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Exact Boundary Controllability on a Planar Tree-Like Network of Vibrating Strings with Dynamical Boundary Conditions

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Abstract. This paper concerns a planar tree-like network of vibrating strings with point masses at the nodes. We use a constructive method with modular structure to get the 'one-sided' exact boundary controllability for this system with dynamical boundary conditions. Moreover, by constructing the 'longest' chain-like subnetwork and its 'midpoint', we divide the whole tree-like network into two tree-like sub-networks, and prove the 'two-sided' exact boundary controllability.

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Key words: A planar tree-like network of vibrating strings, dynamical boundary condition, exact boundary controllability.

1 Introduction

On a planar tree-like network of vibrating strings, the exact boundary controllability for linear wave equations with Dirichlet boundary conditions was studied in [1]- [2] and [8]. In the quasilinear case, by establishing the semi-global piecewise C^2 solution for quasilinear wave equations with Dirichlet, Neumann, Robin and dissipative boundary conditions on a planar tree-like network of strings, Gu & Li ([3]) got the 'one-sided' exact boundary controllability, respectively. Thus, if the network has k simple nodes, then the 'one-sided' exact boundary controllability can be obtained by means of boundary controls acting on $(k-1)$ simple nodes.

For 1-D elastic strings with tip-masses at the ends, based on the theory of semi-global C^2 solution to the corresponding mixed problem, we have established the local exact

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boundary controllability for (a coupled system of) quasilinear wave equations with dynamical boundary conditions in [9, 10] by means of a constructive method with modular schedule (see, e.g., [5, 6]).

In this paper, we consider a planar tree-like network of vibrating strings with point masses at the nodes. Dynamical boundary conditions are encountered in analyzing mechanical behavior of point masses at the simple nodes. While, on the multiple nodes, through the continuity of displacements and the corresponding total stress boundary condition, the interface conditions are obtained. For a tree-like network with k simple nodes, we get its 'one-sided' exact boundary controllability by only $(k-1)$ controls given on simple nodes, which is consistent with the result without point masses in [3]. Moreover, this paper examines the 'two-sided' exact boundary controllability for the first time, that is, the controls are given on all the simple nodes. This result not only leads to a significant reduction of controllability time, but also applies to the case without point masses at nodes, which can be regarded as an important complement to the results in [3].

This paper is organized as follows. The mathematical model and main results of 'one-sided' and 'two-sided' exact boundary controllability on a planar tree-like network are presented in Section 2. To prove these results, in Section 3, we first establish the existence and uniqueness of semi-global piecewise C^2 solution to the corresponding mixed problem on a planar tree-like network. Then based on this, we prove the 'one-sided' exact boundary controllability for this system in Section 4 by constructing the semi-global piecewise C^2 solution, which satisfies simultaneously the initial condition, the final condition, all the given boundary conditions and all the interface conditions. Then, in Section 5, we construct the 'longest' chain-like subnetwork and its 'midpoint', and divide the whole tree-like network into two tree-like sub-networks artificially, then finish the proof of the 'two-sided' boundary controllability.

2 Modeling and main results

In this section, we consider a planar tree-like network which is composed of $N(N>1)$ vibrating strings: $S_1,...,S_N$. Without loss of generality, we suppose that one end of string S_1 is a simple node. We take this simple node as the starting node E of the network. For string S_i, let node d_{i0}(node d_{10} is just E) and node d_{i1} be its two ends, the $x-$ coordinates of which are $x=d_{i0}$ and $x=d_{i1}$, respectively. And $L_i=d_{i1}-d_{i0}$ is the length of string S_i. We always suppose that node d_{i0} is closer to E than node d_{i1} in the network, and all nodes are with unit mass (See Figure 1). The case without masses at nodes can be treated in a similar and simpler way.

Let $\mathcal{M}$ and $\mathcal{S}$ be two subsets of $\{1,...N\}$, $\mathcal{M}\cup\mathcal{S}=\{1,...N\}$. $i\in\mathcal{M}$ if and only if d_{i1} is a multiple node, while, $i\in\mathcal{S}$ if and only if d_{i1} is a simple node. Thus, the collection of simple nodes on the network consists of E and $\{d_{i1},i\in\mathcal{S}\}$, while, the set of multiple nodes is $\{d_{i1},i\in\mathcal{M}\}$.

For $i=1,...,N$, we consider the following quasilinear wave equations on the string S_i

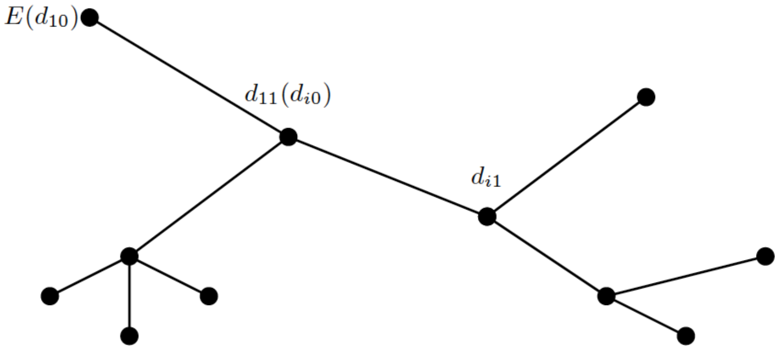

Figure 1: Planar tree-like network with point masses at nodes.

(see [3]):

$$u^i_{tt}-(K^i(u^i,u^i_x))_x=F^i(u^i,u^i_x,u^i_t), \qquad t\geq 0,\ d_{i0}\leq x\leq d_{i1}, \tag{2.1}$$

where $K^i=K^i(u,v)$ is a given C^2 function of u and v, such that

$$K^i_v(u,v)>0, \tag{2.2}$$

and $F^i=F^i(u,v,w)$ is a given C^1 function of u,v,w, satisfying

$$F^i(0,0,0)=0. \tag{2.3}$$

Without loss of generality, we assume that

$$K^i(0,0)=0 \qquad (i=1,\dots,N). \tag{2.4}$$

The initial condition is given by

$$t=0: \qquad u^i=\phi_i(x),\ \ u^i_t=\psi_i(x), \qquad d_{i0}\leq x\leq d_{i1} \quad (i=1,\dots,N). \tag{2.5}$$

At any given simple node E or $d_{i1}\,(i\in\mathcal{S})$, the nonlinear dynamical boundary condition is given as

$$x=d_{10}: \qquad u^1_{tt}=G^1(t,u^1,u^1_x,u^1_t)+h^1(t), \tag{2.6}$$

$$x=d_{i1}: \qquad u^i_{tt}=G^i(t,u^i,u^i_x,u^i_t)+h^i(t) \qquad (i\in\mathcal{S}), \tag{2.7}$$

where $G^i=G^i(t,u^i,u^i_x,u^i_t)$ is a C^1 function of its arguments and $h^i(t)$ is a C^0 function of t ($i=1$ or $i\in\mathcal{S}$), and the conditions of C^2 compatibility are satisfied at the points $(t,x)=(0,d_{10})$ and $(0,d_{i1})(i\in\mathcal{S})$, respectively. Moreover, without loss of generality, we may assume that

$$G^i=G^i(t,0,0,0)\equiv 0, \qquad (i=1 \text{ or } i\,\in\,\mathcal{S}). \tag{2.8}$$

While, on the multiple node $d_{i1}(i\in\mathcal{M})$, we have the interface conditions:

$$\begin{cases} u^j = u^i \qquad (j\in\mathcal{J}_i), \\ u^i_{tt}(t,d_{i1}) = -K^i(u^i,u^i_x) + \sum\limits_{j\in\mathcal{J}_i} K^j(u^j,u^j_x) + H^i(t), \end{cases} \tag{2.9}$$

where $H^i(t)$ is a C^1 function of t, and $\mathcal{J}_i$ denotes the set of all the indices j such that d_{j0} is just the node d_{i1} (see Figure 2).

