Overdetermined Boundary Value Problems in $S^n$

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Abstract. In this paper we use the maximum principle and the Hopf lemma to prove symmetry results to some overdetermined boundary value problems for domains in the hemisphere or star-shaped domains in $S^n$. Our method is based on finding suitable $P$-functions as Weinberger ([26]).

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1 Introduction

In a seminal paper [21], Serrin proved that for a bounded open connected domain $\Omega \subset \mathbb{R}^n$ with sufficient regular boundary $\partial \Omega$, if there exists a solution of the following overdetermined boundary value problem

$$\begin{cases}
\Delta u = n & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} = c & \text{on } \partial \Omega, 
\end{cases} \quad (1.1)$$

where $c$ is a constant, then $\Omega$ must be a ball and $u$ is radially symmetric. Here $\nu$ denotes the outward unit normal of $\partial \Omega$.

The main tool of Serrin’s proof is well-known as the method of moving planes, which is due to Alexandrov. Immediately after Serrin’s paper, Weinberger [26] give an alternative proof of the same result, based on a Rellich-Pohozaev type identity and an interior maximum principle for a subharmonic function (In literatures, it is often referred to as $P$-function). Each of their proofs has its own merits. Serrin’s argument applies to very...

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general partial differential equations if an additional assumption $u > 0$ is added, while Weinberger’s argument is more elementary.

Since the works of Serrin and Weinberger, there have been numerous generalizations to overdetermined problems for general elliptic operators in $\mathbb{R}^n$, the interested readers may refer to [4–6, 8, 11–14, 17, 25] and references therein.

On the other hand, Serrin’s result has been extended to the hemisphere $S^n_+$ and the hyperbolic space $H^n$. Precisely, Molzon [16] considered equation $\Delta u = f(x)$ where $f(x) = \cos r$ (cosh $r$ resp.) in the case $S^n_+$ ($H^n$ resp.) and $r$ is the distance function from a fixed point or $f(x) = n$. Kumaresan and Prajapat [15] considered equation $\Delta u + f(u) = 0$ in $\Omega \subset S^n_+$ or $H^n$, where $f$ is a $C^1$ function. They proved that if $\Delta u + f(u) = 0$ with the boundary condition $u = 0$ and $\frac{\partial u}{\partial \nu} = \text{constant}$ admits a positive solution, then $\Omega$ is a geodesic ball and $u$ is radially symmetric. They used Serrin’s method of moving planes to achieve this, where the positivity of $u$ is an unremovable assumption.

In this paper, we will study an overdetermined problem corresponding to a particular equation on $S^n$: 

$$\begin{cases} 
\Delta u + nu = n & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
\frac{\partial u}{\partial \nu} = c & \text{on } \partial \Omega.
\end{cases} \quad (1.2)$$

Equation (1.2) is related to Schiffer’s problem (See [28] problem 80) on $S^n$. See for instance [1–3, 7, 9, 10, 23, 24, 27] for recent developments of Schiffer’s problem. Previously, Souam [24] showed that for $n = 2$ and $\Omega \subset S^n$ simply connected, if (1.2) admits a solution, then $\Omega$ must be a geodesic ball.

Our first result is the following.

**Theorem 1.1.** Let $\Omega \subset S^n$ be a bounded open connected domain such that $\overline{\Omega}$ is contained in a hemisphere $S^n_+$. If the overdetermined problem (1.2) admits a solution $u$, then $\Omega$ must be a geodesic ball and $u$ is radially symmetric.

We remark that since the first Dirichlet eigenvalue for a domain $\Omega \subset S^n_+$ is strictly larger than $n$, there exists a unique solution for the Dirichlet problem $\Delta u + nu = n$ in $\Omega$ and $u = 0$ on $\partial \Omega$. However, it is not a priori known whether the solution has a definite sign. Therefore, Theorem 1.1 does not follow from the result of Kumaresan and Prajapat [15].

Our approach to Theorem 1.1 is parallel to Weinberger’s, namely, we use a maximum principle for a subharmonic function $P$ and a Rellich-Pohozaev type identity. We remark that our method also applies to equation $\Delta u - nu = n$ in $\Omega \subset H^n$. In this case, $u$ is negative in $\Omega$ by the maximum principle. Hence the conclusion also follows from the result of Kumaresan and Prajapat.

Our next result concerns the same overdetermined problem (1.2) in $\Omega \subset S^n$ without the assumption that $\overline{\Omega}$ is contained in a hemisphere $S^n_+$. Instead, we shall add a star-shapedness assumption on $\Omega$. A domain $\Omega \subset S^n$ is called star-shaped with respect to $p \in S^n$ if $\Omega$ can be written as a graph over a geodesic sphere centered at $p$. It is clear that a