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Weak Convergence Theorems for Mixed Type Total Asymptotically Nonexpansive Mappings in Uniformly Convex Banach Spaces

Gurucharan Singh Saluja*

Department of Mathematics, Govt. Nagarjuna P.G. College of Science, Raipur 492010 (C.G.), India.

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Abstract. In this paper, we study a new two-step iteration scheme of mixed type for two total asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings and establish some weak convergence theorems in the framework of uniformly convex Banach spaces. Our results extend and generalize several results from the current existing literature.

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Key words: Total asymptotically nonexpansive self and non-self mapping, mixed type iteration scheme, common fixed point, uniformly convex Banach space, weak convergence.

1 Introduction and preliminaries

Let C be a nonempty subset of a real Banach space E and $T\colon C\to C$ a nonlinear mapping. $F(T)$ denotes the set of fixed points of the mapping T, that is, $F(T)=\{x\in C: Tx=x\}$, $F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$ denotes the set of common fixed points of the mappings S_1, S_2, T_1 and T_2 and $\mathbb{N}$ denotes the set of all positive integers.

Definition 1.1. A mapping T is said to be total asymptotically nonexpansive [1] if

$$\|T^n(x)-T^n(y)\|\le \|x-y\|+\mu_n\psi(\|x-y\|)+\nu_n, \tag{1.1}$$

for all $x,y\in C$ and $n\in\mathbb{N}$, where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n\to 0$ and $\nu_n\to 0$ as $n\to\infty$ and a strictly increasing continuous function $\psi\colon [0,\infty)\to[0,\infty)$ with $\psi(0)=0$.

*Corresponding author. *Email address:* saluja1963@gmail.com (G. S. Saluja)

From the definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [4] for more details.

Remark 1.1. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\nu_n = 0$, $\mu_n = k_n - 1$ for all $n \ge 1$, $\psi(t) = t$, $t \ge 0$.

Definition 1.2. A subset C of a Banach space E is said to be a retract of E if there exists a continuous mapping $P\colon E \to C$ (called a retraction) such that $P(x) = x$ for all $x \in C$. If, in addition P is nonexpansive, then P is said to be a nonexpansive retract of E.

If $P\colon E \to C$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.3. Let C be a nonempty closed convex subset of a Banach space E. A non-self mapping $T\colon C \to E$ is said to be total asymptotically nonexpansive [18] if there exist sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0,\infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0,\infty) \to [0,\infty)$ with $\psi(0) = 0$ such that

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \le \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{1.2}$$

for all $x,y \in C$ and $n \in \mathbb{N}$.

For the sake of convenience, we restate the following concepts and results.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

Definition 1.4. Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E. The space E has:

(i) G*â*teaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $\mathcal{S}$.

(ii) Fr*é*chet differentiable norm [14] if for each x in $\mathcal{S}$, the above limit exists and is attained uniformly for y in $\mathcal{S}$ and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|) \qquad (*)$$

for all $x,h\in E$, where J is the Fréchet derivative of the functional $\frac{1}{2}\|.\|^2$ at $x\in E$, $\langle ..\rangle$ is the pairing between E and E^*, and b is an increasing function defined on $[0,\infty)$ such that $\lim_{t\to 0}\frac{b(t)}{t}=0$.

(iii) Opial condition [8] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty}\|x_n-x\|<\limsup_{n\to\infty}\|x_n-y\|$ for all $y\in E$ with $y\neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces $l^p(1<p<\infty)$. On the other hand, $L^p[0,2\pi]$ with $1<p\neq 2$ fails to satisfy Opial condition.

Definition 1.5. A mapping $T\colon C\to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in C, the condition x_n converges weakly to $x\in C$ and Tx_n converges strongly to 0 imply $Tx=0$.

Definition 1.6. A Banach space E has the Kadec-Klee property [13] if for every sequence $\{x_n\}$ in E, $x_n\to x$ weakly and $\|x_n\|\to\|x\|$ it follows that $\|x_n-x\|\to 0$.

In 2003, Chidume et al. [2] studied the following iteration process for non-self asymptotically nonexpansive mappings:

$$\begin{aligned}&x_1=x\in C,\\&x_{n+1}=P((1-\alpha_n)x_n+\alpha_n T(PT)^{n-1}x_n),\quad n\geq 1,\end{aligned}\tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and proved some strong and weak convergence theorems in the framework of uniformly convex Banach spaces.

In 2004, Chidume et al. [3] studied the following iteration scheme:

$$\begin{aligned}&x_1=x\in C,\\&x_{n+1}=P((1-\alpha_n)x_n+\alpha_n T(PT)^{n-1}x_n),\quad n\geq 1,\end{aligned}\tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, and C is a nonempty closed convex subset of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto C, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

In 2006, Wang [16] generalized the iteration process (1.4) as follows:

$$\begin{aligned}&x_1=x\in C,\\&x_{n+1}=P((1-\alpha_n)x_n+\alpha_n T_1(PT_1)^{n-1}y_n),\\&y_n=P((1-\beta_n)x_n+\beta_n T_2(PT_2)^{n-1}x_n),\quad n\geq 1,\end{aligned}\tag{1.5}$$

where $T_1,T_2\colon C\to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for asymptotically nonexpansive non-self mappings.

