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Kuramoto-Sivashinsky Equation and Free-interface Models in Combustion Theory

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Abstract. In combustion theory, a thin flame zone is usually replaced by a free interface. A very challenging problem is the derivation of a self-consistent equation for the flame front which yields a reduction of the dimensionality of the system. A paradigm is the Kuramoto-Sivashinsky (K–S) equation, which models cellular instabilities and turbulence phenomena. In this survey paper, we browse through a series of models in which one reaches a fully nonlinear parabolic equation for the free interface, involving pseudo-differential operators. The K–S equation appears to be asymptotically the lowest order of approximation near the threshold of stability.

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Key words: Free interface, combustion theory, Kuramoto-Sivashinsky equation, instability, fully nonlinear parabolic equation.

1 Introduction

Interface phenomena are commonplace in physics, chemistry, biology, and various disciplines bridging these fields (see Fife [18]), such as combustion and flame. The latter domain constitutes an intricate physical system involving fluid dynamics, multistep chemical kinetics, and molecular and radiative heat transfer. In the middle of the 20th century, the Russian School [23] introduced formal asymptotic methods based on large activation energy which have allowed simpler descriptions, especially when a thin flame zone is replaced by a free interface, commonly called the flame front. A very challenging problem is the derivation of a single equation for the free interface, which may capture most of the

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dynamics and, as a consequence, yields a reduction of the effective dimensionality of the system.

In premixed gas combustion, thermal-diffusive instability is a result of the competition between the exothermic reaction and the heat diffusion, which in turn exhibits chaotic dynamics. Near the instability threshold it is possible to (asymptotically) separate the spatial and temporal coordinates, and further reduce the system to a single geometrically invariant surface dynamics equation (see Frankel and Sivashinsky [22]):

$$V_n = 1 + (\alpha - 1)\kappa + \kappa_{ss}, \tag{1.1}$$

where V_n is the normal velocity of the flame sheet, s is the arc-length along the interface, and κ is its curvature. The parameter, α, reflects the physico-chemical characteristics of the combustible; cellular instability occurs when α exceeds unity.

The coordinate-free model (1.1), especially its weakly-nonlinear approximation, the Kuramoto-Sivashinsky (K–S) equation:

$$\Phi_\tau + \nu \Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta} + \frac{1}{2}(\Phi_\eta)^2 = 0, \quad \nu > 0, \tag{1.2}$$

appears in a variety of domains in physics and chemistry which include free interfaces. As it models cellular instabilities (see Sivashinsky [33]), pattern formation, turbulence phenomena (see Kuramoto [27, 28] who independently derived K–S in a study of turbulence in the Belousov-Zhabotinsky reaction), and transition to chaos (see Hyman and Nicolaenko [25]), the K–S equation has received considerable attention from the mathematical community (see Temam [34] and the references therein). Several authors have restricted their attention to a differentiated version of (1.2) which includes a nonlinearity of Burgers type.

The K–S model comprises a balance between several effects: loosely speaking, this equation arises when the competing effects of a destabilizing linear part $\nu\Phi_{\eta\eta\eta\eta} + \Phi_{\eta\eta}$ and a stabilizing nonlinearity $\frac{1}{2}\Phi_\eta^2$ are the dominant processes. The linear instability is itself the result of a competition between two linear operators, $\nu D^2_{\eta\eta}$ (stabilizing) and $D_{\eta\eta}$ (destabilizing). Comparable competition holds in other dissipative systems with similar dynamics, such as the Burgers-Sivashinsky equation (see Berestycki *et al.* [1]) and the Q–S equation (see Section 2.2).

In this survey, we browse through a series of free-interface problems in combustion theory (also a model for supercooling). Our viewpoint is twofold:
(i) First, after a number of simplifications, derive a self-consistent equation for the interface (more precisely for the corrugated perturbation of a planar front) whose general form on an interval $(-\ell/2, \ell/2)$ with periodic boundary conditions reads:

$$\frac{\partial}{\partial t}\mathscr{B}\varphi = \mathscr{S}(\varphi) + \mathscr{F}((\varphi_y)^2), \tag{1.3}$$

or, inverting operator $\mathscr{B}$ whenever it is possible:

$$\varphi_t = \mathscr{L}(\varphi) + \mathscr{G}((\varphi_y)^2). \tag{1.4}$$