

## The Traveling Wave of Auto-Catalytic Systems-Monotone and Multi-Peak Solutions

Yuanwei Qi \*

*Department of Mathematics, University of Central Florida, Orlando,  
Florida 32816, USA*

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**Abstract.** This article studies propagating wave fronts of a reaction-diffusion system modeling an isothermal chemical reaction  $A+2B\rightarrow 3B$  involving two chemical species, a reactant  $A$  and an auto-catalyst  $B$ , whose diffusion coefficients,  $D_A$  and  $D_B$ , are unequal due to different molecular weights and/or sizes. Explicit bounds  $c_*$  and  $c^*$  that depend on  $D_B/D_A$  are derived such that there is a unique travelling wave of every speed  $c\geq c^*$  and there does not exist any travelling wave of speed  $c < c_*$ . Furthermore, the reaction-diffusion system of the Gray-Scott model of  $A+2B\rightarrow 3B$ , and a linear decay  $B\rightarrow C$ , where  $C$  is an inert product is also studied. The existence of multiple traveling waves which have distinctive number of local maxima or peaks is shown. It shows a new and very distinctive feature of Gray-Scott type of models in generating rich and structurally different traveling pulses.

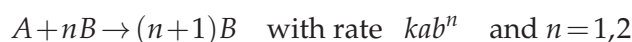
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**Key words:** Cubic autocatalysis, travelling wave, minimum speed, Gray-Scott, multi-peak waves.

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## 1 Introduction

Autocatalytic chemical reaction of the form



between two chemical species  $A$  and  $B$ , appears in many chemical wave models of excitable media from the idealized Brusselator to real-world clock reactions such as Belousov-Zhabotinsky reaction, the Briggs-Rauscher reaction, the Bray-Liebhafsky reaction and the iodine clock reaction. In that setting, their importance was recognized pretty early [13, 14, 29].

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\*Corresponding author. *Email address:* Yuanwei.Qi@ucf.edu (Y. Qi)

More recently, in various models of biological pattern formation of Turing type, for the purpose of replicating experimental results in early 1990s, whether it is CIMA or Gray-Scott [20,22] chemical reaction of the form



with  $C$  an inert chemical species, plays a significant role. In particular, in Gray-Scott model with feeding, self-replicating traveling pulse (traveling wave) is the most exciting and not completely understood phenomenon [9–11, 18].

In this work, we study the traveling wave problem of autocatalytic chemical reaction  $A + nB \rightarrow (n+1)B$ , which, after simple non-dimensionalization results in the reaction-diffusion system,

$$(I) \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^n, \\ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + uv^n, \end{cases}$$

as well as that of chemical reaction  $A + nB \rightarrow (n+1)B$ , and  $B \rightarrow C$ , which has the governing equations

$$(II) \begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^n, \\ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + uv^n - kv^m. \end{cases}$$

Here,  $D$  a positive constant is the ratio of diffusion coefficients of chemical species  $B$  to that of  $A$ ,  $n \geq 1$  is a positive constant not necessarily an integer, and  $kv^m$  describes the rate of  $B \rightarrow C$ , with  $k$  and  $m \geq 1$  both positive constants. We assume throughout that  $1 \leq m \leq n$ .

For a traveling wave solution to (I),  $u(x,t) = u(z)$ ,  $v(x,t) = v(z)$ , where  $z = x - ct$ , the governing ODE system is:

$$\begin{cases} u'' + cu' - uv^n = 0, \\ Dv'' + cv' + uv^n = 0, \end{cases} \quad (1.1)$$

where  $c > 0$  is a constant. Assuming

$$\lim_{z \rightarrow -\infty} (u, v) = (0, a), \quad a > 0,$$

the addition of the two equations and integration on  $(-\infty, z]$  yield

$$u' + Dv' + c(u + v - a) = 0. \quad (1.2)$$