Since there is no loop in the tree-like network, for any given index $i\in\mathcal{S}$, there exists a unique chain-like subnetwork connecting nodes E and $d_{i1}(i\in\mathcal{S})$. $\mathcal{D}_i$ stands for the set of indices corresponding to all the canals in the unique chain-like subnetwork connecting nodes E and $d_{i1}(i\in\mathcal{S})$, and for simplicity, $\mathcal{D}_i$ also stands for this chain-like subnetwork (see Figure 2).

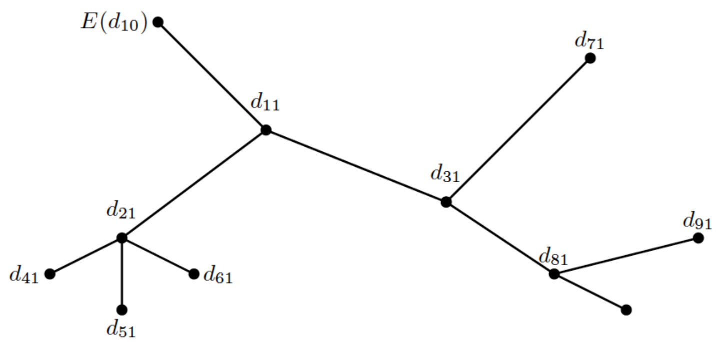

Figure 2: For multiple node d_{11}, $\mathcal{J}_1=\{2,3\}$; for simple node $d_{91}\mathcal{D}_9=\{1,3,8,9\}$.

Theorem 2.1. ***[One-sided exact boundary controllability]*** *Let*

$$T > 2\max_{i\in\mathcal{S}} \sum_{j\in\mathcal{D}_i} \frac{L_j}{\sqrt{K^j_v(0,0)}}, \tag{2.10}$$

for any given initial data (ϕ_i,ψ_i) $(i=1,\dots,N)$ and final data (Φ_i,Ψ_i) $(i=1,\dots,N)$ with small norms

$$\sum_{i=1}^N \|(\phi_i,\psi_i)\|_{C^2[0,L]\times C^1[0,L]} \quad and \quad \sum_{i=1}^N \|(\Phi_i,\Psi_i)\|_{C^2[0,L]\times C^1[0,L]},$$

and for any given function $h^1(t)$ ($h^1(t)\equiv 0$ or $h^1(t)\in C^1[0,T]$ with small C^1 norm) in the boundary condition (2.6) *on the starting node E and any given functions $H^i(t)\in C^1[0,T]$ with small C^1 norms in the interface condition* (2.9) *on all the multiple nodes $d_{i1}(i\in\mathcal{M})$, such that the conditions* *of C^2 compatibility or piecewise C^2 compatibility are satisfied at the points $(t,x)=(0,d_{10}),(T,d_{10})$ and $(0,d_{i1}),(T,d_{i1})(i\in\mathcal{M})$, respectively. Suppose furthermore that*

$$\frac{\partial G^1(t,0,0,0)}{\partial v}\neq 0, \qquad 0\leq t\leq T, \tag{2.11}$$

where v denotes the third argument of function G^1. Then, there exist boundary controls $h_i(t)$ with small norms $\|h^i\|_{C^0[0,T]}(i\in\mathcal{S})$ on all the simple nodes except the starting one, such that on the domain $\mathcal{R}(T)=\cup_{i=1}^N\{(t,x),0\leq t\leq T,d_{i0}\leq x\leq d_{i1}\}$, the mixed initial-boundary value problem for system (2.1) *with* (2.5)-(2.9) *admits a unique piecewise C^2 solution $u^i=u^i(t,x)$ $(i=1,...,N)$ with small piecewise C^2 norm, which exactly satisfies the final condition:*

$$t=T: \qquad u^i=\Phi_i(x),\;\; u^i_t=\Psi_i(x), \qquad d_{i0}\leq x\leq d_{i1} \quad (i=1,...,N). \tag{2.12}$$

The above result can be considered as a generalization of the 'one-sided' exact boundary controllability considered in [3] in the case with masses at the nodes. Furthermore, we can also examine the corresponding 'two-sided' exact boundary controllability. For the convenience of description and proving, we first define a 'longest' chain-like subnetwork, and choose the simple node E as the starting node for this 'longest' one. For this purpose, we assume that the tree-like network has m simple nodes: E_1, E_2, ..., E_m. Let $\tilde{\mathcal{S}}=\{1,...,m\}$. For any given $i,j\in\tilde{\mathcal{S}},i\neq j$, there is a unique chain-like subnetwork connecting simple nodes E_i and E_j. Let $\mathcal{D}_{ij}$ denote not only the set of indices k of all strings S_k contained in this subnetwork, but also the subnetwork itself.

Definition 2.1. The 'length' of the chain-like subnetwork connecting simple nodes E_i and E_j is defined as

$$T_{ij}=\sum_{k\in\mathcal{D}_{ij}}\frac{L_k}{\sqrt{K^k_{v_k}(0,0)}} \qquad (i,j\in\tilde{\mathcal{S}},i\neq j),$$

where v_k is the second argument of function K^k. Noting that $N>1$ and N is finite, we can find $i_0,j_0\in\tilde{\mathcal{S}}(i_0\neq j_0)$, such that

$$T_{i_0j_0}=\max_{i,j\in\tilde{\mathcal{S}},i\neq j}\sum_{k\in\mathcal{D}_{ij}}\frac{L_k}{\sqrt{K^k_{v_k}(0,0)}}.$$

The chain-like subnetwork between E_{i_0} and E_{j_0} is defined as a 'longest' chain-like network in this tree-like network. Then, a simple node of the 'longest' chain-like network is taken as the starting node of the whole network, and we can rewrite the 'length' of the 'longest' chain-like network as

$$\tilde{T}=\max_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}_j}\frac{L_k}{\sqrt{K^k_{v_k}(0,0)}}. \tag{2.13}$$

Theorem 2.2. *[Two-sided Exact Boundary Controllability] Let*

$$T>\tilde{T}, \tag{2.14}$$

where $\tilde{T}$ is given by (2.13). *For any given initial data (ϕ_i,ψ_i) $(i=1,...,N)$ and final data (Φ_i,Ψ_i) $(i=1,...,N)$ with small norms*

$$\sum_{i=1}^{N}\|(\phi_i,\psi_i)\|_{C^2[0,L]\times C^1[0,L]} \text{ and } \sum_{i=1}^{N}\|(\Phi_i,\Psi_i)\|_{C^2[0,L]\times C^1[0,L]},$$

and for any given functions $H^i(t)\in C^1[0,T]$ with small C^1 norms in the interface condition (2.9) *on all the multiple nodes $d_{i1}(i\in\mathcal{M})$, such that the conditions of piecewise C^2 compatibility are satisfied at the points $(0,d_{i1})$ and $(T,d_{i1})(i\in\mathcal{M})$, there exist boundary controls $h_i(t)$ with small norms $\|h^i\|_{C^0[0,T]}(i=1$ or $i\in\mathcal{S})$ on all the simple nodes, such that on the domain $\mathcal{R}(T)=\cup_{i=1}^{N}\{(t,x),0\le t\le T,d_{i0}\le x\le d_{i1}\}$, the mixed initial-boundary value problem for system* (2.1) *with* (2.5)-(2.9) *admits a unique piecewise C^2 solution $u^i=u^i(t,x)$ $(i=1,...,N)$ with small piecewise C^2 norm, which exactly satisfies the final condition:*

$$t=T:\qquad u^i=\Phi_i(x),\ u_t^i=\Psi_i(x),\qquad d_{i0}\le x\le d_{i1}\quad (i=1,...,N). \tag{2.15}$$

3 Existence and uniqueness of semi-global piecewise C^2solution

In order to obtain the exact boundary controllability for system (2.1) with dynamical boundary conditions and interface conditions (2.7)-(2.9) on a tree-like network, we should first prove the existence and uniqueness of semi-global piecewise C^2 solution to the corresponding mixed initial-boundary value problem. To this end, we reduce wave equations (2.1) to a first-order quasilinear hyperbolic system.

