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A Fast Shift-Splitting Method for Singular Generalized Saddle Point Problems

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Abstract. For the iteration solution of singular generalized saddle point problems, a fast shift-splitting iteration method based on shift-splitting technique and symmetric and skew-symmetric splitting with respect to the upper-left block of the system matrix is proposed in this paper. Semi-convergence of the proposed method is carefully studied for singular case, and the conditions guaranteeing the semi-convergence are derived. Numerical experiments of a class of linearized Navier-Stokes equations are implemented to demonstrate the feasibility and effectiveness of the proposed method.

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Key words: Shift-splitting iteration method, singular generalized saddle point problems, semi-convergence.

1 Introduction

Consider the solution of systems of linear equations with the following block 2×2 structure

$$\begin{bmatrix} A & B^T \\ -B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \text{ or } \mathcal{A}u = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a nonsymmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a rectangular matrix with $m \leq n$, $C \in \mathbb{R}^{m\times m}$ is a symmetric positive semi-definite matrix, $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^m$ are given vectors.

The generalized saddle point problems, i.e., the systems of the form (1.1) arise in a variety of scientific computing and engineering applications, including computational fluid

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dynamics [7], mixed finite element approximation of elliptic partial differential equations [17], weighted and equality constrained least squares estimation [9], inversion of geophysical data [20] and others.

If the matrix B is of full column rank, i.e., $\text{rank}(B)=m\leq n$, the generalized saddle point matrix $\mathcal{A}$ is nonsingular. If the matrix B is rank deficient and $\text{null}(C)\cap\text{null}(B^T)\neq\{0\}$, the generalized saddle point matrix $\mathcal{A}$ is singular. Here, $\text{null}(\cdot)$ denotes the null space of the corresponding matrix. In this work, we are particularly interested in the latter. And, for the singular case, we suppose that b is in the range of $\mathcal{A}$, that is, the singular saddle point problems are consistent in this paper.

In many cases A,B and C are large sparse matrices and iterative techniques are preferable for solving (1.1). In recent years, many effective methods have been proposed for solving singular saddle point problems in the literature, for example, the Uzawa-type methods [5,26,27], Hermitian and skew-Hermitian splitting type methods [1–3,14,21,22], and Krylov subspace methods [18,25]. Parameterized Uzawa method was studied in [27], and the semi-convergence of this method was proved when it was applied to solve the singular saddle point problems. Minimum residual and conjugate gradient methods were proposed for solving the rank-deficient saddle point problems in [18, 25], respectively. Inexact Uzawa method, which covers the Uzawa method, the preconditioned Uzawa method, and the parameterized method as special cases, was discussed for singular saddle point problems in [26], and the semi-convergence result under restrictions was proved by verifying two necessary and sufficient conditions. Moreover, sufficient conditions for the semi-convergence of several Uzawa-type methods were also provided in [26].

In this paper, we construct a fast shift-splitting iteration method for singular generalized saddle point problems based on the ideas of the shift-splitting iteration method [6, 12] and the Hermitian and skew-Hermitian splitting technique [4, 21, 28]. The idea of shift-splitting iteration method was first proposed by Bai, Yin and Su in [6] for solving a class of non-Hermitian positive definite linear systems. Then it was extended by Cao, Du and Niu in [11] to solve saddle point problems, and it was generalized by Salkuyeh for saddle point problems in [23]. After that, for nonsymmetric saddle point problems, Cao and Miao in [13] proposed the generalized shift-splitting (GSS) method. Recently, Shen and Shi applied the GSS iteration method to solve a broad class of nonsingular and singular generalized saddle point problems in [24]. In this paper, a fast shift-splitting iteration method is studied. Semi-convergence of this method for singular case is carefully analyzed. Numerical experiments further show that the proposed method is efficient and feasible.

The rest of this paper is organized as follows. In Section 2, a fast shift-splitting iteration method for singular generalized saddle point problems is established. In Section 3, the semi-convergence of the proposed method for singular case is studied. Numerical experiments are presented in Section 4. Finally, a brief conclusion is given in Section 5.

2 The fast shift-splitting iteration method

The coefficient matrix $\mathcal{A}$ can be split as follows

$$\mathcal{A}=\mathcal{M}-\mathcal{N}=\begin{pmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{pmatrix}-\begin{pmatrix}\alpha I-S & 0\\ 0 & \alpha I\end{pmatrix},$$

where $\alpha>0$ is a constant, the matrices H and S are the symmetric (Hermitian) part and skew-symmetric (skew-Hermitian) part of the matrix A, respectively, i.e., $H=\frac{1}{2}(A+A^T)$, $S=\frac{1}{2}(A-A^T)$. I is the identity matrix with appropriate dimension. By this special splitting, the fast shift-splitting iteration method can be defined as follows.

The fast shift-splitting (FSS) iteration method: Given an initial guess $((x^{(0)})^T,(y^{(0)})^T)^T$, for $k=0$, 1, 2, … until $((x^{(k)})^T,(y^{(k)})^T)^T$ converges, compute

$$\begin{pmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{pmatrix}\begin{pmatrix}x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\begin{pmatrix}\alpha I-S & 0\\ 0 & \alpha I\end{pmatrix}\begin{pmatrix}x^{(k)}\\ y^{(k)}\end{pmatrix}+\begin{pmatrix}f\\ g\end{pmatrix}. \tag{2.1}$$

For the singular saddle point problems, the matrix $\mathcal{M}$ is invertible because of

$$\begin{aligned}&\begin{pmatrix}I & 0\\ B(\alpha I+H)^{-1} & I\end{pmatrix}\begin{pmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{pmatrix}\begin{pmatrix}I & -(\alpha I+H)^{-1}B^T\\ 0 & I\end{pmatrix}\\ &=\begin{pmatrix}\alpha I+H & 0\\ 0 & (\alpha I+C)+B(\alpha I+H)^{-1}B^T\end{pmatrix}.\end{aligned}$$

Thus, the iteration scheme (2.1) can be rewritten as

$$\begin{pmatrix}x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\Gamma\begin{pmatrix}x^{(k)}\\ y^{(k)}\end{pmatrix}+\mathcal{M}^{-1}\begin{pmatrix}f\\ g\end{pmatrix}, \tag{2.2}$$

where

$$\Gamma=\begin{pmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{pmatrix}^{-1}\begin{pmatrix}\alpha I-S & 0\\ 0 & \alpha I\end{pmatrix} \tag{2.3}$$

is the iteration matrix of the FSS iteration method.

