J. Math. Study
doi: 10.4208/jms.v49n1.16.05

Vol. **49**, No. 1, pp. 42-49
March 2016

A Note on Burkholder Integrals

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Received 11 April 2015; Accepted (in second revised version) 9 December 2015

Abstract. In this note, for $k-$quasiconformal mappings of a bounded domain into the complex plane, we give an upper bound of Burkholder integral. Moreover, as an application we obtain an upper bound of the L^p-integral of $\sqrt{J_f}$ and $|Df|$ for certain $K-$quasiconformal mappings.

AMS subject classifications: 30C62,30A10

Key words: Quasiconformal mapping, Burkholder function, Morrey's problem.

1 Introduction

Let Ω and Ω' be two bounded simply connected domains of the complex plane $\mathbb{C}$. A homeomorphism $f:\Omega\longrightarrow\Omega'$ is called $k-$quasiconformal, if it belongs locally to the Sobolev class $\mathbb{W}^{1,2}_{loc}(\Omega)$ and satisfies the Beltrami equation

$$\frac{\partial f}{\overline{\partial}z}=\mu_f\frac{\partial f}{\partial z}\quad a.e.\quad z\in\Omega,$$

where the Beltrami coefficient has bounded L_∞ norm: $||\mu_f||_\infty\le k<1$. In particular, a homeomorphism of $\mathbb{C}$ onto itself is called principal solution of the Beltrami equation

$$\frac{\partial f}{\overline{\partial}z}=\mu_f\frac{\partial f}{\partial z},$$

if it satisfies the asymptotical normalization condition

$$f(z)=z+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots,\qquad for\quad |z|\longrightarrow\infty.$$

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We denote the formal partial derivatives of f by

$$\partial f = f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \quad \text{and} \quad \overline{\partial} f = f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right),$$

and use the notation

$$|Df| = |f_z| + |f_{\overline{z}}| \quad \text{and} \quad J_f = |f_z|^2 - |f_{\overline{z}}|^2.$$

Here the value $|Df|$ is the operator norm for Df and J_f is the Jacobian of f.

A continuous function $E: \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}$ is said to be quasiconvex if for every $f \in A + C_0^\infty(\Omega, \mathbb{R}^n)$, we have

$$\mathcal{E}[f] := \int_\Omega E(Df)dx \geq \int_\Omega E(A)dx = E(A)|\Omega|,$$

where A stands for an arbitrary linear mapping (or its matrix) and $\Omega \subset \mathbb{R}^n$ is any bounded domain. In other words, one requires that compactly supported perturbations of linear maps do not decrease the value of the integral. This notion is very important in the calculus of variations [7]. Another notion is that of rank-one convexity, which requires just that $t \to E(A+tX)$ is convex for any fixed matrix A and for any rank one matrix X. E is rank-one concave (resp. quasiconcave) if $-E$ is rank-one convex (resp. quasiconvex). The most famous rank-one concave function in dimension two is the Burkholder functional defined for any 2×2 matrix A by

$$B_P(A) = \left[\frac{p}{2} det A + \left(1 - \frac{p}{2}\right)|A|^2\right]|A|^{p-2}, \quad p \geq 2. \tag{1.1}$$

where $|A|$ is the operator norm of A, see [3]. Morrey's work [8] implies that quasiconvexity implies rank-one convexity. For the dimension n of $\mathbb{R}^n$ is bigger than 2, Šverák's paper [10] showed that the converse is not true. However, for dimension $n=2$, [5] and [7] gave the evidence to the possibility for a different outcome. So in [1], the authors gave the following conjecture in the spirit of Morrey,

Conjecture 1.1. Rank-one convex functions $E: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}$ are quasiconvex.

For the $A = Id$, the authors of [1] showed that the Burkholder function is quasiconcave within quasiconformal perturbations of the identity. They showed that when $f: \Omega \to \Omega$ is a $k-$quasiconformal map of Ω onto itself with extending to the identity on the boundary, then

$$\int_\Omega B_p(Df)dz = \int_\Omega \left(1 - \frac{p|\mu(z)|}{1+|\mu(z)|}\right)(|f_z(z)| + |f_{\overline{z}}(z)|)^p dz \leq \int_\Omega B_p(Id)dz = |\Omega|, \tag{1.2}$$

where $2 \leq p \leq 1 + \frac{1}{k}$.

In this paper, we first use the method learned from [1] to prove the following result: