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On the Benjamin-Bona-Mahony Equation with a Localized Damping

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Abstract. We introduce several mechanisms to dissipate the energy in the Benjamin-Bona-Mahony (BBM) equation. We consider either a distributed (localized) feedback law, or a boundary feedback law. In each case, we prove the global wellposedness of the system and the convergence towards a solution of the BBM equation which is null on a band. If the Unique Continuation Property holds for the BBM equation, this implies that the origin is asymptotically stable for the damped BBM equation.

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Key words: Benjamin-Bona-Mahony equation, unique continuation property, internal stabilization, boundary stabilization.

1 Introduction

The Benjamin-Bona-Mahony equation

$$v_t - v_{xxt} + v_x + vv_x = 0, \tag{1.1}$$

was proposed in [1] as an alternative to the Korteweg-de Vries (KdV) equation as a model for the propagation of one-dimensional, unidirectional, small amplitude long waves in nonlinear dispersive media. In the context of shallow water waves, $v = v(x,t)$ stands for the displacement of the water surface (from rest position) at location x and time t. In the paper, we shall assume that either $x \in \mathbb{R}$, or $x \in (0,L)$ or $x \in \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ (the one-dimensional torus).

The dispersive term $-v_{xxt}$ produces a strong smoothing effect for the time regularity, thanks to which the wellposedness theory of (1.1) is easier than for KdV (see [4,9]). Solutions of (1.1) turn out to be analytic in time. On the other hand, the control theory is at his early stage for BBM (for the control properties of KdV, we refer the reader to the recent

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survey [10]). S. Micu investigated in [5] the boundary controllability of the *linearized* BBM equation, and noticed that the exact controllability fails due to the existence of a *limit point* in the spectrum of the adjoint equation. The author and B.-Y. Zhang introduced in [11] a *moving control* and derived with such a control both the exact controllability and the exponential stability of the full BBM equation. For a distributed control with a *fixed support*, the exact controllability of the linearized BBM equation fails, so that the study of the controllability of the full BBM equation seems hard. However, it is reasonable to expect that some stability properties could be derived by incorporating some dissipation in a fixed subdomain or at the boundary. The aim of this paper is to propose several dissipation mechanisms leading to systems for which one has both the global existence of solutions and a nonincreasing H^1-norm. The conclusion that all the trajectories are indeed attracted by the origin is valid provided that the following conjecture is true:

Unique Continuation Property (UCP) Conjecture: For any $v_0 \in H^1(\mathbb{T})$, if the solution $v = v(x,t)$ of

$$\begin{cases} v_t - v_{xxt} + v_x + vv_x = 0, & x \in \mathbb{T}, \\ v(x,0) = v_0(x), & x \in \mathbb{T} \end{cases} \tag{1.2}$$

satisfies

$$v(x,t) = 0 \quad \forall (x,t) \in \omega \times (0,T) \tag{1.3}$$

for some nonempty open set $\omega \subset \mathbb{T}$ and some time $T > 0$, then $v_0 = 0$ (and hence $v \equiv 0$).

To the best knowledge of the author, the UCP for BBM as stated in the above conjecture is still open. The main difficulty comes from the fact that the lines $x = 0$ are *characteristic* for BBM, so that the "information" does not propagate well in the x-direction. For some UCP for BBM (with additional assumptions) see [11, 12]. See also [6, 7] for control results for some Boussinesq systems of BBM-BBM type.

The following result is a *conditional* UCP in which it is assumed that the initial data is small in the L^∞-norm and it has a *nonnegative* mean value. Its proof was based on the analyticity in time of the trajectories and on the use of some Lyapunov function.

Theorem 1.1. ([11]) *Let $u_0 \in H^1(\mathbb{T})$ be such that*

$$\int_{\mathbb{T}} v_0(x)\, dx \geq 0, \tag{1.4}$$

and

$$\|v_0\|_{L^\infty(\mathbb{T})} < 3. \tag{1.5}$$

Assume that the solution v to (1.2) *satisfies* (1.3) *for some nonempty open set $\omega \subset \mathbb{T}$ and some $T > 0$. Then $v_0 = 0$.*

As it was noticed in [11], the UCP for BBM cannot hold for any state in $L^2(\mathbb{T})$, for any initial data v_0 with values in $\{-2, 0\}$ gives a trivial (stationary) solution of BBM. Thus, either a regularity assumption ($v_0 \in H^1(\mathbb{T})$), or a bound on the norm of the initial data has to be imposed for the UCP to hold.