As a matter of fact, any matrix splitting not only can automatically lead to a splitting iteration method, but also can naturally induce a splitting preconditioner for Krylov subspace methods like GMRES, or its restarted version. The splitting preconditioner corresponds to the FSS iteration method (2.1) is given by

$$\mathcal{M}=\begin{pmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{pmatrix}, \tag{2.4}$$

which is called the FSS preconditioner for the saddle point matrix $\mathcal{A}$.

At each step of the FSS iteration (2.1) or applying preconditioner $\mathcal{M}$ within a Krylov subspace methods, a linear system with $\mathcal{M}$ as the coefficient matrix needs to be solved. That is to say, linear systems of the form

$$\mathcal{M}z=r \tag{2.5}$$

needs to be solved for a given vector r at each step, where $r=(r_1^T,r_2^T)^T$ and $z=(z_1^T,z_2^T)^T$(r_1, $z_1\in\mathbb{R}^n$, r_2, $z_2\in\mathbb{R}^m$). The following algorithmic version can be derived to solve (2.5).

Algorithm 2.1. For a given vector $r=(r_1^T,r_2^T)^T$, the vector $z=(z_1^T,z_2^T)^T$ can be computed according to the following steps:

1. Solve $(\alpha I+C)w=r_2$ for w;
2. Compute $w_1=r_1-B^Tw$;
3. Solve $(\alpha I+H+B^T(\alpha I+C)^{-1}B)z_1=w_1$;
4. Solve $(\alpha I+C)v=Bz_1$ for v;
5. $z_2=v+w$.

In the Algorithm 2.1, a linear system with coefficient matrices $\alpha I+H+B^T(\alpha I+C)^{-1}B$ and $\alpha I+C$ are required to be solved at each iteration. Since the coefficient matrices are symmetric positive definite for any $\alpha>0$, the sub-linear systems with the coefficient matrices $\alpha I+C$ and $\alpha I+H+B^T(\alpha I+C)^{-1}B$ can be solved by the conjugate gradient (CG) method or some direct methods, such as, the Cholesky or LU factorization in combination with AMD or column AMD reordering.

For the singular matrix $\mathcal{A}$, only the semi-convergence of the FSS iteration scheme (2.2) is required [8]. In the following discussion, we discuss the semi-convergence of the FSS method for singular saddle point problems.

3 Semi-convergence analysis

To analyze the semi-convergence, some notations are given firstly. Let $\rho(A)$, $\sigma(A)$, null(A), rank(A) and index(A) be the spectral radius, the spectral set, the null space, the rank and the index of the matrix A, respectively. Next, the semi-convergence about the iteration scheme (2.2) is described in the following lemma.

Lemma 3.1. ([8]) *When $\mathcal{A}$ is singular, then* 1 *is an eigenvalue of the iteration matrix* Γ. *Moreover, when the spectral radius of the iteration matrix* Γ *is equal to* 1, *that is,* $\rho(\Gamma)=1$, *the following two conditions are necessary and sufficient for guaranteeing the semi-convergence of the iteration scheme* (2.2)*:*

(1) The elementary divisors of the iteration matrix Γ *associated with* $\lambda=1\in\sigma(\Gamma)$ *are linear, i.e.,* $\text{rank}(I-\Gamma)^2=\text{rank}(I-\Gamma)$, *or equivalently,* $\text{index}(I-\Gamma)=1$;

(2) If $\lambda\in\sigma(\Gamma)$, *the spectrum of the iteration matrix* Γ, *satisfying* $|\lambda|=1$, *then* $\lambda=1$, *i.e.,*

$\gamma(\Gamma)<1$, *where*

$$\gamma(\Gamma)=\max\{|\lambda|\,|\,\lambda\in\sigma(\Gamma),\,\lambda\neq 1\}.$$

Lemma 3.1 describes the semi-convergence property of the iteration scheme (2.2) when $\mathcal{A}$ is singular. Therefore, to get the semi-convergence property of the FSS iteration method, only the two conditions in Lemma 3.1 need to verify. To begin with, the first condition is considered in the following Theorem 3.1.

3.1 The conditions for $\mathrm{rank}(I-\Gamma)^2=\mathrm{rank}(I-\Gamma)$

Theorem 3.1. *Let A be nonsymmetric positive definite, B be rank deficient, C be symmetric positive semi-definite and* $\mathrm{null}(C)\cap\mathrm{null}(B^T)\neq\{0\}$. *Assume that $\alpha>0$ and Γ is the iteration matrix of the FSS iteration method, then* $\mathrm{rank}(I-\Gamma)^2=\mathrm{rank}(I-\Gamma)$.

