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Cubature Points Based Triangular Spectral Elements: an Accuracy Study

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Abstract. We investigate the cubature points based triangular spectral element method and provide accuracy results for elliptic problems in non polygonal domains using various isoparametric mappings. The capabilities of the method are here again clearly confirmed.

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1 Introduction

Extending the capabilities of the quadrangle based spectral element method to the triangle based one, say from the QSEM to the TSEM, has received a considerable attention in the last twenty years. The relevant choice of interpolation points for the triangle has thus motivated a lot of works, see e.g. the review paper [15] and references herein. Several sets of points have indeed been proposed, some of them showing the advantage of simplicity in their generation, thus allowing an easy extension to the tetrahedron, *e.g.*the so-called warp & blend points [21], whereas some other ones are more satisfactory from the theoretical point of view, *e.g.*the celebrated Fekete points of the triangle, but much more difficult to compute [20].

With respect to the QSEM, based on the use of the tensorial product of Gauss-Lobatto-Legendre (GLL) nodes, such interpolation points are however not quadrature points, so that the so called spectral accuracy gets lost if not using for the quadratures a separate set of points, as *e.g.*done for the Fekete-Gauss approach [14]. Using two separate sets of points shows however some drawbacks, because the mass matrix is then no longer diagonal. Thus, when solving an evolution problem with an explicit time scheme, one

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has at each time step to solve a mass matrix algebraic system. Also, a diagonal mass matrix can be useful to set up high order differentiation operators [12]. This is why it is of interest to design a *T*SEM that makes use of a single set of points showing both nice quadrature and interpolation properties. This was the goal of the cubature points based *T*SEM, as developed in the last few years [2,5,9,13].

Another difficulty with SEMs, and more generally with high order FEMs, is to maintain the accuracy when the computational domain is no longer polygonal. Several strategies have been proposed to handle curved elements, from transfinite mappings [6] to PDEs based transformations, *e.g.*the harmonic extension.

The present work follows two anterior studies:

- In [16], using the Fekete-Gauss *T*SEM we compared different strategies for curved triangular spectral elements, *i.e.* when isoparametric mappings are required to correctly approximate with curved spectral elements the presence of a curved boundary.
- In [17], for computational domains of polygonal shape we compared the cubature *T*SEM, as proposed in [9], and the Fekete-Gauss one. In terms of accuracy for elliptic problems, it turned out that these two *T*SEMs compare well. With non Dirichlet or non homogeneous Neumann conditions, *i.e.* as soon as boundary integrals are involved, some care is however needed if using the cubature *T*SEM.

Here the goal is to extend the results obtained in [16] to the case where the cubature *T*SEM is used. As in [16,17] we only focus on elliptic problems, but keep in mind for future works the inherent difficulties associated to constrained operators, see e.g. [1].

2 Cubature points based *T*SEM

For the sake of completeness, let us recall that [17]:

- If two different sets of points are used for interpolation and quadrature, then the space $\mathbb{P}_N(\hat{T})$ of polynomials of maximal (total) degree N, defined on the reference triangle $\hat{T}=\{(r,s):r\in(-1,1),s\in(-1,-r)\}$, is usually used as approximation space. The cardinality of this space equals $n=(N+1)(N+2)/2$, that can be associated to n interpolation points if using Lagrange polynomials as basis functions. If 3 of these nodes coincide with the vertices of the element, then $3N$ of these n points should belong to the edges of $\hat{T}$ and the remaining $(N-1)(N-2)/2$ are the inner nodes. Usually, the edge nodes proposed in the literature coincide with the GLL points. Since one does not know an explicit formulation of the Lagrange basis functions, say $\varphi_i(r,s), 1\leq i\leq n$, to compute their values or those of their derivatives at given point one generally makes use the orthogonal Kornwinder-Dubiner (KD) basis [4], for which explicit formula exist. Gauss points for the triangle and the corresponding quadrature formula may be found in the literature, up to degree $M\approx 20$ if a