J. Math. Study
doi: 10.4208/jms.v50n1.17.02

Vol. **50**, No. 1, pp. 17-27
March 2017

# Characterization by Symmetry of Solutions of a Nonlinear Subelliptic Equation on the Heisenberg Group

Yazhou Han[1], Xiaodong Wang[2] and Meijun Zhu[3,*]

[1] *Department of Mathematics, College of Science, China Jiliang University, Hangzhou, 310018, P.R. China.*
[2] *Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA.*
[3] *Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA.*

Received March 25, 2016; Accepted February 20, 2017

**Abstract.** In this paper we prove that two calculus lemmas, which are used in the method of moving sphere for classifying certain constant curvature equation, also hold on Heisenberg group $\mathbb{H}^n$.

**AMS subject classifications**: 35H20, 22E30

**Key words**: Classification, moving sphere method, Heisenberg group.

## 1 Introduction

The Heisenberg group $\mathbb{H}^n$ consists of the set

$$\mathbb{C}^n\times\mathbb{R}=\Big\{(z,t):z=(z_1,\cdots,z_n)\in\mathbb{C}^n,\ t\in\mathbb{R}\Big\}$$

with the multiplication law

$$(z,t)\circ(z',t')=(z+z',t+t'+2Im(z\cdot\overline{z'})),$$

where $z\cdot\overline{z'}=\sum_{j=1}^n z_j\overline{z'_j}$. As usual, we write $z_j=x_j+\sqrt{-1}y_j$. The Lie algebra is spanned by the left invariant vector fields

$$T=\frac{\partial}{\partial t},\ X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t},\ Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t},\ j=1,\cdots,n.$$

*Corresponding author. *Email addresses:* `yazhou.han@gmail.com` (Y. Han), `xwang@math.msu.edu` (X. Wang), `mzhu@ou.edu` (M. Zhu)

 

The sub-Laplacian is defined by

$$\triangle_H = \sum_{j=1}^{n} (X_j^2 + Y_j^2).$$

In this paper, we shall study the following semilinear subelliptic equation

$$-\Delta_H u = u^p, \quad u \geq 0, \quad \text{in} \quad \mathbb{H}^n, \tag{1.1}$$

where and throughout the paper we always assume that $p > 1$.

This equation is closely related to the study of the $CR$ Yamabe problem. The $CR$ Yamabe problem on a strictly pseudonconvex $CR$ manifold is analogous to the original Yamabe problem on compact manifolds. As the first step, Jerison and Lee [8] proved a sharp Sobolev inequality on the CR sphere $\mathbb{S}^{2n+1}$. Moreover they classified all extremal functions of the sharp Sobolev inequality. Since $\mathbb{S}^{2n+1}$ is CR equivalent to the Heisenberg group via the Cayley transform, their classification can be stated as following.

**Theorem 1.1.** *Let $u \in C^2(\mathbb{H}^n)$ be a solution to equation* (1.1) *with $p = \frac{Q+2}{Q-2}$ (where $Q = 2n+2$ is the homogeneous dimension), and suppose $u \in L^{1+p}$. Then*

$$u(z,t) = |t + \sqrt{-1}|z|^2 + \mu \cdot z + \lambda|^{-n}, \tag{1.2}$$

*where $\mu \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$, and $Im\lambda > |\mu|^2/4$.*

The proof hinges on a complicated and remarkable identity they discovered with the help of a computer program. It is motivated by Obata's classic work [16] in the Riemannian case. When restricted on $\mathbb{R}^n$, Obata's theomem has the following version.

**Theorem 1.2.** *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution for the following equation*

$$-\Delta u = u^{\frac{n+2}{n-2}}, \tag{1.3}$$

*and suppose $u(x) = O(|x|^{2-n})$ for large $|x|$. Then*

$$u(x) = \left( \frac{\lambda}{|x-x_0|^2 + \lambda^2} \right)^{(n-2)/2},$$

*for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$.*

We recall that there is another completely different approach to such classification problems. This is the method of moving plane initiated by Gidas, Ni and Nirenberg [6]. In particular they reproved the above theorem using this method. Later, Caffarelli, Gidas and Spruck [4] classified all positive solutions to equation (1.3) by sharpening the method of moving plane. Their result plays a crucial role in establishing the compactness results for the solution set to Yamabe problem, see, e.g., Schoen [17], Li and Zhu [14],