm{Coker}d^{n-2}_A \arrow[uur, "a^{n-1}"'] & & & \mathrm{Coker}d^{n-1}_A \arrow[uur, "a^n"'] & &
\end{tikzcd} \tag{2.1}$$

with determinate $a^n$ and $a^{n-1}$. Evidently, Coker($a^{n-1}\cdot$coker$d^{n-2}$)=Coker$a^{n-1}$ and Ker ((ker$d^{n+1}$)$\cdot a^n$)=Ker$a^n$. We denote $\mathrm{H}^n_-(A^{\cdot})=\mathrm{Coker}a^{n-1}$ and $\mathrm{H}^n_+(A^{\cdot})=\mathrm{Ker}a^n$. We call $\mathrm{H}^n_-(A^{\cdot})$ the $n$th *right cohomology of complex* $A^{\cdot}$ and $\mathrm{H}^n_+(A^{\cdot})$ the $n$th *left cohomology of complex* $A^{\cdot}$.