Recently, Guo et al. [7] generalized the iteration process (1.5) as follows:

$$\begin{aligned}&x_1=x\in C,\\&x_{n+1}=P((1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n),\\&y_n=P((1-\beta_n)S_2^n x_n+\beta_n T_2(PT_2)^{n-1}x_n),\quad n\geq 1,\end{aligned}\tag{1.6}$$

where $S_1,S_2\colon C\to C$ are two asymptotically nonexpansive self mappings and $T_1,T_2\colon C\to E$ are two asymptotically nonexpansive non-self mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1)$, and proved some strong and weak convergence theorems for mixed type asymptotically nonexpansive mappings.

Now, we define the mixed type iteration scheme.

Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $P\colon E\to C$ is a nonexpansive retraction of E onto C. Let $S_1,S_2\colon C\to C$ be two total asymptotically nonexpansive self mappings and $T_1,T_2\colon C\to E$ are two total asymptotically nonexpansive non-self mappings. Then the mixed type iteration scheme for the mentioned mappings is as follows:

$$\begin{aligned}&x_1=x\in C,\\&x_{n+1}=P((1-\alpha_n)S_1^n x_n+\alpha_n T_1(PT_1)^{n-1}y_n),\\&y_n=P((1-\beta_n)S_2^n x_n+\beta_n T_2(PT_2)^{n-1}x_n),\quad n\geq 1,\end{aligned}\tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$.

Next we state the following useful lemmas to prove our main results.

Lemma 1.1. ([15]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1}\leq(1+\beta_n)\alpha_n+r_n,\quad \forall n\geq 1.$$

If $\sum_{n=1}^{\infty}\beta_n<\infty$ and $\sum_{n=1}^{\infty}r_n<\infty$, then

(i) $\lim_{n\to\infty}\alpha_n$ *exists.*

(ii) In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges strongly to zero, then

$$\lim_{n\to\infty}\alpha_n=0.$$

Lemma 1.2. ([12]) *Let E be a uniformly convex Banach space and $0<\alpha\leq t_n\leq\beta<1$ for all $n\in\mathbb{N}$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n\to\infty}\|x_n\|\leq a,\ \limsup_{n\to\infty}\|y_n\|\leq a,\ \lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=a$$

hold for some $a\geq 0$. Then

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Lemma 1.3. ([13]) *Let E be a real reflexive Banach space with its dual E^* has the Kadec-Klee property. Let $\{x_n\}$ be a bounded sequence in E and $p,q\in w_w(x_n)$ (where $w_w(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty}\|tx_n+(1-t)p-q\|$ exists for all $t\in[0,1]$. Then $p=q$.*

Lemma 1.4. ([13]) *Let K be a nonempty convex subset of a uniformly convex Banach space E. Then there exists a strictly increasing continuous convex function $\phi\colon [0,\infty)\to[0,\infty)$ with $\phi(0)=0$ such that for each Lipschitzian mapping $T\colon C\to C$ with the Lipschitz constant L,*

$$\|tTx+(1-t)Ty-T(tx+(1-t)y)\|\leq L\phi^{-1}\left(\|x-y\|-\frac{1}{L}\|Tx-Ty\|\right)$$

for all $x,y\in K$ and all $t\in[0,1]$.

The purpose of this paper is to study newly define mixed type iteration scheme (1.7) and establish some weak convergence theorems in the setting of uniformly convex Banach spaces.

2 Main results

In this section, we prove some weak convergence theorems of iteration scheme (1.7) for two total asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings in the framework of uniformly convex Banach spaces. First, we shall need the following lemmas.

Lemma 2.1. *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $S_1,S_2\colon C\to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1'}\},\{\mu_{n_1''}\},\{\nu_{n_1'}\},\{\nu_{n_1''}\}\in[0,\infty)$ with $\mu_{n_1'},\mu_{n_1''},\nu_{n_1'},\nu_{n_1''}\to 0$ as $n\to\infty$ and $T_1,T_2\colon C\to E$ are two total asymptotically nonexpansive non-self mappings with sequences $\{\mu_{n'}\},\{\mu_{n''}\},\{\nu_{n'}\},\{\nu_{n''}\}\in[0,\infty)$ with $\mu_{n'},\mu_{n''},\nu_{n'},\nu_{n''}\to 0$ as $n\to\infty$ and*

$$F=F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)\neq\emptyset.$$

Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1) and the following conditions are satisfied:

(i) $\sum_{n=1}^{\infty}\mu_{n_1'}<\infty$, $\sum_{n=1}^{\infty}\mu_{n_1''}<\infty$, $\sum_{n=1}^{\infty}\mu_{n'}<\infty$, $\sum_{n=1}^{\infty}\mu_{n''}<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1'}<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1''}<\infty$, $\sum_{n=1}^{\infty}\nu_{n'}<\infty$, $\sum_{n=1}^{\infty}\nu_{n''}<\infty$;

(ii) there exists a constant $M>0$ such that $\psi(t)\leq Mt$, $t\geq 0$.