Proof. Since $\Gamma=\mathcal{M}^{-1}\mathcal{N}=I-\mathcal{M}^{-1}\mathcal{A}$, $\mathrm{rank}(I-\Gamma)^2=\mathrm{rank}(I-\Gamma)$ holds if

$$\mathrm{null}((\mathcal{M}^{-1}\mathcal{A})^2)=\mathrm{null}(\mathcal{M}^{-1}\mathcal{A}).$$

It is obvious that $\mathrm{null}((\mathcal{M}^{-1}\mathcal{A})^2)\supseteq\mathrm{null}(\mathcal{M}^{-1}\mathcal{A})$. Now, we only need to show

$$\mathrm{null}((\mathcal{M}^{-1}\mathcal{A})^2)\subseteq\mathrm{null}(\mathcal{M}^{-1}\mathcal{A}). \tag{3.1}$$

Let $p=\begin{bmatrix}p_1\\p_2\end{bmatrix}\in\mathrm{null}((\mathcal{M}^{-1}\mathcal{A})^2)$ with $p_1\in\mathbb{R}^n$ and $p_2\in\mathbb{R}^m$. Denote by $q=\mathcal{M}^{-1}\mathcal{A}p$. Let $q=\begin{bmatrix}q_1\\q_2\end{bmatrix}\in\mathbb{R}^{n+m}$. It must be satisfied

$$(\mathcal{M}^{-1}\mathcal{A})^2p=(\mathcal{M}^{-1}\mathcal{A})q=0. \tag{3.2}$$

To prove (3.1), we only need to prove $q=0$, i.e., $q_1=0$ and $q_2=0$. On one hand, since the matrix $\mathcal{M}$ is nonsingular, from (3.2) we have

$$\begin{cases} Aq_1+B^Tq_2=0,\\ -Bq_1+Cq_2=0. \end{cases} \tag{3.3}$$

Since A is nonsymmetric positive definite, from the first equality of (3.3) we can easily get $q_1=-A^{-1}B^Tq_2$. Then substituting this relationship into the second equality of (3.3), we obtain $(C+BA^{-1}B^T)q_2=0$ and $q_2^TCq_2+(B^Tq_2)^TA^{-1}B^Tq_2=0$. Owing to the symmetric positive semi-definiteness of the matrix C and the positive definiteness of the matrix A^{-1}, we obtain $Cq_2=0$ and $B^Tq_2=0$. Taking $B^Tq_2=0$ into the first equality of (3.3), we get $q_1=0$. On the other hand, from the relationship between the vectors p and q, we have

$$\begin{bmatrix}A & B^T\\ -B & C\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}=\begin{bmatrix}\alpha I+H & B^T\\ -B & \alpha I+C\end{bmatrix}\begin{bmatrix}q_1\\q_2\end{bmatrix}. \tag{3.4}$$

From above discussion, we know that $q_1=0$ and $q_2\in\text{null}(C)\cap\text{null}(B^T)$. Then (3.4) can be equivalently rewritten as

$$\begin{cases} Ap_1+B^Tp_2=0, \\ -Bp_1+Cp_2=\alpha q_2. \end{cases} \tag{3.5}$$

Solving p_1 from the first equality of (3.5) and substituting it into the second equality of (3.5) gives

$$q_2=\frac{1}{\alpha}(C+BA^{-1}B^T)p_2.$$

Owing to the positive definiteness of the matrix A^{-1} and $q_2\in\text{null}(C)\cap\text{null}(B^T)$, we have $(C+BA^{-T}B^T)q_2=0$ and

$$\frac{1}{\alpha^2}p_2^T(C+BA^{-1}B^T)^T(C+BA^{-1}B^T)p_2=0,$$

which implies $(C+BA^{-1}B^T)p_2=0$ and therefore $q_2=0$. Thus, the proof is completed. □

3.2 The condition for $\gamma(\Gamma)<1$

Now we consider the second condition $\gamma(\Gamma)<1$ for semi-convergence. For this purpose, some analyses of iteration matrix Γ are first given. The matrix Γ is equivalent to the following form:

$$\Gamma=\begin{pmatrix}\alpha I+H & B^T \\ B & -\alpha I-C\end{pmatrix}^{-1}\begin{pmatrix}\alpha I-S & 0 \\ 0 & -\alpha I\end{pmatrix}.$$

Denote

$$\Omega=\begin{pmatrix}\alpha I & \\ & -\alpha I\end{pmatrix},\ \mathcal{H}=\begin{pmatrix}H & B^T \\ B & -C\end{pmatrix},\ \mathcal{S}=\begin{pmatrix}S & 0 \\ 0 & 0\end{pmatrix}.$$

Without loss of generality, we assume that the symmetric positive semi-definite matrix C has rank r and can be factorized as

$$C=\begin{bmatrix}E & F\end{bmatrix}\begin{bmatrix}D & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}E^T \\ F^T\end{bmatrix},$$

where $D\in\mathbb{R}^{r\times r}$ is a diagonal matrix, whose elements are nonzero eigenvalues of the matrix C. $[E\ \ F]\in\mathbb{R}^{m\times m}$ is orthogonal and columns of the matrix $F\in\mathbb{R}^{m\times(m-r)}$ span the null space of C.

Consider additionally the orthogonal matrix

$$Q_1=\begin{bmatrix}I & 0 & 0 \\ 0 & E & F\end{bmatrix}.$$

Then iteration matrix Γ is similar to the following matrix

$$\begin{aligned}\hat{\Gamma} &= Q_1^T \Gamma Q_1 \\ &= Q_1^T (\Omega+\mathcal{H})^{-1}(\Omega-\mathcal{S})Q_1 \\ &= (Q_1^T(\Omega+\mathcal{H})Q_1)^{-1}(Q_1^T(\Omega-\mathcal{S})Q_1) \\ &= \begin{bmatrix} \alpha I+H & B^TE & B^TF \\ E^TB & -\alpha I-D & 0 \\ F^TB & 0 & -\alpha I \end{bmatrix}^{-1} \begin{bmatrix} \alpha I-S & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & -\alpha I \end{bmatrix} \\ &\triangleq \begin{bmatrix} \Omega+\hat{\mathcal{H}} & \hat{B}^T \\ \hat{B} & -\alpha I \end{bmatrix}^{-1} \begin{bmatrix} \Omega-\hat{\mathcal{S}} & 0 \\ 0 & -\alpha I \end{bmatrix},\end{aligned}$$

where

$$\hat{\mathcal{H}} = \begin{bmatrix} H & B^TE \\ E^TB & -D \end{bmatrix},\ \hat{\mathcal{S}} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+r)\times(n+r)},\ \hat{B}^T = \begin{bmatrix} B^TF \\ 0 \end{bmatrix} \in \mathbb{R}^{(n+r)\times(m-r)}.$$