Let

$$v^i=u_x^i,\quad w^i=u_t^i\qquad (i=1,...,N).$$

For $i=1,...,N$, system (2.1) can be reduced to

$$\begin{cases}\dfrac{\partial u^i}{\partial t}=w^i,\\ \dfrac{\partial v^i}{\partial t}-\dfrac{\partial w^i}{\partial x}=0,\\ \dfrac{\partial w^i}{\partial t}-K_v^i(u^i,v^i)\dfrac{\partial v^i}{\partial x}=F^i(u^i,v^i,w^i)+K_u^i(u^i,v^i)v^i\triangleq\tilde{F}^i(u^i,v^i,w^i),\end{cases} \tag{3.1}$$

where $\tilde{F}^i$ is still a C^1 function of u^i,v^i,w^i, satisfying

$$\tilde{F}^i(0,0,0)=0. \tag{3.2}$$

Noting (2.2), (3.1) is a distinct hyperbolic system with $3N$ real eigenvalues:

$$\lambda_i^- = -\sqrt{K^i_{v^i}(u^i,v^i)}, \quad \lambda_i^0 = 0, \quad \lambda_i^+ = \sqrt{K^i_{v^i}(u^i,v^i)} \qquad (i=1,\dots,N),$$

and the corresponding left eigenvectors can be taken as

$$l_i^- = (0,\sqrt{K^i_{v^i}},1), \quad l_i^0 = (1,0,0), \quad l_i^+ = (0,-\sqrt{K^i_{v^i}},1) \qquad (i=1,\dots,N).$$

Let

$$\begin{cases} v_i^- = \sqrt{K^i_v(u^i,v^i)}v^i + w^i, \\ v_i^0 = u^i, \\ v_i^+ = -\sqrt{K^i_v(u^i,v^i)}v^i + w^i, \end{cases} \qquad (i=1,\dots,N). \tag{3.3}$$

Noting (2.2), in a neighborhood of $(u^i,v^i,w^i) = (0,0,0)$, by the Implicit Function Theorem, (3.3) can be rewritten as

$$\begin{cases} u_i = v_i^0, \\ v_i = p^i(v_i^0,v_i^-,v_i^+), \\ w_i = \dfrac{v_i^- + v_i^+}{2}, \end{cases} \qquad (i=1,\dots,N), \tag{3.4}$$

where p^i is a C^1 function of its arguments.

The initial condition (2.5) now becomes

$$t=0: \ (u^i,v^i,w^i) = (\phi_i(x),\phi_i'(x),\psi_i(x))^{\mathrm{T}}, \qquad d_{i0} \le x \le d_{i1} \quad (i=1,\dots,N). \tag{3.5}$$

On simple node E and $d_{i1} (i \in \mathcal{S})$, the dynamical boundary conditions (2.6) and (2.7) can be correspondingly replaced by the following nonlocal boundary conditions [9],

$x = d_{10}$:

$$w^1(t,d_{10}) = \psi^1(d_{10}) + \int_0^t G^1(\tau,u^1(\tau,d_{10}),v^1(\tau,d_{10}),w^1(\tau,d_{10}))\,\mathrm{d}\tau + \int_0^t h^1(\tau)\,\mathrm{d}\tau, \tag{3.6}$$

$x = d_{i1}$ (for $i \in \mathcal{S}$):

$$w^i(t,d_{i1}) = \psi^i(d_{i1}) + \int_0^t G^i(\tau,u^i(\tau,d_{i1}),v^i(\tau,d_{i1}),w^i(\tau,d_{i1}))\,\mathrm{d}\tau + \int_0^t h^i(\tau)\,\mathrm{d}\tau. \tag{3.7}$$

Noting (3.3)-(3.4), the nonlocal boundary conditions above can be rewritten as,

$x = d_{10}$:

$$v_1^- = -v_1^+ + 2\psi^1(d_{10}) + 2\int_0^t g^1(\tau,v_1^-(\tau,d_{10}),v_1^0(\tau,d_{10}),v_1^+(\tau,d_{10}))\,\mathrm{d}\tau + 2\int_0^t h^1(\tau)\,\mathrm{d}\tau, \tag{3.8}$$

$x = d_{i1}$ (for $i \in \mathcal{S}$):

$$v_i^- = -v_i^+ + 2\psi^i(d_{i1}) + 2\int_0^t g^i(\tau,v_i^-(\tau,d_{i1}),v_i^0(\tau,d_{i1}),v_i^+(\tau,d_{i1}))\,\mathrm{d}\tau + 2\int_0^t h^i(\tau)\,\mathrm{d}\tau, \tag{3.9}$$

where

$$g^i(\tau,v_i^-,v_i^0,v_i^+)=G^i(\tau,v_i^0,p^i(v_i^0,v_i^-,v_i^+),v_i^-+v_i^+),\qquad i=1 \text{ or } i\in\mathcal{S}. \tag{3.10}$$

While, noting the conditions of compatibility at multiple node $d_{i1}\,(i\in\mathcal{M})$, the interface condition (2.9) can be replaced by

$$\begin{cases} w^i(t,d_{i1})=\psi^i(d_{i1})+\displaystyle\int_0^t\Big(-K^i(u^i,u_x^i)(\tau,d_{i1})+\sum_{j\in\mathcal{J}_i}K^j(u^j,u_x^j)(\tau,d_{i1})\Big)\mathrm{d}\tau+\int_0^t H^i(\tau)\mathrm{d}\tau,\\ w^j=w^i,\qquad\qquad j\in\mathcal{J}_i,\end{cases} \tag{3.11}$$

then it can be rewritten as

$$\begin{cases} v_i^-=-v_i^++2\psi^i(d_{i1})+2\displaystyle\int_0^t\Big[-K^i\big(v_i^0,p^i(v_i^0,v_i^-,v_i^+)\big)(\tau,d_{i1})\\ \qquad\qquad+\displaystyle\sum_{j\in\mathcal{J}_i}K^j\big(v_j^0,p^j(v_j^0,v_j^-,v_j^+)\big)(\tau,d_{i1})\Big]\mathrm{d}\tau+2\int_0^t H^i(\tau)\mathrm{d}\tau,\\ v_j^-=-v_j^++2\psi^i(d_{i1})+2\displaystyle\int_0^t\Big[-K^i\big(v_i^0,p^i(v_i^0,v_i^-,v_i^+)\big)(\tau,d_{i1})\\ \qquad\qquad+\displaystyle\sum_{j\in\mathcal{J}_i}K^j\big(v_j^0,p^j(v_j^0,v_j^-,v_j^+)\big)(\tau,d_{i1})\Big]\mathrm{d}\tau+2\int_0^t H^i(\tau)\mathrm{d}\tau,\quad j\in\mathcal{J}_i.\end{cases} \tag{3.12}$$

Applying the result on the semi-global C^1 solution to a first-order quasilinear hyperbolic system with zero eigenvalues associated with a kind of nonlocal boundary condition (see [9]) , we can obtain the following

Lemma 3.1. *Under the assumptions given at the beginning of Section 2, for any given $T>0$, suppose that $\|(\phi_i,\psi_i)\|_{C^2[d_{i0},d_{i1}]\times C^1[d_{i0},d_{i1}]}$ $(i=1,...,N)$, $\|h^i\|_{C^0[0,T]}$ $(i=1$ or $i\in\mathcal{S})$ and $\|H^i\|_{C^1[0,T]}$ $(i\in\mathcal{M})$ are small enough (depending on T), and the conditions of C^2 compatibility or piecewise C^2 compatibility are satisfied at the points $(t,x)=(0,d_{10})$, $(0,d_{i1})$ $(i\in\mathcal{S})$ and $(0,d_{i1})$ $(i\in\mathcal{M})$, respectively. Then, the forward mixed initial-boundary value problems* (2.1) *and* (2.5)-(2.9) *admit a unique piecewise C^2 solution $u^i=u^i(t,x)$ $(i=1,...,N)$ with small piecewise C^2 norm on the domain $\mathcal{R}(T)=\cup_{i=1}^N\{(t,x)|0\le t\le T,d_{i0}\le x\le d_{i1}\}\triangleq\cup_{i=1}^N\mathcal{R}_i(T)$.*

Similar result can be obtained for the backward mixed problem (2.1)(2.7)-(2.9) and (2.12) with the final condition

$$t=T:\ (u^i,u_t^i)=(\Phi_i(x),\Psi_i(x)),\qquad d_{i0}\le x\le d_{i1}\qquad (i=1,...,N) \tag{3.13}$$

as follows.