Then $\lim_{n\to\infty}\|x_n-q\|$ and $\lim_{n\to\infty}d(x_n,F)$ both exist for all $q\in F$.

Proof. Let $q\in F$ and let $\mu_{n_1}=\max\{\mu_{n_1'},\mu_{n_1''}\}$, $\mu_n=\max\{\mu_{n'},\mu_{n''}\}$, $\nu_{n_1}=\max\{\nu_{n_1'},\nu_{n_1''}\}$, $\nu_n=\max\{\nu_{n'},\nu_{n''}\}$ with $\sum_{n=1}^{\infty}\mu_{n_1}<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1}<\infty$ and $\sum_{n=1}^{\infty}\nu_n<\infty$. Again let $h_n=\max\{\mu_{n_1},\mu_n\}$ and $m_n=\max\{\nu_{n_1},\nu_n\}$ for all $n\in\mathbb{N}$ with $\sum_{n=1}^{\infty}h_n<\infty$ and $\sum_{n=1}^{\infty}m_n<\infty$.

From (1.7), we have

$$
\begin{aligned}
&\|y_n-q\|=\|P((1-\beta_n)S_2^nx_n+\beta_nT_2(PT_2)^{n-1}x_n)-P(q)\|\\
&\le \|(1-\beta_n)S_2^nx_n+\beta_nT_2(PT_2)^{n-1}x_n-q\|\\
&= \|(1-\beta_n)(S_2^nx_n-q)+\beta_n(T_2(PT_2)^{n-1}x_n-q)\|\\
&\le (1-\beta_n)\|S_2^nx_n-q\|+\beta_n\|T_2(PT_2)^{n-1}x_n-q\| \qquad (2.1)\\
&\le (1-\beta_n)[\|x_n-q\|+\mu_{n_1}\psi(\|x_n-q\|)+\nu_{n_1}]+\beta_n[\|x_n-q\|+\mu_n\psi(\|x_n-q\|)+\nu_n]\\
&\le (1-\beta_n)[\|x_n-q\|+h_nM\|x_n-q\|+m_n]+\beta_n[\|x_n-q\|+h_nM\|x_n-q\|+m_n]\\
&\le (1-\beta_n)[(1+h_nM)\|x_n-q\|+m_n]+\beta_n[(1+h_nM)\|x_n-q\|+m_n]\\
&\le (1+h_nM)\|x_n-q\|+m_n. \qquad (2.2)
\end{aligned}
$$

Again using (1.7), we have

$$
\begin{aligned}
&\|x_{n+1}-q\|=\|P((1-\alpha_n)S_1^nx_n+\alpha_nT_1(PT_1)^{n-1}y_n)-P(q)\|\\
&\le \|(1-\alpha_n)S_1^nx_n+\alpha_nT_1(PT_1)^{n-1}y_n-q\|\\
&= \|(1-\alpha_n)(S_1^nx_n-q)+\alpha_n(T_1(PT_1)^{n-1}y_n-q)\|\\
&\le (1-\alpha_n)\|S_1^nx_n-q\|+\alpha_n\|T_1(PT_1)^{n-1}y_n-q\|\\
&\le (1-\alpha_n)[\|x_n-q\|+\mu_{n_1}\psi(\|x_n-q\|)+\nu_{n_1}]+\alpha_n[\|y_n-q\|+\mu_n\psi(\|y_n-q\|)+\nu_n]\\
&\le (1-\alpha_n)[\|x_n-q\|+h_nM\|x_n-q\|+m_n]+\alpha_n[\|y_n-q\|+h_nM\|y_n-q\|+m_n]\\
&= (1-\alpha_n)[(1+h_nM)\|x_n-q\|+m_n]+\alpha_n[(1+h_nM)\times\|y_n-q\|+m_n]\\
&= (1-\alpha_n)(1+h_nM)\|x_n-q\|+\alpha_n(1+h_nM)\|y_n-q\|+m_n. \qquad (2.3)
\end{aligned}
$$

Using equation (2.2) in (2.3), we obtain

$$
\begin{aligned}
&\|x_{n+1}-q\|\le(1-\alpha_n)(1+h_nM)\|x_n-q\|+\alpha_n(1+h_nM)[(1+h_nM)\|x_n-q\|+m_n]+m_n\\
&\le (1+h_nM)^2\|x_n-q\|+(2+h_nM)m_n\\
&= (1+t_n)\|x_n-q\|+s_n, \qquad (2.4)
\end{aligned}
$$

where $t_n=2h_nM+h_n^2M^2$ and $s_n=(2+h_nM)m_n$. Since $\sum_{n=1}^{\infty}h_n<\infty$ and $\sum_{n=1}^{\infty}m_n<\infty$, it follows that $\sum_{n=1}^{\infty}t_n<\infty$ and $\sum_{n=1}^{\infty}s_n<\infty$. Hence from Lemma 1.1 that $\lim_{n\to\infty}\|x_n-q\|$ exists.

