J. Math. Study
doi: 10.4208/jms.v49n1.16.10

Vol. **49**, No. 1, pp. 82-92
March 2016

Some Improvements on Hermite-Hadamard's Inequalities for s-convex Functions

Yujiao Li and Tingsong Du*

Department of Mathematics, Science College, China Three Gorges University, Yichang 443002, Hubei, P.R. China.

Received 30 May 2015; Accepted (in second revised version) 15 February 2016

Abstract. Using an integral identity for a once differentiable mapping, this paper establishes Hadamard's integral inequalities for s-convex and s-concave mappings. In particular, our results improve and extend some known ones in the literature. Finally, these inequalities are applied to special means.

AMS subject classifications: 26D15, 26A51, 26E60, 41A55

Key words: Convex function, s-convex function, Hadamard's inequality.

1 Introduction

Throughout the present paper, we use $I \subseteq \mathbb{R}$ to denote the real interval, I° to denote the interior of I.

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a,b \in I$ with $a<b$, then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This remarkable result is well known in the literature as Hermite-Hadamard's inequality for convex mapping. Both inequalities hold in the reversed direction if f is concave.

We know two kinds of s-convexity/concavity $(0<s\le 1)$ of real valued functions are famous in the literature.

A function $f: \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = [0,+\infty)$ is said to be s-convex function in the first sense, if the inequality

$$f(\alpha\mu+\beta\nu) \le \alpha^s f(\mu) + \beta^s f(\nu)$$

holds for all $\mu,\nu \in \mathbb{R}_+$, and all $\alpha,\beta \ge 0$ with $\alpha^s+\beta^s=1$.

*Corresponding author. *Email addresses:* `yujiaolictgu@163.com` (Y. J. Li), `tingsongdu@ctgu.edu.cn` (T. S. Du)

Definition 1.1. ([7]) The function $f: I\subseteq[0,\infty)\to\mathbb{R}$ is said to be s-convex function in the second sense on I, if the inequality

$$f(\lambda x+(1-\lambda)y)\leq\lambda^s f(x)+(1-\lambda)^s f(y) \tag{1.2}$$

holds for all $x,y\in I$, $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.

In this paper we mainly study Hadamard's integral inequalities for s-convex and s-concave mappings in the second sense. Kavurmaci et al. proved the following result connected with the right part of (1.1) in [9].

Lemma 1.1. ([9] Lemma 1) *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where $a,b\in I$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\\&=\frac{(x-a)^2}{b-a}\int_0^1(t-1)f'(tx+(1-t)a)\mathrm{d}t+\frac{(b-x)^2}{b-a}\int_0^1(1-t)f'(tx+(1-t)b)\mathrm{d}t.\end{aligned} \tag{1.3}$$

In recent years, a lot of inequalities of Hermite-hadamard type for convex and s-convex functions were presented, some of them can be reformulated as the following theorems.

Theorem 1.1. ([6]) *Suppose that $f:[0,\infty)\to[0,\infty)$ is an s-convex function in the second sense, where $s\in(0,1]$, and let $a,b\in[0,\infty)$, $a<b$. If $f\in L[a,b]$, then the following inequalities hold:*

$$2^{s-1}f\Big(\frac{a+b}{2}\Big)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

Theorem 1.2. ([11] Theorem 2.1) *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, where $a,b\in I^\circ$ with $a<b$ and let $q>1$. If $|f'|^q$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\leq\left(\frac{3^{1-\frac{1}{q}}}{8}\right)(b-a)\Big(|f'(a)|+|f'(b)|\Big). \tag{1.5}$$

Theorem 1.3. ([1] Theorem 2.5 and [10] Theorem 2) *Let $f: I\to\mathbb{R}, I\subseteq\mathbb{R}$ be a differentiable mapping on I° such that $f'\in L[a,b]$, where $a,b\in I$, $a<b$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q>1$, then the following inequalities hold:*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\leq\frac{b-a}{4}\Big(\frac{q-1}{2q-1}\Big)^{\frac{q-1}{q}}\left[\left|f'\Big(\frac{a+3b}{4}\Big)\right|+\left|f'\Big(\frac{3a+b}{4}\Big)\right|\right] \tag{1.6}$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\leq\frac{b-a}{4}\Big(\frac{q-1}{2q-1}\Big)^{\frac{q-1}{q}}\left[\left|f'\Big(\frac{a+3b}{4}\Big)\right|+\left|f'\Big(\frac{3a+b}{4}\Big)\right|\right]. \tag{1.7}$$