Lemma 3.2. *Under the assumptions given at the beginning of Section 2, for any given $T>0$, suppose that $\|(\Phi_i,\Psi_i)\|_{C^2[d_{i0},d_{i1}]\times C^1[d_{i0},d_{i1}]}$ $(i=1,...,N)$, $\|h^i\|_{C^0[0,T]}$ $(i=1$ or $i\in\mathcal{S})$ and $\|H^i\|_{C^1[0,T]}$ $(i\in\mathcal{M})$ are small enough (depending on T), and the conditions of C^2 compatibility or piecewise C^2*

compatibility are satisfied at the points $(t,x)=(T,d_{10})$, (T,d_{i1}) $(i\in\mathcal{S})$ *and* (T,d_{i1}) $(i\in\mathcal{M})$, *respectively. Then, the backward mixed initial-boundary value problems* (2.1), (2.7)-(2.9) *and* (3.13) *admit a unique piecewise* C^2 *solution* $u^i=u^i(t,x)$ $(i=1,...,N)$ *with small piecewise* C^2 *norm on the domain* $\mathcal{R}(T)=\cup_{i=1}^N\mathcal{R}_i(T)$.

Remark 3.1. [Hidden Regularity] For the piecewise C^2 solution $u^i=u^i(t,x)$ $(i=1,...,N)$ given in Lemma 3.1 (or Lemma 3.2), if $h^i(t)\equiv 0$ or more generally, $h^i(t)\in C^1[0,T]$ with small $C^1[0,T]$ norm on all the simple nodes E and $d_{i1}(i\in\mathcal{S})$, then there is a hidden regularity on $x=d_{10}$ and $x=d_{i1}(i\in\mathcal{S})$ that both $u^1(t,d_{10})$ and $u^i(t,d_{i1})(i\in\mathcal{S})\in C^3[0,T]$ with small C^3 norms.

Similarly, since it is assumed that $H^i(t)\in C^1[0,T]$ with small C^1 norm on all the multiple nodes $d_{i1}(i\in\mathcal{M})$, there is a hidden regularity of the piecewise C^2 solution $u^i=u^i(t,x)$ $(i=1,...,N)$ on $x=d_{i1}$ $(i\in\mathcal{M})$ that $u^i(t,d_{i1})\in C^3[0,T]$ $(i\in\mathcal{M})$ with small C^3 norm.

Remark 3.2. If the types of boundary conditions on the simple nodes E and $d_{i1}(i\in\mathcal{S})$ are mixed, namely, at $x=d_{10}$ and $x=d_{i1}(i\in\mathcal{S})$, for each index $i=1$ or $i\in\mathcal{S}$, the corresponding boundary condition can be taken as anyone of the following boundary conditions:

$$u^i=h^i(t) \qquad \text{(Dirichlet Type)}, \tag{3.14a}$$

$$u^i_x=h^i(t) \qquad \text{(Neumann Type)}, \tag{3.14b}$$

$$u^i_x+b_i(u^i)u^i=h^i(t) \qquad \text{(Third Type)}, \tag{3.14c}$$

$$u^i_{tt}=G^i(t,u^i,u^i_x,u^i_t)+h^i(t) \qquad \text{(Dynamical Type)}, \tag{3.14d}$$

where $b_i>0$ is a C^1 function of u^i, $G^i=G^i(t,u^i,u^i_x,u^i_t)$ is a C^1 function of its arguments, while $h^i(t)$ is C^2 (for Dirichlet Type) or C^1 (for Neumann Type or Third Type) or C^0 (for Dynamical Type) function of t. The interface condition on multiple node d_{i1} $(i\in\mathcal{M})$ is still given by (2.9) such that the conditions of piecewise C^2 compatibility at the points $(t,x)=(0,d_{i1})$ $(i\in\mathcal{M})$(resp. (T,d_{i1}) $(i\in\mathcal{M})$) are still satisfied. Then Lemma 3.1 and Lemma 3.2 still hold, provided that the corresponding conditions of C^2 compatibility at the points $(0,d_{10})$ and $(0,d_{i1})$ $(i\in\mathcal{S})$ (resp.(T,d_{10}) and (T,d_{i1}) $(i\in\mathcal{S})$) are satisfied, respectively.

It is worth mentioning that the hidden regularity can be only realized in the case that the boundary condition at simple node is taken as a dynamical boundary condition (3.14d) and the interface condition at multiple node is given as (2.9), satisfying the additional assumptions given in Remark 3.1.

4 Proof of 'one-sided' exact boundary controllability

In order to prove Theorem 2.1, it suffices to prove the following:

Lemma 4.1. *Under the assumptions of Theorem* 2.1, *with the interface conditions* (2.9) *at all the multiple nodes and boundary condition* (2.6) *(*$h^1(t)\equiv 0$ *or* $h^1(t)\in C^1[0,T]$ *with small* C^1 *norm)* *at the starting simple node E* (d_{10})*, system* (2.1) *admits a piecewise* C^2 *solution* $u^i = u^i(t,x)$ $(i = 1,\ldots,N)$ *with small piecewise* C^2 *norm on the domain* $\mathcal{R}(T) = \cup_{i=1}^N \{(t,x)|0 \le t \le T,\ d_{i0} \le x \le d_{i1}\} \triangleq \cup_{i=1}^N \mathcal{R}_i(T)$*, which exactly satisfies the initial condition* (2.5) *and the final condition* (2.12)*.*

Proof. The proof is based on Lemma 3.1 and Lemma 3.2 and by means of the constructive method given in [4], [5] and [9]. Noting (2.10), there exists an $\varepsilon_0 > 0$ so small that

$$T > 2\max_{i\in\mathcal{S}} \sum_{j\in\mathcal{D}_i} \left(\sup_{|u^j|+|v^j|\le\varepsilon_0} \frac{L_j}{\sqrt{K^j_{v^j}(u^j,v^j)}} \right). \tag{4.1}$$

Let

$$T_0 = \max_{i\in\mathcal{S}} \sum_{j\in\mathcal{D}_i} \left(\sup_{|u^j|+|v^j|\le\varepsilon_0} \frac{L_j}{\sqrt{K^j_{v^j}(u^j,v^j)}} \right), \tag{4.2}$$

and for $i \in \mathcal{S}$, let

$$T_j = \sup_{|u^j|+|v^j|\le\varepsilon_0} \frac{L_j}{\sqrt{K^j_{v^j}(u^j,v^j)}}, \qquad j \in \mathcal{D}_i. \tag{4.3}$$

For convenience, we suppose that there are k multiple nodes and m simple nodes.

(i) We first consider the forward mixed initial-boundary value problem for system (2.1) with the initial condition (2.5), the interface conditions (2.9) on k multiple nodes, the boundary condition

$$x = d_{10}: \;\; u^1_{tt} = G^1(t,u^1,u^1_x,u^1_t) + h^1(t) \tag{4.4}$$

on the simple node E, and the following artificial boundary conditions on other $(m-1)$ simple nodes:

$$x = d_{i1}: \; u^i = f^i(t) \qquad (i \in \mathcal{S}),$$

where $f^i(t)(i\in\mathcal{S})$ are any given C^2 functions of t with small $C^2[0,T_0]$ norm, such that the conditions of C^2 compatibility are satisfied at the point $(t,x) = (0,d_{i1})(i \in \mathcal{S})$. By Lemma 3.1 and Remark 3.2, on the domain

$$\mathcal{R}_f(T_0) = \cup_{i=1}^N \{(t,x)|0 \le t \le T_0, d_{i0} \le x \le d_{i1}\},$$

there exists a unique piecewise C^2 solution $\mathbf{u}_f = \mathbf{u}_f(t,x) = (u^1_f(t,x),\ldots,u^N_f(t,x))$ with small piecewise C^2 norm. In particular, we have

$$\left|\left(\mathbf{u}_f, \frac{\partial \mathbf{u}_f}{\partial x}\right)\right| \le \varepsilon_0, \qquad \forall (t,x) \in \mathcal{R}_f(T_0).$$

Thus, we can determine the value of (u^1_f, u^1_{fx}) at $x = d_{10}$ as follows (see Figure 3(a)):

$$x = d_{10}: \quad (u^1_f, u^1_{fx}) = (a(t), \bar{a}(t)), \qquad 0 \le t \le T_0, \tag{4.5}$$

and $(u^1,u^1_t,u^1_{tt},u^1_x)=(a(t),a'(t),a''(t),\bar{a}(t))$ satisfies the boundary condition (4.4). Noting the hidden regularity of u^1_f at $x=d_{10}$, $\|(a,\bar{a})\|_{C^3[0,T_0]\times C^1[0,T_0]}$ is small enough.