Now, taking the infimum over all $q\in F$ in (2.4), we have

$$d(x_{n+1},F)\le (1+t_n)d(x_n,F)+s_n \qquad (2.5)$$

for all $n\in\mathbb{N}$, it follows from $\sum_{n=1}^{\infty}t_n<\infty$, $\sum_{n=1}^{\infty}s_n<\infty$ and Lemma 1.1 that $\lim_{n\to\infty}d(x_n,F)$ exists. This completes the proof. □

Lemma 2.2. *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E. Let $S_1,S_2\colon C\to C$ be two total asymptotically nonexpansive self mappings with sequences $\{\mu_{n_1'}\},\{\mu_{n_1''}\},\{\nu_{n_1'}\},\{\nu_{n_1''}\}\in[0,\infty)$ with $\mu_{n_1'},\mu_{n_1''},\nu_{n_1'},\nu_{n_1''}\to 0$ as $n\to\infty$ and $T_1,T_2\colon C\to E$ be two* *total asymptotically nonexpansive non-self mappings with sequences* $\{\mu_{n'}\},\{\mu_{n''}\},\{\nu_{n'}\},\{\nu_{n''}\}\in [0,\infty)$ *with* $\mu_{n'},\mu_{n''},\nu_{n'},\nu_{n''}\to 0$ *as* $n\to\infty$ *and*

$$F=F(S_1)\bigcap F(S_2)\bigcap F(T_1)\bigcap F(T_2)\neq\emptyset.$$

Let $\{x_n\}$ *be the sequence defined by* (1.7). *If the following conditions hold:*

(i) $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are real sequences in [a,b] for all* $n\in\mathbb{N}$ *and for some* $a,b\in(0,1)$;

(ii) $\mu_{n_1}=\max\{\mu_{n_1'},\mu_{n_1''}\}$, $\mu_n=\max\{\mu_{n'},\mu_{n''}\}$, $\nu_{n_1}=\max\{\nu_{n_1'},\nu_{n_1''}\}$, $\nu_n=\max\{\nu_{n'},\nu_{n''}\}$ *with* $\sum_{n=1}^{\infty}\mu_{n_1}<\infty$, $\sum_{n=1}^{\infty}\mu_n<\infty$, $\sum_{n=1}^{\infty}\nu_{n_1}<\infty$ *and* $\sum_{n=1}^{\infty}\nu_n<\infty$, $h_n=\max\{\mu_{n_1},\mu_n\}$ *and* $m_n=\max\{\nu_{n_1},\nu_n\}$ *for all* $n\in\mathbb{N}$ *with* $\sum_{n=1}^{\infty}h_n<\infty$ *and* $\sum_{n=1}^{\infty}m_n<\infty$;

(iii) *For all* $x,y\in C$, $\|x-T_1(PT_1)^{n-1}y\|\leq\|S_1^n x-T_1(PT_1)^{n-1}y\|$ *and* $\|x-T_2(PT_2)^{n-1}x\|\leq \|S_2^n x-T_2(PT_2)^{n-1}x\|$;

(iv) *there exists a constant* $M>0$ *such that* $\psi(t)\leq Mt,\, t\geq 0$.

Then

$$\lim_{n\to\infty}\|x_n-S_ix_n\|=\lim_{n\to\infty}\|x_n-T_ix_n\|=0 \ for\ i=1,2.$$

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-q\|$ exists for all $q\in F$ and therefore $\{x_n\}$ is bounded. Thus there exists a real number $\varepsilon>0$ such that $\{x_n\}\subseteq C'=\overline{B_\varepsilon(0)}\cap C$, so that C' is a closed convex subset of C. Let $\lim_{n\to\infty}\|x_n-q\|=r$. Then $r>0$ otherwise there is nothing to prove.

