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The Traveling Wave of Auto-Catalytic Systems-Monotone and Multi-Peak Solutions

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Abstract. This article studies propagating wave fronts of a reaction-diffusion system modeling an isothermal chemical reaction $A+2B \to 3B$ involving two chemical species, a reactant A and an auto-catalyst B, whose diffusion coefficients, D_A and D_B, are unequal due to different molecular weights and/or sizes. Explicit bounds c_* and c^* that depend on D_B/D_A are derived such that there is a unique travelling wave of every speed $c \geqslant c^*$ and there does not exist any travelling wave of speed $c<c_*$. Furthermore, the reaction-diffusion system of the Gray-Scott model of $A+2B \to 3B$, and a linear decay $B \to C$, where C is an inert product is also studied. The existence of multiple traveling waves which have distinctive number of local maxima or peaks is shown. It shows a new and very distinctive feature of Gray-Scott type of models in generating rich and structurally different traveling pulses.

AMS subject classifications: 34C20, 34C25, 92E20

Key words: Qubic autocatalysis, travelling wave, minimum speed, Gray-Scott, multi-peak waves.

1 Introduction

Autocatalytic chemical reaction of the form

$$A+nB \to (n+1)B \quad \text{with rate} \quad kab^n \quad \text{and } n=1,2$$

between two chemical species A and B, appears in many chemical wave models of excitable media from the idealized Brusselator to real-world clock reactions such as Belousov-Zhabotinsky reaction, the Briggs-Rauscher reaction, the Bray-Liebhafsky reaction and the iodine clock reaction. In that setting, their importance was recognized pretty early [13,14,29].

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More recently, in various models of biological pattern formation of Turing type, for the purpose of replicating experimental results in early 1990s, whether it is CIMA or Gray-Scott [20, 22] chemical reaction of the form

$$A+2B\rightarrow 3B \quad \text{and} \quad B\rightarrow C,$$

with C an inert chemical species, plays a significant role. In particular, in Gray-Scott model with feeding, self-replicating traveling pulse (traveling wave) is the most exciting and not completely understood phenomenon [9–11, 18].

In this work, we study the traveling wave problem of autocatalytic chemical reaction $A+nB\rightarrow(n+1)B$, which, after simple non-dimensionalization results in the reaction-diffusion system,

$$(I)\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}-uv^n, \\ \dfrac{\partial v}{\partial t}=D\dfrac{\partial^2 v}{\partial x^2}+uv^n, \end{cases}$$

as well as that of chemical reaction $A+nB\rightarrow(n+1)B$, and $B\rightarrow C$, which has the governing equations

$$(II)\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}-uv^n, \\ \dfrac{\partial v}{\partial t}=D\dfrac{\partial^2 v}{\partial x^2}+uv^n-kv^m. \end{cases}$$

Here, D a positive constant is the ratio of diffusion coefficients of chemical species B to that of A, $n\geq 1$ is a positive constant not necessarily an integer, and kv^m describes the rate of $B\rightarrow C$, with k and $m\geq 1$ both positive constants. We assume throughout that $1\leq m\leq n$.

For a traveling wave solution to (I), $u(x,t)=u(z)$, $v(x,t)=v(z)$, where $z=x-ct$, the governing ODE system is:

$$\begin{cases} u''+cu'-uv^n=0, \\ Dv''+cv'+uv^n=0, \end{cases} \tag{1.1}$$

where $c>0$ is a constant. Assuming

$$\lim_{z\to-\infty}(u,v)=(0,a), \qquad a>0,$$

the addition of the two equations and integration on $(-\infty,z]$ yield

$$u'+Dv'+c(u+v-a)=0. \tag{1.2}$$