J. Math. Study
doi: 10.4208/jms.v50n3.17.06

Vol. **50**, No. 3, pp. 291-306
September 2017

Optimal Error Estimate of Fourier Spectral Method for the Kawahara Equation

Zhenguo Deng*

School of Mathematics and Information Science, Guangxi University, Nanning 530004, Guangxi, P.R. China.

Received February 13, 2017; Accepted (revised) June 20, 2017

Abstract. An optimal error estimate in L^2-norm for Fourier spectral method is presented for the Kawahara equation with periodic boundary conditions. A numerical example is provided to confirm the theoretical analysis. The method and proving skills are also applicable to the periodic boundary problems for some nonlinear dispersive wave equations provided that the dispersive operator is bounded and antisymmetric and commutes with differentiation.

AMS subject classifications: 65M70, 76B15

Key words: Fourier spectral method, Kawahara equation, error estimate.

1 Introduction

We will analyze Fourier spectral method for the Kawahara equation with periodic boundary conditions:

$$\begin{cases} \partial_t U+\partial_x F(U)+\partial_x^3 U-\partial_x^5 U=0, & x\in\mathbb{R},\ t\in(0,T], \\ U(x+2\pi,t)=U(x,t), & x\in\mathbb{R},\ t\in(0,T], \\ U(x,0)=U_0(x), & x\in\mathbb{R}, \end{cases} \tag{1.1}$$

where U_0 is 2π-periodic in space and $F(U)=\alpha U+U^2/2$, α is a non-negative real constant. The Kawahara equation, also known as fifth-order Korteweg-de Vries equation, arises in the study of several physical phenomena, such as water waves and plasma physics [1–4]. The Fourier spectral methods for the initial- and periodic boundary-value problems of the Kawahara equation have been studied together with time-stepping methods, e.g., the mixture of integrating factor with fourth-order Runge-Kutta method [5], leapfrog

*Corresponding author. *Email address:* `deng@gxu.edu.cn` (Z. G. Deng)

method [6] and the mixture of exponential time differencing with fourth-order Runge-Kutta method [7]. For non-periodic boundary-value problems of the equation, a fully discrete Crank-Nicolson leapfrog dual-Petrov-Galerkin scheme was used in [8,9].

The semi-discrete Fourier spectral method for (1.1) is to find $u_N(t)\in V_N$ such that for any $v\in V_N$ and $t\in(0,T]$,

$$\begin{cases} (\partial_t u_N(t)+\partial_x P_N F(u_N(t))+\partial_x^3 u_N(t)-\partial_x^5 u_N(t),v)=0, \\ (u_N(0),v)=(P_N U_0,v). \end{cases} \tag{1.2}$$

Here $(\cdot,\cdot)$ is the inner product $L^2(I)$, $I=(-\pi,\pi)$, $P_N:L^2(I)\to V_N$ is the Fourier orthogonal projection operator, i.e.,

$$(P_N u-u,v)=0, \qquad v\in V_N,$$

and the approximation space V_N of the real trigonometric polynomials of degree N is defined by

$$V_N=\left\{u(x)=\sum_{k=-N}^{N} a_k \mathrm{e}^{ikx} : \overline{a_k}=a_{-k},\ -N\le k\le N\right\}.$$

This semidiscrete scheme was considered in [6] and an optimal error estimate was claimed. The estimate in [6] was based on the optimal estimate of the projection (2.6). However, an optimal estimate of the numerical solution cannot be obtained even though the error estimate for the projection is optimal, see Remark 2.1. Here we present a new projection, see (2.4) and obtain optimal error estimates for both our projection and the numerical solution.

We will focus on optimal error estimate in L^2-norm of the semi-discrete Fourier spectral method for (1.1). A fully discrete scheme can be obtained when we discretize in time the semi-discrete scheme using the second-order leapfrog-Crank-Nicolson method [10, 11] and optimal error estimate can also be obtained in space. We will not elaborate on the estimate for the fully discrete scheme since the estimate can be done similarly as in [12].

In Section 2, we give some lemmas and theorems needed in the error estimate. In Section 3, we analyze the stability and convergence of the semi-discrete Fourier spectral method. In Section 4, we present a numerical example for the Kawahara equation showing the accuracy of the fully discrete Fourier spectral method in space and time.

2 Preliminaries

In this section, we give some lemmas and theorems needed in the error estimate. Throughout this article, C denotes a generic positive constant, independent of N.

Let $I=(-\pi,\pi)$. The inner product and norm of $L^2(I)$ are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively. For any non-negative real number r, we denote the usual Sobolev space by $H^r(I)$. The subspace of $H^r(I)$ consisting of all periodic functions of period 2π is denoted