Now (2.2) implies that

$$\limsup_{n\to\infty}\|y_n-q\|\leq r. \tag{2.6}$$

Also, we have

$$\begin{aligned}
&\|S_2^n x_n-q\|\leq(1+h_nM)\|x_n-q\|+m_n, && \forall n\in\mathbb{N},\\
&\|T_2(PT_2)^{n-1}x_n-q\|\leq(1+h_nM)\|x_n-q\|+m_n, && \forall n\in\mathbb{N},\\
&\|S_1^n x_n-q\|\leq(1+h_nM)\|x_n-q\|+m_n, && \forall n\in\mathbb{N}.
\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\|S_2^n x_n-q\|\leq r, \tag{2.7}$$

$$\limsup_{n\to\infty}\|T_2(PT_2)^{n-1}x_n-q\|\leq r, \tag{2.8}$$

$$\limsup_{n\to\infty}\|S_1^n x_n-q\|\leq r. \tag{2.9}$$

Next,

$$\|T_1(PT_1)^{n-1}y_n-q\|\leq\,(1+h_nM)\|y_n-q\|+m_n$$

gives by virtue of (2.6) that

$$\limsup_{n\to\infty}\|T_1(PT_1)^{n-1}y_n-q\|\leq r. \tag{2.10}$$

Also, it follows from

$$\begin{aligned}r&=\lim_{n\to\infty}\|x_{n+1}-q\|\\&=\lim_{n\to\infty}\|(1-\alpha_n)S_1^nx_n+\alpha_nT_1(PT_1)^{n-1}y_n-q\|\\&=\lim_{n\to\infty}\|(1-\alpha_n)[S_1^nx_n-q]+\alpha_n[T_1(PT_1)^{n-1}y_n-q]\|\end{aligned}$$

and Lemma 1.2 that

$$\lim_{n\to\infty}\|S_1^nx_n-T_1(PT_1)^{n-1}y_n\|=0. \tag{2.11}$$

By condition (iv), it follows that

$$\|x_n-T_1(PT_1)^{n-1}y_n\|\leq\|S_1^nx_n-T_1(PT_1)^{n-1}y_n\|$$

and so, from (2.11), we have

$$\lim_{n\to\infty}\|x_n-T_1(PT_1)^{n-1}y_n\|=0. \tag{2.12}$$

From (1.7) and (2.11), we have

$$\begin{aligned}\|x_{n+1}-S_1^nx_n\|&\leq\alpha_n\|S_1^nx_n-T_1(PT_1)^{n-1}y_n\|\\&\leq b\|S_1^nx_n-T_1(PT_1)^{n-1}y_n\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{2.13}$$

Hence from (2.11) and (2.13), we have

$$\begin{aligned}&\|x_{n+1}-T_1(PT_1)^{n-1}y_n\|\\&\leq\|x_{n+1}-S_1^nx_n\|+\|S_1^nx_n-T_1(PT_1)^{n-1}y_n\|\to 0 \text{ as } n\to\infty.\end{aligned} \tag{2.14}$$

Now

$$\begin{aligned}\|x_{n+1}-q\|&\leq\|x_{n+1}-T_1(PT_1)^{n-1}y_n\|+\|T_1(PT_1)^{n-1}y_n-q\|\\&\leq\|x_{n+1}-T_1(PT_1)^{n-1}y_n\|+(1+h_nM)\|y_n-q\|+m_n,\end{aligned} \tag{2.15}$$

which gives from (2.14) that

$$r\leq\liminf_{n\to\infty}\|y_n-q\|. \tag{2.16}$$

From (2.6) and (2.16), we obtain

$$r=\|y_n-q\|=\|(1-\beta_n)(S_2^nx_n-q)+\beta_n(T_2(PT_2)^{n-1}x_n-q)\|. \tag{2.17}$$

It follows from Lemma 1.2 that

$$\lim_{n\to\infty}\|S_2^nx_n-T_2(PT_2)^{n-1}x_n\|=0. \tag{2.18}$$

By condition (iv), it follows that

$$\|x_n - T_2(PT_2)^{n-1}x_n\| \le \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$$

and so, from (2.18), we have

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{2.19}$$

Again note that

$$\begin{aligned}
\|y_n - x_n\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n) - P(x_n)\| \\
&\le \|(1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n - x_n\| \\
&= \beta_n \|T_2(PT_2)^{n-1}x_n - S_2^n x_n\| \\
&\le b\|T_2(PT_2)^{n-1}x_n - S_2^n x_n\|.
\end{aligned}$$

Hence from (2.18), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.20}$$

Now, note that

$$\|S_1^n x_n - x_n\| \le \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\|.$$

Hence from (2.11) and (2.12), we obtain

$$\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0. \tag{2.21}$$

Also note that

$$\begin{aligned}
&\|x_{n+1} - x_n\| = \|P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(x_n)\| \\
&\le \|(1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n - x_n\| \\
&= \|(S_1^n x_n - x_n) + \alpha_n(S_1^n x_n - T_1(PT_1)^{n-1}y_n)\| \\
&\le \|S_1^n x_n - x_n\| + \alpha_n \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| \\
&\le \|S_1^n x_n - x_n\| + b\|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| \to 0 \text{ as } n \to \infty,
\end{aligned} \tag{2.22}$$

so that

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0 \text{ as } n \to \infty. \tag{2.23}$$