Similarly, on the multiple node d_{11}, we can determine the corresponding value of (u^1_f,u^1_{fx}) and $(u^j_f,u^j_{fx})(j\in\mathcal{J}_1)$ as follows:

$$x=d_{11}: \quad (u^j_f,u^j_{fx})=(a_j(t),\bar{a}_j(t)), \qquad 0\le t\le T_0 \quad (j=1 \text{ and } j\in\mathcal{J}_1), \tag{4.6}$$

and $(u^j,u^j_t,u^j_{tt},u^j_x)=(a_j(t),a_j'(t),a_j''(t),\bar{a}_j(t))(j=1 \text{ and } j\in\mathcal{J}_1)$ satisfy the interface conditions (2.9) (in which we take $i=1$) on the interval $0\le t\le T_0$. Noting the hidden regularity of u^j_f at $x=d_{11}$, $\|(a_j,\bar{a}_j)\|_{C^3[0,T_0]\times C^1[0,T_0]}$is small enough $(j=1$ and $j\in\mathcal{J}_1)$.

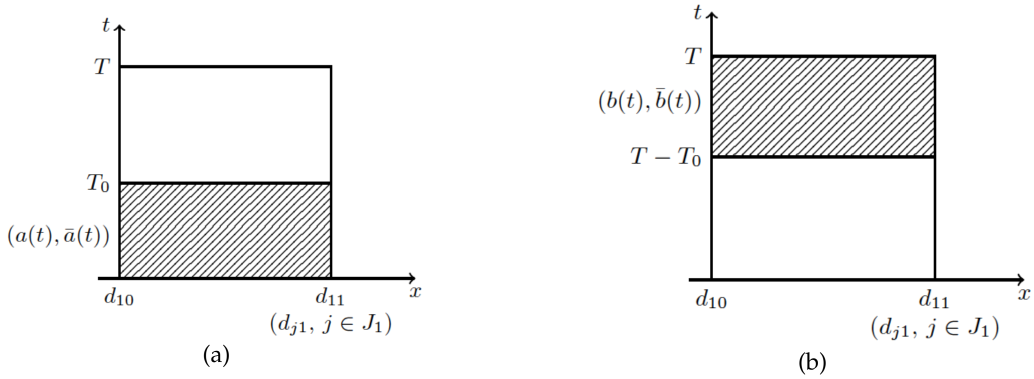

Figure 3: (a) Forward mixed problem; (b) backward mixed problem.

(ii) Similarly, we consider the backward mixed initial-boundary value problem for system (2.1) with the final condition (2.12), the interface conditions (2.9) on k multiple nodes, the boundary condition (4.4) on the simple node E and the following artificial boundary conditions on other $(m-1)$ simple nodes:

$$x=d_{i1}: u^i=\bar{f}^i(t) \qquad (i\in\mathcal{S})$$

where $\bar{f}^i(t)(i\in\mathcal{S})$ are any given C^2 functions of t with small $C^2[T-T_0,T]$ norm, such that the conditions of C^2 compatibility are satisfied at the point $(t,x)=(T,d_{i1})(i\in\mathcal{S})$. By Lemma 3.2 and Remark 3.2, on the domain

$$\mathcal{R}_b(T_0)=\cup_{i=1}^N\{(t,x)|T-T_0\le t\le T,d_{i0}\le x\le d_{i1}\},$$

there exists a unique piecewise C^2 solution $\mathbf{u}_b=\mathbf{u}_b(t,x)=(u^1_b(t,x),\dots,u^N_b(t,x))$ with small piecewise C^2 norm. In particular, we have

$$\left|\left(\mathbf{u}_b,\frac{\partial\mathbf{u}_b}{\partial x}\right)\right|\le\varepsilon_0, \qquad \forall(t,x)\in\mathcal{R}_b(T_0).$$

Thus, we can determine the value of (u_b^1, u_{bx}^1) at $x=d_{10}$ as follows (see Figure 3(b)):

$$x=d_{10}: \quad (u_b^1,u_{bx}^1)=(b(t),\bar{b}(t)), \qquad T-T_0\le t\le T, \tag{4.7}$$

and $(u^1,u_t^1,u_{tt}^1,u_x^1)=(b(t),b'(t),b''(t),\bar{b}(t))$ satisfies the boundary condition (4.4) on the interval $T-T_0\le t\le T$. Noting the hidden regularity of u_b^1 at $x=d_{10}$, $\|(b,\bar{b})\|_{C^3[0,T_0]\times C^1[0,T_0]}$ is small enough.

Similarly, on the multiple node d_{11}, we can determine the corresponding value of (u_b^1,u_{bx}^1) and $(u_b^j,u_{bx}^j)(j\in\mathcal{J}_1)$ as follows:

$$x=d_{11}: \quad (u_f^j,u_{fx}^j)=(b_j(t),\bar{b}_j(t)), \qquad T-T_0\le t\le T\ (j=1 \text{ and } j\in\mathcal{J}_1), \tag{4.8}$$

and $(u^j,u_t^j,u_{tt}^j,u_x^j)=(b_j(t),b_j'(t),b_j''(t),\bar{b}_j(t))$ $(j=1$ and $j\in\mathcal{J}_1)$ satisfy the interface conditions (2.9) (in which we take $i=1$) on the interval $T-T_0\le t\le T$. Noting the hidden regularity of u_b^j at $x=d_{11}$, $\|(b_j,\bar{b}_j)\|_{C^3[T-T_0,T]\times C^1 T-T_0,T]}$ is small enough $(j=1$ and $j\in\mathcal{J}_1)$.

(iii) Since $\mathcal{R}_f(T_0)$ and $\mathcal{R}_b(T_0)$ do not overlap, we want to construct $(c(t),\bar{c}(t))$ as a choice of the value of (u^1,u_x^1) at $x=d_{10}$ with small $C^3[0,T]\times C^1[0,T]$ norm, such that

$$(c(t),\bar{c}(t))=\begin{cases}(a(t),\bar{a}(t)), & 0\le t\le T_0,\\ (b(t),\bar{b}(t)), & T-T_0\le t\le T,\end{cases} \tag{4.9}$$

and $(u^1,u_t^1,u_{tt}^1,u_x^1)=(c(t),c'(t),c''(t),\bar{c}(t))$ satisfies the boundary condition (4.4) on the whole interval $[0,T]$ at $x=d_{10}$.

To this end, we first find $c(t)\in C^3[0,T]$ with small $C^3[0,T]$ norm, such that

$$c(t)=\begin{cases}a(t), & 0\le t\le T_0,\\ b(t), & T-T_0\le t\le T.\end{cases} \tag{4.10}$$

Thus, at $x=d_{10}$, we have $(u^1,u_t^1,u_{tt}^1)=(c(t),c'(t),c''(t))$.

Noting (2.11), by the Implicit Function Theorem, the boundary condition (4.4) at $x=d_{10}$ can be uniquely rewritten on $0\le t\le T$ as follows:

$$x=d_{10}: \quad u_x^1=\tilde{G}(t,u^1,u_t^1,u_{tt}^1), \tag{4.11}$$

where $\tilde{G}$ is a C^1 function of its arguments.