Since the matrix B is rank deficient, $\text{null}(C)\cap\text{null}(B^T)\neq\{0\}$ and the columns of F span the null space of the matrix C, the matrix B^TF is also rank deficient. Without loss of generality, we further assume that the rank of the matrix B^TF is $t(<m-r)$. Let the singular value decomposition of the matrix $\hat{B}$ be $\hat{B} = U\begin{bmatrix}\hat{B}_t \\ 0\end{bmatrix}V^T$ where $U\in\mathbb{R}^{(m-r)\times(m-r)}$ and $V\in\mathbb{R}^{(n+r)\times(n+r)}$ are orthogonal matrices, $\hat{B}_t = [\Sigma_t\ 0]\in\mathbb{R}^{t\times(n+r)}$ with $\Sigma_t = \text{diag}(\sigma_1,\sigma_2,\ldots,\sigma_t)\in\mathbb{R}^{t\times t}$ being a diagonal matrix.

Consider the block diagonal matrix

$$Q_2 = \begin{pmatrix} V & 0 \\ 0 & U \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)},$$

which is an orthogonal matrix. Then $\hat{\Gamma}$ is similar to

$$\begin{aligned}\tilde{\Gamma} &= Q_2^T \hat{\Gamma} Q_2 \\ &= \begin{bmatrix} \Omega+V^T\hat{\mathcal{H}}V & \hat{B}_t^T & 0 \\ \hat{B}_t & -\alpha I & 0 \\ 0 & 0 & -\alpha I \end{bmatrix}^{-1} \begin{bmatrix} \Omega-V^T\hat{\mathcal{S}}V & 0 & 0 \\ 0 & -\alpha I & 0 \\ 0 & 0 & -\alpha I \end{bmatrix} \\ &= \begin{bmatrix} Q_3^T(\Omega+\tilde{\mathcal{H}})^{-1}(\Omega-\tilde{\mathcal{S}})Q_3 & 0 \\ 0 & I \end{bmatrix},\end{aligned}$$

where

$$Q_3 = \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix},\ \tilde{\mathcal{H}} = \begin{bmatrix} \hat{\mathcal{H}} & \hat{B}_t^T \\ \hat{B}_t & 0 \end{bmatrix},\ \tilde{\mathcal{S}} = \begin{bmatrix} \hat{\mathcal{S}} & 0 \\ 0 & 0 \end{bmatrix}$$

are all $(n+r+t)\times(n+r+t)$ matrices. Since the generalized shift-splitting matrix Γ is similar to $\tilde{\Gamma}$, $m-r-t$ eigenvalues of Γ are equal to one. At the same time, verifying the

condition for $\gamma(\Gamma)<1$ is equivalent to studying the conditions for the tenability of the inequality $\rho((\Omega+\tilde{\mathcal{H}})^{-1}(\Omega-\tilde{\mathcal{S}}))<1$.

And, the matrix $(\Omega+\tilde{\mathcal{H}})^{-1}(\Omega-\tilde{\mathcal{S}})$ is equivalent to the iteration matrix of iteration scheme

$$\begin{bmatrix}\alpha I+\hat{\mathcal{H}} & \hat{B}_t^T\\ -\hat{B}_t & \alpha I\end{bmatrix}\begin{bmatrix}\hat{x}_{k+1}\\ \hat{y}_{k+1}\end{bmatrix}=\begin{bmatrix}\alpha I-\hat{\mathcal{S}} & 0\\ 0 & \alpha I\end{bmatrix}\begin{bmatrix}\hat{x}_k\\ \hat{y}_k\end{bmatrix}+\begin{bmatrix}\hat{f}\\ \hat{g}\end{bmatrix}, \tag{3.6}$$

which can be used to solve the following nonsingular saddle point problems

$$\begin{bmatrix}\hat{A} & \hat{B}_t^T\\ -\hat{B}_t & 0\end{bmatrix}\begin{bmatrix}\hat{x}\\ \hat{y}\end{bmatrix}=\begin{bmatrix}\hat{f}\\ \hat{g}\end{bmatrix}, \tag{3.7}$$

where

$$\hat{A}=\begin{bmatrix}A & B^TE\\ E^TB & -D\end{bmatrix}\in\mathbb{R}^{(n+r)\times(n+r)},$$

and the iteration matrix of the iteration scheme (3.6) is

$$\mathcal{T}=\begin{bmatrix}\alpha I+\hat{\mathcal{H}} & \hat{B}_t^T\\ -\hat{B}_t & \alpha I\end{bmatrix}^{-1}\begin{bmatrix}\alpha I-\hat{\mathcal{S}} & 0\\ 0 & \alpha I\end{bmatrix}. \tag{3.8}$$

Hence, studying the tenable conditions of the inequality $\rho(\mathcal{T})<1$ is transformed into discussing the convergence of the iteration scheme (3.6) used to solve the saddle point problems (3.7).