Since $\|x_n - T_1(PT_1)^{n-1}y_n\| \le \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|$ by condition (iv) and

$$\begin{aligned}
&\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| \\
&\le \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\
&\le \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + (1+h_n M)\|y_n - x_n\| + m_n.
\end{aligned}$$

Using (2.11), (2.20) and $m_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1} x_n\| = 0. \tag{2.24}$$

Now, we have

$$\|x_n - T_1(PT_1)^{n-1} x_n\| \leq \|x_n - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1} x_n\|.$$

Hence from (2.21) and (2.24), we obtain

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1} x_n\| = 0. \tag{2.25}$$

In addition, we have

$$\|x_{n+1} - T_1(PT_1)^{n-1} y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1} y_n\|.$$

Using (2.11) and (2.13), we have

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1} y_n\| = 0. \tag{2.26}$$

It follows from (2.19), (2.21) and the inequality

$$\|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1} x_n\|$$

that

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| = 0. \tag{2.27}$$

Since

$$\begin{aligned}
&\|x_{n+1} - T_2(PT_2)^{n-1} y_n\| \\
&\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| + \|T_2(PT_2)^{n-1} x_n - T_2(PT_2)^{n-1} y_n\| \\
&\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1} x_n\| + (1 + h_n M)\|x_n - y_n\| + m_n,
\end{aligned}$$

from (2.13), (2.20), (2.27) and $m_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1} y_n\| = 0. \tag{2.28}$$

Since T_i for $i = 1,2$ is continuous and P is nonexpansive retraction, it follows from (2.27) that

$$\begin{aligned}
&\|T_i(PT_i)^{n-1} y_{n-1} - T_i x_n\| \\
&= \|T_i[(PT_i)(PT)^{n-2}) y_{n-1}] - T_i(P x_n)\| \to \ 0 \text{ as } n \to \infty,
\end{aligned} \tag{2.29}$$

for $i=1,2$. In addition, we have

$$\begin{aligned}\|x_n-T_1x_n\| \le\ & \|x_n-T_1(PT_1)^{n-1}x_n\|+\|T_1(PT_1)^{n-1}x_n-T_1(PT_1)^{n-1}y_{n-1}\| \\ & +\|T_1(PT_1)^{n-1}y_{n-1}-T_1x_n\| \\ \le\ & \|x_n-T_1(PT_1)^{n-1}x_n\|+(1+h_nM)\|x_n-y_{n-1}\|+m_n \\ & +\|T_1(PT_1)^{n-1}y_{n-1}-T_1x_n\|.\end{aligned}$$

Thus, it follows from (2.23), (2.25), (2.29) and $m_n\to 0$ as $n\to\infty$, that

$$\lim_{n\to\infty}\|x_n-T_1x_n\|=0. \tag{2.30}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|x_n-T_2x_n\|=0. \tag{2.31}$$

Finally, by using cond. (iv), we have

$$\begin{aligned}\|x_n-S_1x_n\| \le\ & \|x_n-T_1(PT_1)^{n-1}x_n\|+\|S_1x_n-T_1(PT_1)^{n-1}x_n\| \\ \le\ & \|x_n-T_1(PT_1)^{n-1}x_n\|+\|S_1^nx_n-T_1(PT_1)^{n-1}x_n\|.\end{aligned}$$

Thus, it follows from (2.24) and (2.25) that

$$\lim_{n\to\infty}\|x_n-S_1x_n\|=0. \tag{2.32}$$

Similarly, we can prove that

$$\lim_{n\to\infty}\|x_n-S_2x_n\|=0. \tag{2.33}$$

This completes the proof. □

Lemma 2.3. *Under the assumptions of Lemma* 2.1*, for all* $p_1,p_2\in F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$*, the limit*

$$\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|$$

exists for all $t\in[0,1]$*, where* $\{x_n\}$ *is the sequence defined by* (1.7).

Proof. By Lemma 2.1, $\lim_{n\to\infty}\|x_n-z\|$ exists for all $z\in F$ and therefore $\{x_n\}$ is bounded. Let

$$a_n(t)=\|tx_n+(1-t)p_1-p_2\|$$

for all $t\in[0,1]$. Then $\lim_{n\to\infty}a_n(0)=\|p_1-p_2\|$ and $\lim_{n\to\infty}a_n(1)=\|x_n-p_2\|$ exists by Lemma 2.1. It, therefore, remains to prove the Lemma 2.3 for $t\in(0,1)$. For all $x\in C$, we define the mapping $W_n\colon C\to C$ by:

$$\begin{aligned}&R_n(x)=P((1-\beta_n)S_2^nx+\beta_nT_2(PT_2)^{n-1}x), \\ &W_n(x)=P((1-\alpha_n)S_1^nx+\alpha_nT_1(PT_1)^{n-1}R_n(x)).\end{aligned}$$