Let

$$\bar{c}(t)=\tilde{G}(t,c(t),c'(t),c''(t)). \tag{4.12}$$

Noting the hidden regularity given in Remark 3.1, $\bar{c}(t)\in C^1[0,T]$ with small $C^1[0,T]$ norm, and it is easy to see that

$$\bar{c}(t)=\begin{cases}\bar{a}(t), & 0\le t\le T_0,\\ \bar{b}(t), & T-T_0\le t\le T.\end{cases} \tag{4.13}$$

Moreover, $(u^1,u^1_t,u^1_{tt},u^1_x)=(c(t),c'(t),c''(t),\bar{c}(t))$ satisfies the boundary condition (4.4) on the whole interval $[0,T]$.

(iv) Noting (2.2), we now change the status of t and x, and consider the rightward mixed initial-boundary value problem (see Figure 4(a)) for system (2.1) with the initial condition

$$x=d_{10}: \quad u^1=c(t), \quad u^1_x=\bar{c}(t), \qquad 0\le t\le T \tag{4.14}$$

and Dirichlet boundary conditions

$$t=0: \quad u^1=\phi_1(x), \qquad d_{10}\le x\le d_{11}, \tag{4.15}$$

$$t=T: \quad u^1=\Phi_1(x), \qquad d_{10}\le x\le d_{11}, \tag{4.16}$$

where $\phi_1(x)$ and $\Phi_1(x)$ are given by (2.5) and (2.12), respectively. Obviously, the conditions of C^2 compatibility at the points $(t,x)=(0,d_{10})$ and (T,d_{10}) are satisfied. Then by Lemma 3.1 and Remark 3.2, there exists a unique C^2 solution $u^1=u^1(t,x)$ with small C^2 norm on the domain $\mathcal{R}_1(T)=\{(t,x)|0\le t\le T,d_{10}\le x\le d_{11}\}$ and

$$|u^1(t,x)|+|\frac{\partial u^1}{\partial x}(t,x)|\le\varepsilon_0, \qquad \forall(t,x)\in\mathcal{R}_1(T). \tag{4.17}$$

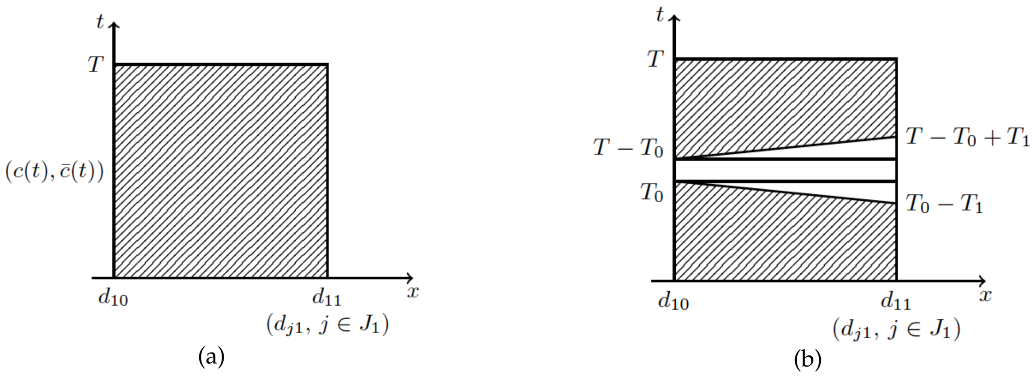

Figure 4: (a) Rightward mixed problem; (b) uniqueness of solutions on the shaded domain.

The C^2 solutions u^1 and u^1_f satisfy simultaneously the same system (2.1) (in which we take $i=1$), the same initial condition

$$x=d_{10}: \quad u^1=c(t), \quad u^1_x=\bar{c}(t), \qquad 0\le t\le T_0 \tag{4.18}$$

and the same boundary condition (4.15), by uniqueness of C^2 solutions to the one-sided mixed initial boundary value problem (see [7]), it is easy to see that $u^1\equiv u^1_f$ on the domain (see Figure 4(b))

$$\Big\{(t,x)\,\Big|\,0\le t\le T_0-A_1(x-d_{10}),d_{10}\le x\le d_{11}\Big\}, \tag{4.19}$$

where

$$A_1 = \sup_{|u^1|+|v^1|\le\varepsilon_0} \frac{1}{\sqrt{K^1_{v^1}(u^1,v^1)}}.$$

In particular, on the interval $[d_{10},d_{11}]$ on the x-axis, u^1 satisfies the initial condition (2.5) (in which we take $i=1$) and

$$x=d_{11}:(u^1,u^1_x)=(a_1(t),\bar{a}_1(t)), \qquad 0\le t\le T_0-T_1, \tag{4.20}$$

where $(a_1(t),\bar{a}_1(t))$ is given by (4.6) (in which we take $j=1$).

In a similar manner, we obtain that $u^1\equiv u^1_b$ on the domain (see Figure 4(b))

$$\{(t,x)|T-T_0+A_1(x-d_{10})\le t\le T,\ d_{10}\le x\le d_{11}\}. \tag{4.21}$$

In particular, on the interval $[d_{10},d_{11}]$ on the x-axis, u^1 satisfies the final condition (2.12) (in which we take $i=1$) and

$$x=d_{11}:(u^1,u^1_x)=(b_1(t),\bar{b}_1(t)), \qquad T-T_0+T_1\le t\le T, \tag{4.22}$$

where $(b_1(t),\bar{b}_1(t))$ is given by (4.8) (in which we take $j=1$).

(v) On the multiple node d_{11}, we now construct $(c_j(t),\bar{c}_j(t))\in C^3[0,T]\times C^1[0,T]$ ($j=1$ and $j\in\mathcal{J}_1$) with small $C^3[0,T]\times C^1[0,T]$ norm, such that

$$(c_j(t),\bar{c}_j(t))=\begin{cases}(a_j(t),\bar{a}_j(t)), & 0\le t\le T_0-T_1,\\ (b_j(t),\bar{b}_j(t)), & T-T_0+T_1\le t\le T,\end{cases} \qquad (j=1 \text{ and } j\in\mathcal{J}_1), \tag{4.23}$$

where $(a_j(t),\bar{a}_j(t))$ and $(b_j(t),\bar{b}_j(t))$ are given by (4.6),(4.8) (in which we take $j\in\mathcal{J}_1$) and (4.20), (4.22) (in which we take $j=1$), respectively. Furthermore, $(u^j,u^j_t,u^j_{tt},u^j_x)=(c_j(t),c'_j(t),c''_j(t),\bar{c}_j(t))$ ($j=1$ and $j\in\mathcal{J}_1$) should satisfy the interface conditions (2.9) at $x=d_{11}$ on the whole interval $[0,T]$.

Firstly, the C^2 solution $u^1=u^1(t,x)$ given in step (iv) determines $(c_1(t),\bar{c}_1(t))$ as the value of (u^1,u^1_x) on the whole interval $[0,T]$ at $x=d_{11}$. Thus, $(c_1(t),\bar{c}_1(t))\in C^3[0,T]\times C^1[0,T]$ with small $C^3[0,T]\times C^1[0,T]$ norm, and by (4.20) and (4.22) we have

$$(c_1(t),\bar{c}_1(t))=\begin{cases}(a_1(t),\bar{a}_1(t)), & 0\le t\le T_0-T_1,\\ (b_1(t),\bar{b}_1(t)), & T-T_0+T_1\le t\le T.\end{cases} \tag{4.24}$$

Then, by the continuity of displacements given by the interface conditions (2.9), setting

$$x=d_{11}:\ c_j(t)=c_1(t), \qquad 0\le t\le T\ \ (j\in\mathcal{J}_1). \tag{4.25}$$

we have $c_j(t)\in C^3[0,T]$ with small $C^3[0,T]$ norm and

$$c_j(t)=\begin{cases}a_j(t), & 0\le t\le T_0-T_1,\\ b_j(t), & T-T_0+T_1\le t\le T,\end{cases} \qquad (j\in\mathcal{J}_1), \tag{4.26}$$

where $a_j(t)$ and $b_j(t)$ are given by (4.6) and (4.8) (in which we take $j\in\mathcal{J}_1$), respectively.