Lemma 3.2. *Let λ be an eigenvalue of the matrix $\mathcal{T}$ and $[\hat{u}^*,\hat{v}^*]^*$ be the corresponding eigenvector, then $\lambda\neq 1$.*

Proof. From (3.8) we have

$$\begin{bmatrix}\alpha I-\hat{\mathcal{S}} & 0\\ 0 & \alpha I\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}=\begin{bmatrix}\alpha I+\hat{\mathcal{H}} & \hat{B}_t^T\\ -\hat{B}_t & \alpha I\end{bmatrix}\begin{bmatrix}\lambda\hat{u}\\ \lambda\hat{v}\end{bmatrix}, \tag{3.9}$$

If $\lambda=1$, then from (3.9) we have

$$\begin{bmatrix}\hat{A} & \hat{B}_t^T\\ -\hat{B}_t & 0\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{v}\end{bmatrix}=0.$$

Since the coefficient matrix is nonsingular, we have $\hat{u}=0$ and $\hat{v}=0$, which contradicts with the assumption that $[\hat{u}^*,\hat{v}^*]^*$ is an eigenvector of the iteration matrix $\mathcal{T}$. So $\lambda\neq 1$. □

Lemma 3.3. ([5]) *Both roots of the complex quadratic equation $\lambda^2+\phi\lambda+\varphi=0$ have modulus less than one if and only if $|\phi-\bar{\phi}\varphi|+|\varphi|^2<1$, where $\bar{\phi}$ denotes the conjugate number of ϕ.*

By Theorem 3.1 in [15], the condition theorem of $\gamma(\Gamma)<1$ can be given as follow.

Theorem 3.2. *Let A be nonsymmetric positive definite, B be rank deficient, C be symmetric positive semi-definite and* $\mathrm{null}(C)\cap\mathrm{null}(B^T)\neq\{0\}$. *Suppose that $\alpha>0$ is a given constant. Let λ be an eigenvalue of the matrix $\mathcal{T}$ and $[\hat{u}^*,\hat{v}^*]^*$ be the corresponding eigenvector. Denote*

$$a=\frac{\hat{u}^*\hat{\mathcal{H}}\hat{u}}{\hat{u}^*\hat{u}},\quad b=\frac{\hat{u}^*\hat{B}_t^T\hat{B}_t\hat{u}}{\hat{u}^*\hat{u}},\quad is=\frac{\hat{u}^*\hat{\mathcal{S}}\hat{u}}{\hat{u}^*\hat{u}},\tag{3.10}$$

where i is the imaginary unit. Then the pseudo-spectral radius of the matrix Γ is less than 1, *i.e., $\gamma(\Gamma)<1$ if and only if the parameter α satisfies the following conditions*

$$\begin{cases}\alpha^2s^2<\alpha^2a^2+b^2+2a\alpha^3+2b\alpha^2+2ab\alpha,\\ 4a^2\alpha^4+2a\alpha^3(a^2+4b-s^2)+(5a^2b+4b^2-3bs^2)\alpha^2+4ab^2\alpha+b^3>0.\end{cases}\tag{3.11}$$

Proof. From (3.9) we have

$$\begin{cases}(\lambda-1)\alpha\hat{u}+\lambda\hat{B}_t^T\hat{v}+\lambda\hat{\mathcal{H}}\hat{u}+\hat{\mathcal{S}}\hat{u}=0,\\ \lambda\hat{B}_t\hat{u}+(1-\lambda)\alpha\hat{v}=0.\end{cases}\tag{3.12}$$

By Lemma 3.2, we know that $\lambda\neq1$. In addition, we can get $\hat{u}\neq0$. Otherwise, by (3.12) we have $(1-\lambda)\alpha\hat{v}=0$. Then, it follows that $\hat{v}=0$, which contradicts with the assumption that $[\hat{u}^*,\hat{v}^*]^*$ is an eigenvector. Now, solving $\hat{v}$ from the second equation of (3.12) and substituting it into the first one, we have

$$\alpha^2(\lambda-1)^2\hat{u}+\alpha(\lambda-1)(\lambda\hat{\mathcal{H}}+\hat{\mathcal{S}})\hat{u}+\lambda^2\hat{B}_t^T\hat{B}_t\hat{u}=0.\tag{3.13}$$

Multiplying $\frac{\hat{u}^*}{\hat{u}^*\hat{u}}$ to the two sides of the equation (3.13) from the left yields

$$(\alpha^2+a\alpha+b)\lambda^2+(-a\alpha-2\alpha^2+is\alpha)\lambda+\alpha^2-is\alpha=0.\tag{3.14}$$

(I) If $\alpha^2+a\alpha+b=0$, then $\lambda=\frac{a\alpha+b+is\alpha}{a\alpha+2b+is\alpha}$, then $|\lambda|<1$ due to $b>0$.
(II) If $\alpha^2+a\alpha+b\neq0$, the quadratic equation (3.14) can be written as $\lambda^2+\phi\lambda+\varphi=0$, where

$$\phi=\frac{-a\alpha-2\alpha^2+is\alpha}{\alpha^2+a\alpha+b},\ \ \varphi=\frac{\alpha^2-is\alpha}{\alpha^2+a\alpha+b}.$$

By Lemma 3.3, we know that $|\lambda|<1$ if and only if $|\phi-\bar{\phi}\varphi|+|\varphi|^2<1$, Define three auxiliary functions

$$\begin{aligned}h_1(\alpha)&=\alpha^4+\alpha^2s^2,\\ h_2(\alpha)&=(\alpha^2+a\alpha+b)^2,\\ h_3(\alpha)&=(a^2\alpha^2+ab\alpha+2a\alpha^3+2b\alpha^2-s^2\alpha^2)^2+\alpha^2s^2b^2,\end{aligned}$$

then we have

$$|\phi-\bar{\phi}\varphi|+|\varphi|^2=\frac{h_1(\alpha)+\sqrt{h_3(\alpha)}}{h_2(\alpha)},$$

and $|\phi-\bar{\phi}\varphi|+|\varphi|^2<1$ if and only if

$$h_1(\alpha)-h_2(\alpha)<0,\ \text{ and }\ (h_1(\alpha)-h_2(\alpha))^2-h_3(\alpha)>0.$$

By careful calculation, we obtain that the iteration scheme (3.6) is convergent if and only if the parameter α satisfies the inequality (3.11). Hence, $\gamma(\Gamma)<1$ if and only if the parameter α satisfies the inequality (3.11). □

Finally, the semi-convergence theorem of the FSS iteration method for singular saddle point problems readily follows from Lemma 3.1, Theorem 3.1 and Theorem 3.2.

