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High-Order Local Artificial Boundary Conditions for the Fractional Diffusion Equation on One-Dimensional Unbounded Domain

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Abstract. In this paper we consider the numerical solutions of the fractional diffusion equation on the unbounded spatial domain. With the application of Laplace transformation, we obtain one-way equations which absorb the wave touching on the artificial boundaries. By using Padé expansion to approximate the frequency in Laplace space and introducing auxiliary variables to reduce the order of the derivatives with respect to time t, we achieve a system of ODEs within the artificial boundaries. This system of ODEs, called high-order local absorbing boundary conditions (LABCs), reformulate the fractional diffusion problem on the unbounded domain to an initial-boundary-value (IBV) problem on a bounded computational domain. A fully discrete implicit difference scheme is constructed for the reduced problem. The stability and convergence rate are established for a finite difference scheme. Finally, numerical experiments are given to demonstrate the efficiency and accuracy of our approach.

AMS subject classifications: 35R11, 65M06, 65M12

Key words: Fractional subdiffusion equation, high-order absorbing boundary conditions, Laplace transform, Padé expansion, artificial boundary methods.

1 Introduction

In the past two decades, anomalous diffusion phenomena have been observed in a wide range of complex systems ranging from financial markets, movement of active particles in biological systems, to the diffusion in porous medium [1, 2]. The anomalous subdiffusion process, also referred as non-Gaussian phenomena, can be described by the time

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fractional diffusion equation arose in various important physics phenomena, such as ordinary diffusion, dispersive anomalous diffusion [3,4], Pinkin's viscoelasticity [5], porous materials in fractals percolation clusters [6], biological systems [7], random and disorder media [8]. In this paper we consider the numerical solutions of the fractional subdiffusion equation on unbounded domain:

$$
\begin{aligned}
&{}_0^C D_t^\alpha u(x,t) = \kappa_\alpha u_{xx}(x,t) + f(x,t), && x\in\mathbb{R}, 0<t\le T, && (1.1)\\
&u(x,0) = u_0(x), && x\in\mathbb{R}, && (1.2)\\
&u\to 0, && \text{when } |x|\to\infty, && (1.3)
\end{aligned}
$$

where κ_α is the positive diffusion coefficient, the initial value u_0 and the source $f(x,t)$ are the given compactly supported functions, and the Caputo fractional derivative ${}_0^C D_t^\alpha$ with order α is defined by

$$
{}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_s(x,s)}{(t-s)^\alpha} ds, \quad 0<\alpha<1. \tag{1.4}
$$

Recently, much attention has been received on how to solve the fractional diffusion equations in both analytical and numerical viewpoints. The analytical solution of the fractional diffusion equation is generally found by the integral transforms, such as Laplace transform, Fourier transform and Mellin transform. In the literatures, Schneider and Wyss [9] consider a n-dimensional time fractional diffusion in the form of integro-differential equation. Gorenflo *et al.* [10] present a mapping between solutions of fractional diffusion-wave equation in form of a linear integral operator. Mainardi *et al.* [11] gave the fundamental solution to the fractional diffusion-wave equation in one dimensional spatial domain. Barkai [12] discussed an integral transformation which maps a Gaussian type of diffusion onto a fractional diffusion for fractional Fokker-Planck equation. In view of the asymptotic properties of Fox-H functions, Eidelman and Kochubei [13] studied the asymptotic behavior of time regularized fractional diffusion equations. Kilbas *et al.* [14] investigated the Cauchy-type problem for diffusion-wave equations with Riemann-Liouville time-fractional derivative. One can refer to the reviews [1, 2, 12, 15] for various applications and the analytical methods of solving fractional diffusion-type equations.

The analytical solution of a fractional diffusion equation can be expressed in the form of special functions such as Fox-H function, Mittag-Leffler function, Wright function and hyperbolic geometry function. In general, it is difficult to compute the exact solutions, especially for long time, because of the slow convergence of those special functions. On the other hand, it is usually impossible to obtain the exact solution for the general case. These reasons motivate us to enhance the research of numerical methods for fractional differential equations which are valuable tools in exploring numerous phenomenons.

Due to the significant importance of the fractional equations in applications, the numerical solutions received immense interest in recent years. A variety of numerical