Noting (2.2), in particular, there exists at least one index $j_0\in\mathcal{J}_1$, such that

$$K_v^{j_0}(0,0)>0. \tag{4.27}$$

Then, we can first construct $\bar{c}_j(t)\in C^1[0,T]$ $(j\in\mathcal{J}_1, j\neq j_0)$ with small $C^1[0,T]$ norm, such that

$$\bar{c}_j(t)=\begin{cases}\bar{a}_j(t), & 0\le t\le T_0-T_1,\\ \bar{b}_j(t), & T-T_0+T_1\le t\le T\end{cases}\qquad (j\in\mathcal{J}_1, j\neq j_0), \tag{4.28}$$

where $\bar{a}_j(t)$ and $\bar{b}_j(t)$ are given by (4.6) and (4.8) $(j\in\mathcal{J}_1, j\neq j_0)$, respectively. Suppose that $\mathcal{J}_1=\{j_0,j_1,\ldots,j_l\}$, noting (4.27), by the Implicit Function Theorem, the second formula of the interface conditions (2.9) can be uniquely rewritten as follows:

$$x=d_{11}:\ \ u_x^{j_0}=\tilde{f}(t,u^1,u_x^1,u_{tt}^1,u^{j_0},u^{j_1},\ldots,u^{j_l},u_x^{j_1},\ldots,u_x^{j_l}), \tag{4.29}$$

where $\tilde{f}$ is a C^1 function of its arguments.

Setting

$$\bar{c}_{j_0}(t)=\tilde{f}(t,c_1,\bar{c}_1,c_1'',c_{j_0},c_{j_1},\ldots,c_{j_l},\bar{c}_{j_1},\ldots,\bar{c}_{j_l}), \tag{4.30}$$

we get that $c_{j_0}(t)\in C^1[0,T]$ with small C^1 norm, and

$$\bar{c}_{j_0}(t)=\begin{cases}\bar{a}_{j_0}(t), & 0\le t\le T_0-T_1,\\ \bar{b}_{j_0}(t), & T-T_0+T_1\le t\le T,\end{cases} \tag{4.31}$$

where $\bar{a}_{j_0}(t)$ and $\bar{b}_{j_0}(t)$ are given by (4.6) and (4.8)(in which we take $j=j_0$), respectively.

Thus, we find $(c_j(t),c_j'(t),c_j''(t),\bar{c}_j(t))$ as the value of $(u^j,u_t^j,u_{tt}^j,u_x^j)(j=1$ and $\in\mathcal{J}_1)$, satisfying the interface conditions (2.9) on the whole interval $[0,T]$ at $x=d_{11}$.

(vi) For $j\in\mathcal{J}_1$ (namely, $d_{j0}=d_{11}$), we can regard the node d_{j0} as a new starting node and do step (iv) again, namely, consider the rightward mixed initial-boundary value problem for system (2.1) with the initial condition

$$x=d_{j0}:\quad u^j=c_j(t),\quad u_x^1=\bar{c}_j(t),\qquad 0\le t\le T \tag{4.32}$$

and Dirichlet boundary conditions

$$t=0:\quad u_j=\phi_j(x),\qquad d_{j0}\le x\le d_{j1}, \tag{4.33}$$

$$t=T:\quad u_j=\Phi_j(x),\qquad d_{j0}\le x\le d_{j1}, \tag{4.34}$$

where $\phi_j(x)$ and $\Phi_j(x)$ are given by (2.5) and (2.12) (in which we take $i=j$), respectively. Obviously, the conditions of C^2 compatibility at the points $(t,x)=(0,d_{j0})$ and (T,d_{j0}) are satisfied, respectively. Then by Lemma 3.1 and Remark 3.2, there exists a unique C^2

solution $u^j=u^j(t,x)$ with small C^2 norm on the domain $\mathcal{R}_j(T)=\{(t,x)|0\le t\le T,\ d_{j0}\le x\le d_{j1}\}$, and

$$|u^j(t,x)|+\left|\frac{\partial u^j}{\partial x}(t,x)\right|\le\varepsilon_0,\qquad\forall(t,x)\in\mathcal{R}_j(T).\tag{4.35}$$

The C^2 solutions u^j and u_f^j satisfy simultaneously the same system (2.1) (in which we take $i=j$), the same initial condition

$$x=d_{j0}:\quad u^j=c_j(t),\quad u_x^j=\bar{c}_j(t),\qquad 0\le t\le T_0-T_1\tag{4.36}$$

and the same boundary condition (4.33). By uniqueness of C^2 solutions to the one-sided mixed initial boundary value problem (see [7]), it is easy to see that $u^j\equiv u_f^j$ on the domain

$$\big\{(t,x)|0\le t\le T_0-T_1-A_j(x-d_{j0}),d_{j0}\le x\le d_{j1}\big\},\tag{4.37}$$

where

$$A_j=\sup_{|u^j|+|v^j|\le\varepsilon_0}\frac{1}{\sqrt{K_{v^j}^j(u^j,v^j)}}.$$

In particular, on the interval $[d_{j0},d_{j1}]$ on the x-axis, u^j satisfies the initial condition (2.5) (in which we take $i=j$) and

$$x=d_{j1}:\ (u^j,u_x^j)=(a_j(t),\bar{a}_j(t)),\qquad 0\le t\le T_0-T_1-T_j,\tag{4.38}$$

where $(a_j(t),\bar{a}_j(t))$ is given by (4.6).

Similarly, we have that $u^j\equiv u_b^j$ on the domain

$$\big\{(t,x)|T-T_0+T_1+A_j(x-d_{j0})\le t\le T,d_{j0}\le x\le d_{j1}\big\}.\tag{4.39}$$

In particular, on the interval $[d_{j0},d_{j1}]$ on the x-axis, u^j satisfies the final condition (2.12) (in which we take $i=j$) and

$$x=d_{j1}:\ (u^j,u_x^j)=(b_j(t),\bar{b}_j(t)),\qquad T-T_0+T_1+T_j\le t\le T,\tag{4.40}$$

where $(b_j(t),\bar{b}_j(t))$ is given by (4.8).

The constructive process is over when $j\in\mathcal{S}$, namely, d_{j1} is a simple node. Otherwise, if $j\in\mathcal{M}$, namely, d_{j1} is still a multiple node, we do step (v) again, namely, consider $d_{k0}(k\in J_j)$ as a new starting node and do step (iv). This process can be stopped in limited steps.

(vii) Noting the definition of T_0 given by (4.2), for any given $i\in\mathcal{S}$, we have

$$T_0-\sum_{j\in\mathcal{D}_i}T_j\ge 0,$$

which promises that the piecewise C^2 solutions $u^i = u^i(t,x)(i=1,\dots,N)$ obtained above satisfies

$$t=0:\quad u^i \equiv u^i_f,\quad d_{i0}\le x\le d_{i1}\qquad (i=1,\dots,N),$$
$$t=T:\quad u^i \equiv u^i_b,\quad d_{i0}\le x\le d_{i1}\qquad (i=1,\dots,N),$$

thus, $\mathbf{u}=(u^1,\dots,u^N)=(u^1(t,x),\dots,u^N(t,x))$ is a solution required by Lemma 4.1. □

5 The proof of 'two-sided' exact boundary controllability

In order to proof Theorem 2.2, it suffices to proof the following

Lemma 5.1. *Under the assumptions of Theorem* 2.2*, system* (2.1) *with the interface conditions* (2.9) *at all the multiple nodes admits a piecewise* C^2 *solution* $u^i=u^i(t,x)$ $(i=1,\dots,N)$ *with small piecewise* C^2 *norm on the domain* $\mathcal{R}(T)=\cup_{i=1}^N\{(t,x)|0\le t\le T, d_{i0}\le x\le d_{i1}\}\triangleq\cup_{i=1}^N\mathcal{R}_i(T)$*, which exactly satisfies the initial condition* (2.5) *and the final condition* (2.15).

