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On a New Class of Projectively Flat Finsler Metrics

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Abstract. A class of Finsler metrics with three parameters is constructed. Moreover, a sufficient and necessary condition for this Finsler metrics to be projectively flat was obtained.

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1 Introduction

Finsler geometry is more colorful than Riemannian geometry because there are several non-Riemannian quantities on a Finsler manifold besides the Riemannian quantities. One of the important problems in Finsler geometry is to study and characterize the projectively flat metrics on an open domain $U \subset \mathbb{R}^n$. Projectively flat metrics on U are Finsler metrics whose geodesics are straight lines. This is the Hilbert's 4th problem in the regular case [5]. In 1903, Hamel [4] found a system of partial differential equations

$$F_{x^k y^l} y^k = F_{x^l}, \tag{1.1}$$

which can characterize the projectively flat metrics $F = F(x,y)$ on an open subset $U \subset \mathbb{R}^n$. And we know that Riemannian metrics form a special and important class in Finsler geometry. Beltrami's theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature [10]. The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. Besides, every locally projectively flat Finsler metric F on a manifold M is of scalar flag curvature, i.e., the flag curvature $K = K(x,y)$ is a scalar function on $TM \backslash \{0\}$. Many projectively flat Finsler metrics with constant flag curvature are obtained in [8], [1], [12], [2]. Besides, there are a lot of locally projectively flat Finsler metrics which are not of constant flag curvature [9], [13], [6]. Thus, the Beltrami's theorem is no longer true for Finsler metrics.

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Recently, Huang and Mo discussed a class of interesting Finsler metrics [13], [6] satisfying

$$F(Ax,Ay)=F(x,y), \tag{1.2}$$

for all $A\in O(n)$. A Finsler metric F is said to be spherically symmetric if F satisfies (1.2) for all $A\in O(n)$. Besides, it was pointed out in [7] that a Finsler metric F on $\mathbb{B}^n(r)$ is a spherically symmetric if and only if there is a function $\phi:[0,r)\times\mathbb{R}\longrightarrow\mathbb{R}$ such that

$$F(x,y)=|y|\phi\left(|x|,\frac{\langle x,y\rangle}{|y|}\right), \tag{1.3}$$

where$(x,y)\in T\mathbb{R}^n(r)\backslash\{0\}$.

In this paper, we construct a new class of Finsler metrics with three parameters and obtain the formula of the flag curvature of this kind of metrics.

Let ζ be an arbitrary constant and $\Omega=\mathbb{B}^n(r)\subset\mathbb{R}^n$ where $r=\frac{1}{\sqrt{-\zeta}}$ if $\zeta<0$ and $r=+\infty$ if $\zeta\geqslant 0$, $|\cdot|$ and $\langle,\rangle$ be the standard Euclidean norm and inner product in $\mathbb{R}^n$, respectively.

Define $F:T\Omega\to[0,+\infty)$ by

$$F=\frac{\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2}+\frac{\kappa\langle x,y\rangle}{(1+\zeta|x|^2)^{\frac{3}{2}}}, \tag{1.4}$$

where ϵ is an arbitrary positive constant, κ is an arbitrary constant.

As a natural prolongation, we obtain the following results

Theorem 1.1. *Let $F:T\Omega\to[0,+\infty)$ be a function given by* (1.4). *Then, it has the following properties.*

(1) F is a Finsler metric.

(2) F is projectively flat Finsler metric if and only if $\kappa^2+\epsilon\zeta=0$.

(3) When $\kappa^2+\epsilon\zeta=0$, the flag curvature of the Finsler metrics (1.4) *is given by*

$$K=\frac{\kappa^2}{\epsilon^2F^2}\left[\frac{\Delta\kappa\langle x,y\rangle}{F(1+\zeta|x|^2)^{\frac{7}{2}}}-\frac{\Delta}{(1+\zeta|x|^2)^2}-\frac{\Delta^2+6\Delta\kappa^2\langle x,y\rangle^2+6\Delta^{\frac{3}{2}}\kappa\langle x,y\rangle(1+\zeta|x|^2)^{\frac{1}{2}}}{4F^2(1+\zeta|x|^2)^5}\right],$$

where $\Delta=\epsilon|y|^2(1+\zeta|x|^2)+\kappa^2\langle x,y\rangle^2$.

2 Preliminaries

A Minkowski norm $\Psi(y)$ on a vector space V is a C^∞ function on $V\backslash\{0\}$ with the following properties:

(1) $\Psi(y)\geqslant 0$ and $\Psi(y)=0$ if and only if $y=0$;