Theorem 3.3. *Let A be nonsymmetric positive definite, B be rank deficient, C be symmetric positive semi-definite and* $\text{null}(C)\cap\text{null}(B^T)\neq\{0\}$. *Suppose that $\alpha>0$ is a given constant. Then the FSS iteration method is semi-convergent for solving singular saddle point problems if the inequality* (3.11) *is valid.*

Remark 3.1. The FSS iteration method is a stationary iteration. Even with the optimal choice of the parameter, the convergence of the stationary iteration (2.1) is typically too slow for the method to be competitive. For this reason, we propose using a nonsymmetric Krylov subspace method like GMRES, or its restarted version GMRES(m), to accelerate the convergence of the iteration. A clustered spectrum of the preconditioned matrix $\mathcal{M}^{-1}\mathcal{A}$ often translates in rapid convergence of GMRES. Since $\mathcal{M}^{-1}\mathcal{A}=I-\Gamma$, from Theorem 3.3 we can see that the nonzero eigenvalues of the preconditioned matrix $\mathcal{M}^{-1}\mathcal{A}$ are located in a circle centered at $(1,0)$ with the radius strictly less than 1 under some conditions, which is a desirable property for Krylov subspace acceleration.

4 Numerical experiments

In this section, some numerical experiments are presented to illustrate the feasibility and effectiveness of the FSS iteration method for singular generalized saddle point problems (1.1). The GSS iteration method studied recently [24] are compared with the FSS iteration method from aspects of iteration steps (denoted by 'IT') and the elapsed CPU time in seconds (denoted by 'CPU'). In addition, numerical results of well-known GMRES method, the GSS and the FSS preconditioned GMRES methods are also given, which can further show that the induced FSS preconditioner is much better than the induced GSS preconditioner for the generalized saddle point problems. The GSS iteration scheme is defined as follows

$$\frac{1}{2}\begin{pmatrix}\alpha I+A & B^T\\ -B & \beta I+C\end{pmatrix}\begin{pmatrix}x^{(k+1)}\\ y^{(k+1)}\end{pmatrix}=\frac{1}{2}\begin{pmatrix}\alpha I-A & -B^T\\ B & \beta I-C\end{pmatrix}\begin{pmatrix}x^{(k)}\\ y^{(k)}\end{pmatrix}+\begin{pmatrix}f\\ g\end{pmatrix}, \tag{4.1}$$

where α and β are two positive constants. The GSS preconditioner, which is induced by the iteration (4.1), is defined by

$$\mathcal{P}_{GSS}=\frac{1}{2}\begin{pmatrix}\alpha I+A & B^T\\ -B & \beta I+C\end{pmatrix}. \tag{4.2}$$

All experiments are performed in MATLAB on a personal computer with Intel Core i5 CPU 2.50 GHz, 4.0GB memory.

Table 1: Numerical results of different iteration methods (ν=1).

Methods		Grids			
		8×8	16×16	32×32	64×64
GSS	α	0.8	0.4	0.2	0.2
	β	0.1	0.01	0.001	0.0001
	IT	30	55	111	227
	CPU	0.03	0.21	2.33	22.44
	RES	7.87e-7	9.89e-7	9.73e-7	8.11e-7
FSS	α	0.001	0.001	0.001	0.0001
	IT	4	5	8	4
	CPU	0.01	0.02	0.17	0.42
	RES	3.46e-8	1.88e-7	8.00e-7	5.83e-7

Table 2: Numerical results of different preconditioned GMRES methods (ν=1).

Preconditioner		Grids			
		8×8	16×16	32×32	64×64
$\mathcal{I}$	IT	50	108	209	386
	CPU	0.04	0.28	3.03	33.16
	RES	9.73e-7	8.51e-7	9.91e-7	9.73e-7
$\mathcal{P}_{GSS}$ (β=0.25)	α	0.2968	0.0764	0.0192	0.0048
	IT	18	26	29	50
	CPU	0.01	0.04	0.49	7.82
	RES	5.43e-7	6.59e-7	5.59e-7	9.02e-7
$\mathcal{P}_{FSS}$	α	0.001	0.001	0.001	0.0001
	IT	4	5	8	6
	CPU	0.00	0.01	0.16	1.58
	RES	6.25e-8	3.28e-7	2.22e-7	4.89e-7

Consider the following two dimensional linearized Navier-Stokes equation, namely, Oseen equation [17]

$$\begin{cases} -\nu\triangle u+(\omega\cdot\bigtriangledown)u+\bigtriangledown p=f, \\ \bigtriangledown\cdot u=0, \end{cases} \tag{4.3}$$

in a bounded domain. Here the vector field ω is the approximation of u from the previous Picard iteration. The parameter $\nu>0$ represents viscosity. Various approximation schemes can be used to discretize the Oseen problem (4.3), which can lead to a generalized saddle point system of type (1.1). A is the discretization of the convection and diffusion terms, which is nonsymmetric positive definite. B^T is the discrete gradient, B is the discrete divergence, C is the local stabilization matrix, which is symmetric positive semi-definite. f and g contain forcing and boundary terms. The test problem using the IFISS software package [16] is the leaky-lid driven cavity problem. The discretization used is stabilized Q1-P0 finite elements. In all cases the default value of the stabiliza-

Table 3: Numerical results of different iteration methods (ν=0.1).