Then it follows that $x_{n+1}=W_nx_n$, $W_np=p$ for all $p\in F$. Now from (2.2) and (2.4) of Lemma 2.1, we see that

$$\begin{aligned}&\|R_n(x)-R_n(y)\|\leq(1+h_nM)\|x-y\|+m_n,\\&\|W_n(x)-W_n(y)\|\leq[1+t_n]\|x-y\|+s_n=f_n\|x-y\|+s_n,\end{aligned}\tag{2.34}$$

where $t_n=2h_nM+h_n^2M^2$ and $s_n=(2+h_nM)m_n$ with $\sum_{n=1}^{\infty}t_n<\infty$ and $\sum_{n=1}^{\infty}s_n<\infty$ and $f_n=1+t_n$. Since $\sum_{n=1}^{\infty}t_n<\infty$, it follows that $f_n\to 1$ as $n\to\infty$. Set

$$\begin{aligned}&S_{n,m}=W_{n+m-1}W_{n+m-2}\dots W_n,\ \ m\in\mathbb{N}\\&b_{n,m}=\|S_{n,m}(tx_n+(1-t)p_1)-(tS_{n,m}x_n+(1-t)S_{n,m}p_2)\|.\end{aligned}\tag{2.35}$$

From (2.34) and (2.35), we have

$$\begin{aligned}&\|S_{n,m}(x)-S_{n,m}(y)\|\\=\;&\|W_{n+m-1}W_{n+m-2}\dots W_n(x)-W_{n+m-1}W_{n+m-2}\dots W_n(y)\|+s_{n+m-1}\\\leq\;&f_{n+m-1}f_{n+m-2}\|W_{n+m-3}\dots W_n(x)-W_{n+m-3}\dots W_n(y)\|+s_{n+m-1}+s_{n+m-2}\\&\vdots\\\leq\;&\Big(\prod_{i=n}^{n+m-1}f_i\Big)\|x-y\|+\sum_{i=n}^{n+m-1}s_i\\=\;&G_n\|x-y\|+\sum_{i=n}^{n+m-1}s_i\end{aligned}\tag{2.36}$$

for all $x,y\in C$, where $G_n=\prod_{i=n}^{n+m-1}f_i$ and $S_{n,m}x_n=x_{n+m}$ and $S_{n,m}p=p$ for all $p\in F$. Thus

$$\begin{aligned}a_{n+m}(t)&=\|tx_{n+m}+(1-t)p_1-p_2\|\\&\leq b_{n,m}+\|S_{n,m}(tx_n+(1-t)p_1)-p_2\|\\&\leq b_{n,m}+G_na_n(t)+\sum_{i=n}^{n+m-1}s_i.\end{aligned}\tag{2.37}$$

By using Theorem 2.3 in [5], we have

$$\begin{aligned}b_{n,m}&\leq\varphi^{-1}(\|x_n-u\|-\|S_{n,m}x_n-S_{n,m}u\|)\\&\leq\varphi^{-1}(\|x_n-u\|-\|x_{n+m}-u+u-S_{n,m}u\|)\\&\leq\varphi^{-1}(\|x_n-u\|-(\|x_{n+m}-u\|-\|S_{n,m}u-u\|))\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m}\to 0$ as $n\to\infty$. Since $\lim_{n\to\infty}G_n=1$ and $\lim_{n\to\infty}s_n=0$, therefore from (2.37), we have

$$\limsup_{n\to\infty}a_n(t)\leq\lim_{n,m\to\infty}b_{n,m}+\liminf_{n\to\infty}a_n(t)=\liminf_{n\to\infty}a_n(t).$$

This shows that $\lim_{n\to\infty}a_n(t)$ exists, that is, $\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|$ exists for all $t\in[0,1]$. This completes the proof. □

Lemma 2.4. *Under the assumptions of Lemma* 2.1*, if E has a Frěchet differentiable norm, then for all $p_1,p_2\in F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$, the limit*

$$\lim_{n\to\infty}\langle x_n,J(p_1-p_2)\rangle$$

exists, where $\{x_n\}$ is the sequence defined by (1.7)*, if $W_w(\{x_n\})$ denotes the set of all weak subsequential limits of $\{x_n\}$, then*

$$\langle q_1-q_2,J(p_1-p_2)\rangle=0$$

for all $p_1,p_2\in F$ and $q_1,q_2\in W_w(\{x_n\})$.

