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Complex Deformation of Critical Kähler Metrics

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Abstract. In this paper, we use Pacard-Xu's methods to discuss the complex deformation of constant scalar curvature metrics in the case of fixed and varying complex structures. Moreover, we also discuss the complex deformation of Kähler-Ricci solitons.

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Key words: Complex deformation, constant scalar curvature metrics, Kähler Ricci solitons, extremal solitons.

1 Introduction

In [5, 6], Calabi introduced the extremal Kähler metrics, which is the citical point of the L^2 norm of the scalar curvature in the Kähler class. The existence and uniqueness of the extremal Kähler metrics have been intensively studied during past decades([2,7] and reference therein). By Kodaira-Spencer's work [15], every Kähler manifold admits Kähler metrics under small perturbation of the complex structure. A natural question is whether Kähler-Einstein metrics or extremal Kähler metrics still exist when the complex structures varies. In [17], Koiso showed that the Kähler-Einstein metrics can be perturbed under the complex deformation of the complex structure when the first Chern class is zero or negative. When the first Chern class is positive, Koiso showed this result if the manifold has no nontrivial holomorphic vector fields. In [11, 12], Lebrun-Simanca systematically studied the deformation theory of extremal Kähler metrics and constant scalar curvature metrics and they proved that on a Kähler manifold, the set of Kähler classes which admits extremal metrics is open and the constant scalar curvature metrics can be perturbed under some extra restrictions. Based on Lebrun-Simanca's results, Apostolov-Calderbank-Gauduchon-T. Friedman [1], Rollin-Simanca-Tipler [19, 20] further discussed extremal metrics under the deformation of complex structures.

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The main goal of this paper is to give an alternative proof on the deformation of constant scalar curvature metrics, which was discussed by [11] in the case of fixed complex structure, and later by [1, 19] in the case of varying complex structures. Here we use the method of Pacard-Xu in [18] in the context of constant mean curvature problems, which is quite different from [11] in analysis. We will also discuss the deformation of Kähler-Ricci solitons.

First we consider the case of fixed complex structure. The main difficulty of the deformation problems of the Kähler-Einstein metrics or constant scalar curvature metrics is that the linearized equation has nontrivial kernel so that we cannot use the implicit function theorem directly. For this reason, Koiso in [17] assumed that the manifold has no nontrivial holomorphic vector fields, and Lebrun-Simanca in [11] used the surjective version of the implicit function theorem so that the nondegeneracy of the Futaki invariant must be assumed. The same difficulty appears in some other geometrical equations such as the constant mean curvature equation. In [18], Pacard-Xu constructed a new functional to solve the constant mean curvature equation and they removed the nondegeneracy condition of Ye's result in [24]. We observe that Pacard-Xu's method can be applied in our situation and we have the result:

Theorem 1.1. *Let (M,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g. There exists $\epsilon_0>0$ and a smooth function*

$$\Phi:(0,\epsilon_0)\times\mathcal{H}^{1,1}(M)\to\mathbb{R}$$

such that if $\beta\in\mathcal{H}^{1,1}(M)$ has unit norm and satisfies $\Phi(t,\beta)=0$ for some $t\in(0,\epsilon_0)$ then M admits a constant scalar curvature metric in the Kähler class $[\omega_g+t\beta]$. Moreover,

(1) If $\beta\in\mathcal{H}^{1,1}(M)$ is traceless, Φ has the expansion:

$$\Phi(t,\beta)=t^2\int_M(\Pi_g(R_{i\bar{j}}\beta_{j\bar{i}}))^2\omega_g^n+O(t^3).$$

(2) If $\beta\in\mathcal{H}^{1,1}(M)$ is traceless and ω_g is a Kähler-Einstein metric, then Φ has the expansion:

$$\Phi(t,\beta)=t^4\int_M(\Pi_g(\beta_{i\bar{j}}\beta_{j\bar{i}}))^2\omega_g^n+O(t^5).$$

Here the operator Π_g is the projection to the space of Killing potentials with respect to ω_g.

Theorem 1.1 gives us some information in which directions we can find the constant scalar curvature metrics. The function Φ is constructed by the Futaki invariant, and it is automatically zero when the Futaki invariant vanishes. Thus, a direct corollary of Theorem 1.1 is the following result, which was proved by Lebrun-Simanca using the deformation theory of the extremal Kähler metrics and a result of Calabi in [6]:

Corollary 1.1. (Lebrun-Simanca [11]) Let (M,ω_g) be a compact Kähler manifold with a constant scalar curvature metric ω_g. For any $\beta\in\mathcal{H}^{1,1}(M)$, there is a $\epsilon_0>0$ such that if the