Methods		Grids			
		8×8	16×16	32×32	64×64
GSS	α	0.1	0.1	0.05	0.05
	β	0.25	0.06	0.01	0.001
	IT	69	103	161	418
	CPU	0.06	0.32	2.52	32.02
	RES	9.53e-7	9.47e-7	9.57e-7	8.28e-7
FSS	α	0.001	0.001	0.001	0.0001
	IT	6	5	6	4
	CPU	0.02	0.02	0.11	0.31
	RES	2.29e-7	1.53e-7	4.32e-7	3.24e-7

Table 4: Numerical results of different preconditioned GMRES methods (ν=0.1).

Preconditioner		Grids			
		8×8	16×16	32×32	64×64
$\mathcal{I}$	IT	71	118	189	335
	CPU	0.09	0.30	1.81	17.53
	RES	8.93e-7	9.07e-7	9.66e-7	9.84e-7
$\mathcal{P}_{GSS}$ ($\beta=0.25$)	α	0.0299	0.0079	0.002	0.0005
	IT	11	17	23	22
	CPU	0.01	0.02	0.36	5.80
	RES	9.30e-7	6.94e-7	4.67e-7	9.34e-7
$\mathcal{P}_{FSS}$	α	0.001	0.001	0.001	0.0001
	IT	4	5	7	6
	CPU	0.00	0.01	0.10	1.58
	RES	6.25e-8	6.04e-7	2.99e-7	3.79e-7

tion parameter τ=0.25 is used. It should be mentioned that the generalized saddle point matrix generated by this package is actually singular, since B has rank $m-2$ and the condition $\text{null}(C)\cap\text{null}(B^T)\neq\{0\}$ is satisfied.

To get test problems, three viscosity values $\nu=1$, 0.1, 0.01 and four grids, 8×8, 16×16, 32×32, 64×64, are considered. To implement these iteration methods efficiently and get fast convergence rate of the corresponding preconditioned GMRES methods, the parameters involved in these methods should be chosen appropriately. In this paper, for the FSS iteration, the parameters are the experimentally found optimal ones, which result in the least iteration steps. For the GSS preconditioner, the parameter α is chosen to be the same as the parameter in [24]. In all the test problems discussed in this section, the sub-linear systems are solved by direct methods, which corresponds to computing the Cholesky or LU factorization in combination with AMD or column AMD reordering in MATLAB. In all experiments, the initial vector x_0 is set to be the zero vector and the

Table 5: Numerical results of different iteration methods (ν=0.01).

Methods		Grids			
		8×8	16×16	32×32	64×64
GSS	α	0.05	0.03	0.02	0.01
	β	1.0	0.04	0.008	0.002
	IT	151	283	416	693
	CPU	0.11	0.85	6.48	54.29
	RES	9.47e-7	9.82e-7	9.92e-7	9.91e-7
FSS	α	0.03	0.03	0.02	0.005
	IT	26	45	180	159
	CPU	0.02	0.14	3.77	12.29
	RES	8.48e-7	9.17e-7	9.69e-7	9.90e-7

Table 6: Numerical results of different preconditioned GMRES methods (ν=0.01).

Preconditioner		Grids			
		8×8	16×16	32×32	64×64
$\mathcal{I}$	IT	123	279	489	784
	CPU	0.26	1.87	13.90	113.68
	RES	9.37e-7	9.86e-7	9.94e-7	9.89e-7
$\mathcal{P}_{GSS}$ ($\beta=0.25$)	α	0.0026	0.0008	0.0002	0.0001
	IT	8	13	25	47
	CPU	0.01	0.07	0.40	12.49
	RES	3.83e-7	3.67e-7	7.07e-7	7.92e-7
$\mathcal{P}_{FSS}$	α	0.001	0.0001	0.0001	0.0001
	IT	14	19	20	23
	CPU	0.01	0.02	0.30	6.01
	RES	6.25e-8	6.04e-7	2.99e-7	3.79e-7

relative residual error (denoted by 'RES') is defined by $\text{RES} := \|b-\mathcal{A}u_k\|_2/\|b\|_2$. Iterations are terminated if the relative residual error satisfies $\text{RES} < 10^{-6}$ or the prescribed iteration step is larger than 2500. Note that in the following given tables, '$\mathcal{I}$' denotes the GMRES method without preconditioning.

Table 1 gives the iteration steps, the CPU time and the relative error of the GSS and FSS methods with $\nu = 1$. The optimal parameters of the FSS method are also listed in Table 1. From Table 1, it can be seen that the iteration steps and the elapsed CPU time of the proposed FSS iteration method are much less than those of the GSS method. These results show that the proposed FSS iteration method is much efficient.

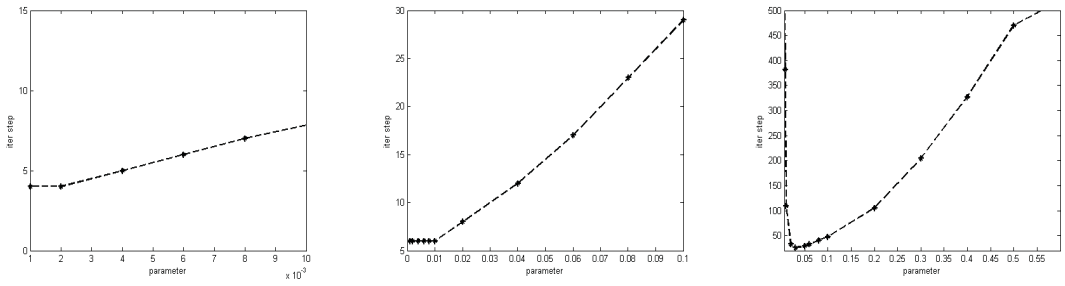
In Table 2, numerical results of the GMRES, the GSS and FSS preconditioned GMRES methods are given for $\nu = 1$. The optimal parameters of the FSS preconditioned method are also presented. Numerical results show that both GSS and FSS preconditioners can accelerate the convergence rate of the GMRES greatly. Besides, both the iteration steps

and the elapsed CPU time show that the proposed FSS preconditioner has advantages over the GSS preconditioner.