Proof. Suppose that $x=p_1-p_2$ with $p_1\neq p_2$ and $h=t(x_n-p_1)$ in inequality $(*)$. Then, we get

$$\begin{aligned}&\langle x_n-p_1,J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2\\ \le&\frac{1}{2}\|tx_n+(1-t)p_1-p_2\|^2\\ \le&t\langle x_n-p_1,J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2+b(t\|x_n-p_1\|).\end{aligned}$$

Since $\sup_{n\ge1}\|x_n-p_1\|\le K^*$ for some $K^*>0$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_n-p_1,J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2\\ \le&\frac{1}{2}\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|^2\\ \le&t\liminf_{n\to\infty}\langle x_n-p_1,J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2+b(tK^*).\end{aligned}$$

That is,

$$\begin{aligned}&\limsup_{n\to\infty}\langle x_n-p_1,J(p_1-p_2)\rangle\\ \le&\liminf_{n\to\infty}\langle x_n-p_1,J(p_1-p_2)\rangle+\frac{b(tK^*)}{tK^*}K^*.\end{aligned}$$

If $t\to0$, then $\lim_{n\to\infty}\langle x_n-p_1,J(p_1-p_2)\rangle$ exists for all $p_1,p_2\in F$; in particular, we have $\langle q_1-q_2,J(p_1-p_2)\rangle=0$ for all $q_1,q_2\in W_w(\{x_n\})$. This completes the proof. □

Theorem 2.1. *Under the assumptions of Lemma* 2.2*, if E has Frěchet differentiable norm, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. By Lemma 2.4, $\langle q_1-q_2,J(p_1-p_2)\rangle=0$ for all $q_1,q_2\in W_w(\{x_n\})$. Therefore

$$\|q^*-p^*\|^2=\langle q^*-p^*,J(q^*-p^*)\rangle=0$$

implies $q^*=p^*$. Consequently, $\{x_n\}$ converges weakly to a common fixed point in $F=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)$. This completes the proof. □

Theorem 2.2. *Under the assumptions of Lemma* 2.2*, if the dual space E^* of E has the Kadec-Klee (KK) property and the mappings $I-S_i$ and $I-T_i$ for $i=1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. By Lemma 2.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q_* \in C$. By Lemma 2.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k} = 0\| \text{ and } \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i=1,2$. Since by hypothesis the mappings $I-S_i$ and $I-T_i$ for $i=1,2$ are demiclosed at zero, therefore $S_i q_* = q_*$ and $T_i q_* = q_*$ for $i=1,2$, which means $q_* \in F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$. Now, we show that $\{x_n\}$ converges weakly to q_*. Suppose $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $p_* \in C$. By the same method as above, we have $p_* \in F$ and $q_*, p_* \in W_w(\{x_n\})$. By Lemma 2.3, the limit

$$\lim_{n\to\infty} \|t x_n + (1-t) q_* - p_*\|$$

exists for all $t \in [0,1]$ and so $q_* = p_*$ by Lemma 1.3. Thus, the sequence $\{x_n\}$ converges weakly to $q_* \in F$. This completes the proof. □

Theorem 2.3. *Under the assumptions of Lemma* 2.2*, if E satisfies Opial's condition and the mappings $I-S_i$ and $I-T_i$ for $i=1,2$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point of S_1, S_2, T_1 and T_2.*

Proof. Let $u_* \in F$, from Lemma 2.1 the sequence $\{\|x_n - u_*\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $f^* \in C$. From Lemma 2.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - S_i x_{n_k}\| = 0 \text{ and } \lim_{k\to\infty} \|x_{n_k} - T_i x_{n_k}\| = 0$$

for $i=1,2$. Since the mappings $I-S_i$ and $I-T_i$ for $i=1,2$ are demiclosed at zero, therefore $S_i f^* = f^*$ and $T_i f^* = f^*$ for $i=1,2$, which means $f^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to f^*. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $g^* \in C$ and $f^* \neq g^*$. Then by the same method as given above, we can also prove that $g^* \in F$. From Lemma 2.1 the limits $\lim_{n\to\infty} \|x_n - f^*\|$ and $\lim_{n\to\infty} \|x_n - g^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - f^*\| &= \lim_{n_k\to\infty} \|x_{n_k} - f^*\| < \lim_{n_k\to\infty} \|x_{n_k} - g^*\| \\
&= \lim_{n\to\infty} \|x_n - g^*\| = \lim_{n_j\to\infty} \|x_{n_j} - g^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - f^*\| = \lim_{n\to\infty} \|x_n - f^*\|
\end{aligned}$$

which is a contradiction, so $f^* = g^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of S_1, S_2, T_1 and T_2. This completes the proof. □

3 Concluding remarks

In this paper, we study mixed type iteration scheme for two total asymptotically nonexpansive self mappings and two total asymptotically nonexpansive non-self mappings and establish some weak convergence theorems using the following conditions: (a) the space E has a Fre̍chet differentiable norm (b) dual space E^* of E has the Kadec-Klee (KK) property (c) the space E satisfies Opial's condition. Our results extend and generalize the corresponding results of [2, 6, 7, 9–12, 15–17] and many others.

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