In Tables 3 and 5, the numerical results and the optimal parameters of the GSS and FSS iteration methods for $\nu=0.1$ and $\nu=0.01$ are given, respectively. From Tables 3 and 5, it can be seen that FSS iteration method with the optimal iteration parameters succeed to quickly produce approximate solutions. Moreover, the FSS method always outperforms the GSS method considerably in iteration steps, CPU time.

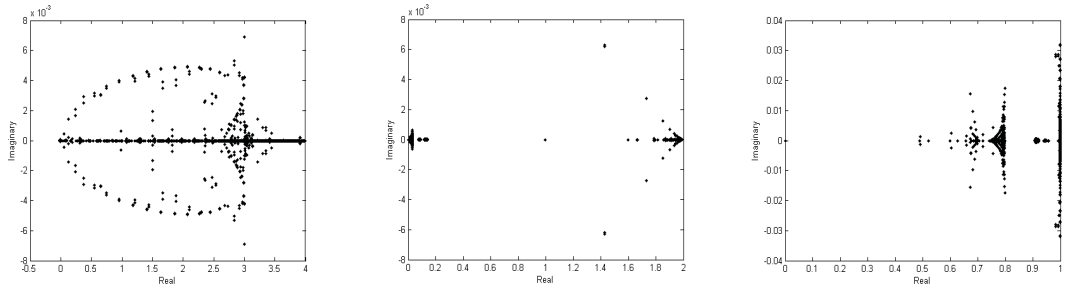
Tables 4 and 6 give the iteration steps, the CPU time and the relative error for the GMRES and two preconditioned GMRES methods for $\nu=0.1$ and $\nu=0.01$, respectively. From Tables 4 and 6, it can be seen that the iteration steps and CPU time of the FSS preconditioned GMRES method are less than those of the GSS preconditioned GMRES method.

In order to investigate the dependence of the FSS iteration method on the iteration parameter α, we illustrate the changing of its iteration steps with respect to different α in Figure 1. Here, we choose three case with $\nu=1$, 0.1 and 0.01 on a 8×8 grid. From Figure 1, it is seen that iteration steps get larger as the parameter increases when $\nu=1$ and 0.1, while $\nu=0.01$ its iteration steps change dramatically with the varying of α. Besides, it is



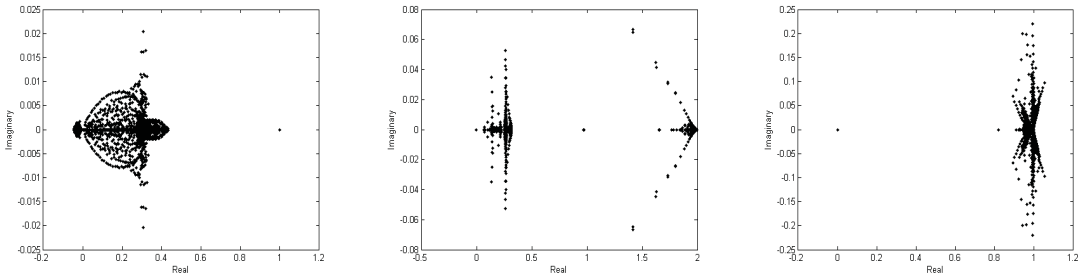
(a) $\nu=1$ (b) $\nu=0.1$ (c) $\nu=0.01$

Figure 1: Iteration steps of the FSS iteration method versus parameter α on a 8×8 grid.



(a) Original matrix (b) GSS preconditioned matrix (c) FSS preconditioned matrix

Figure 2: Eigenvalue of the original matrix and preconditioned matrices on a 32×32 grid($\nu=1$).



(a) Original matrix (b) GSS preconditioned matrix (c) FSS preconditioned matrix

Figure 3: Eigenvalue of the original matrix and preconditioned matrices on a 32×32 grid($\nu=0.1$).

observed that the iteration steps of the FSS method increase greatly when the viscosity parameter ν becomes small. The main reason is the Oseen problem usually becomes harder to solve as the viscosity gets smaller.

Figure 2 shows the eigenvalue distribution of the original matrix and the preconditioned matrices for ν=1 on a 32×32 grid. The graph on the left corresponds to the original matrix, the one in the middle to the GSS preconditioned matrix and the one on the right to the FSS preconditioned matrix. The eigenvalues of the original matrix and the preconditioned matrices for $\nu=0.1$ on a 32×32 grid are displayed in Figure 3. From Figures 2 and 3, it is observed that all preconditioned matrices have much clustered eigenvalues compared with original matrix. Note the strongest clustering of the eigenvalues for the case with the FSS preconditioned matrix, which in turn leads to the fastest convergence rate. In addition, the optimal parameter α is chosen to approximate to zero, and then the condition of the semi-convergence is tenable, which means that the numerical results verify the theoretical analyses.

5 Conclusion

In this paper, a fast shift-splitting iteration method is proposed for singular generalized saddle point problems. Semi-convergence conditions are presented. Numerical results show the proposed FSS iteration method is feasible and effective.

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