Proof. Based on Lemma 3.1 and Lemma 3.2, referring to and adjusting the constructive method given in [4], [5] and [9], we first find a 'longest' chain-like subnetwork and its midpoint x_0. Then, starting from x_0, we solve a leftward and a rightward mixed problem on the corresponding half tree-like networks, respectively, and construct a desired piecewise C^2 solution.

Noting the definition of control time T given by (2.13)-(2.14), there exists an $\varepsilon_0>0$ so small that

$$T>\max_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}_j}\left(\sup_{|u^k|+|v^k|\le\varepsilon_0}\frac{L_k}{\sqrt{K^k_{v^k}(u^k,v^k)}}\right). \tag{5.1}$$

Let

$$T_0=\frac{1}{2}\max_{j\in\mathcal{S}}\sum_{k\in\mathcal{D}_j}\left(\sup_{|u^k|+|v^k|\le\varepsilon_0}\frac{L_k}{\sqrt{K^k_{v^k}(u^k,v^k)}}\right), \tag{5.2}$$

and for $j\in\mathcal{S}$, let

$$T_k=\sup_{|u^k|+|v^k|\le\varepsilon_0}\frac{L_k}{\sqrt{K^k_{v^k}(u^k,v^k)}},\qquad k\in\mathcal{D}_j. \tag{5.3}$$

Similarly to the proof of Lemma 4.1, we want to construct a piecewise C^2 solution $\mathbf{u}=(u^1,\dots,u^N)$ required by Lemma 5.1 on the tree-like network.

(i) On the domain

$$\mathcal{R}_f(T_0)=\cup_{i=1}^N\{(t,x)|0\le t\le T_0, d_{i0}\le x\le d_{i1}\},$$

we consider the forward mixed initial-boundary value problem for system (2.1) with the initial condition (2.5), the interface conditions (2.9) on k multiple nodes and the following artificial boundary conditions on all the simple nodes:

$$\begin{aligned} x=d_{10}: &\quad u^i=f^0(t),\\ x=d_{i1}: &\quad u^i=f^i(t) \qquad (i\in\mathcal{S}), \end{aligned} \tag{5.4}$$

where $f^0(t)$ and $f^i(t)(i\in\mathcal{S})$ are any given C^2 functions of t with small $C^2[0,T_0]$ norms, such that the conditions of C^2 compatibility are satisfied at the point $(t,x)=(0,d_{10})$ and $(0,d_{i1})(i\in\mathcal{S})$, respectively. By Lemma 3.1 and Remark 3.2, there exists a unique piecewise C^2 solution $\mathbf{u}_f=(u_f^1,\dots,u_f^N)$ with small piecewise C^2 norm on the domain $\mathcal{R}_f(T_0)$.

(ii) On the domain

$$\mathcal{R}_b(T_0)=\cup_{i=1}^N\{(t,x)|T-T_0\le t\le T,d_{i0}\le x\le d_{i1}\}$$

we consider the backward mixed initial-boundary value problem for system (2.1) with the final condition (2.15), the interface condition (2.9) on k multiple nodes and the following artificial boundary conditions on all the simple nodes:

$$\begin{aligned} x=d_{10}: &\quad u^i=\bar f^0(t),\\ x=d_{i1}: &\quad u^i=\bar f^i(t) \qquad (i\in\mathcal{S}), \end{aligned} \tag{5.5}$$

where $\bar f^0(t)$ and $\bar f^i(t)(i\in\mathcal{S})$ are any given C^2 functions of t with small $C^2[T-T_0,T]$ norms, such that the conditions of C^2 compatibility are satisfied at the point $(t,x)=(T,d_{10})$ and $(T,d_{i1})(i\in\mathcal{S})$, respectively. By Lemma 3.2 and Remark 3.2, there exists a unique piecewise C^2 solution $\mathbf{u}_b=(u_b^1,\dots,u_b^N)$ with small piecewise C^2 norm on the domain $\mathcal{R}_b(T_0)$.

(iii) Consider one of the simple nodes on the 'longest' chain-like subnetwork as the starting node $E(d_{10})$. Thus, there exists $j_0\in\mathcal{S}$ such that

$$T_0=\frac{1}{2}\sum_{k\in\mathcal{D}_{j_0}}\left(\sup_{|u^k|+|v^k|\le\varepsilon_0}\frac{L_k}{\sqrt{K^k_{v^k}(u^k,v^k)}}\right), \tag{5.6}$$

and $\mathcal{D}_{j0}$ is a 'longest' chain-like subnetwork on the tree-like network.

Noting (5.6), we can find $\bar k,\bar{\bar k}$ satisfying $d_{\bar k1}=d_{\bar{\bar k}0}$, namely, strings $S_{\bar k}$ and $S_{\bar{\bar k}}$ are connected end-to-end, and we have

$$\sum_{l\in\mathcal{D}_{\bar k}}\sup_{|u^l|+|v^l|\le\varepsilon_0}\frac{L_l}{\sqrt{K^l_{v_l}(u^l,v^l)}}\le T_0<\sum_{l\in\mathcal{D}_{\bar{\bar k}}}\sup_{|u^l|+|v^l|\le\varepsilon_0}\frac{L_l}{\sqrt{K^l_{v_l}(u^l,v^l)}}, \tag{5.7}$$

where $\mathcal{D}_{\bar{k}}$(resp. $\mathcal{D}_{\bar{\bar{k}}}$) stands for the set of indices l of string S_l, which are contained in the chain-like network between E and node $d_{\bar{k}1}$ (resp. $d_{\bar{\bar{k}}1}$). Hence, we can find $x_0 \in [d_{\bar{\bar{k}}0}, d_{\bar{\bar{k}}1})$ such that

$$T_0 = \sum_{l\in\mathcal{D}_{\bar{k}}} \sup_{|u^l|+|v^l|\le\varepsilon_0} \frac{L_l}{\sqrt{K^l_{v_l}(u^l,v^l)}} + \frac{x_0 - d_{\bar{\bar{k}}0}}{\sqrt{K^{\bar{\bar{k}}}_{v_{\bar{\bar{k}}}}(u^l,v^l)}}. \tag{5.8}$$

Thus, We can regard x_0 as a 'midpoint' of this 'longest' chain-like subnetwork (see Figure 5).

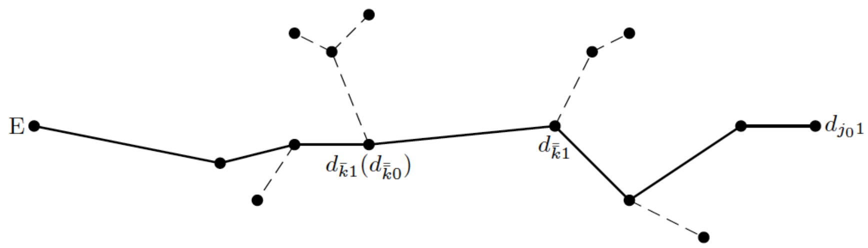

Figure 5: The 'longest' chain-like subnetwork.

The following discussion will be divided into two cases.

We first discuss the case that $d_{\bar{\bar{k}}0} < x_0 < d_{\bar{\bar{k}}1}$, namely, x_0 is not a node on the tree-like network.

(iv) Thus, respectively, we can determine the value of $(u_f^{\bar{\bar{k}}}, u_{fx}^{\bar{\bar{k}}})$ and $(u_b^{\bar{\bar{k}}}, u_{bx}^{\bar{\bar{k}}})$ at $x = x_0$ by the results given in (i) and (ii) as follows :

$$x = x_0: \begin{cases} (u_f^{\bar{\bar{k}}}, u_{fx}^{\bar{\bar{k}}}) = (a(t), \bar{a}(t)), & 0 \le t \le T_0, \\ (u_b^{\bar{\bar{k}}}, u_{bx}^{\bar{\bar{k}}}) = (b(t), \bar{b}(t)), & T - T_0 \le t \le T, \end{cases} \tag{5.9}$$

and $\|(a,\bar{a})\|_{C^2[0,T_0]\times C^1[0,T_0]}$ and $\|(b,\bar{b})\|_{C^2[T-T_0,T]\times C^1[T-T_0,T]}$ are both small enough.

Noting (5.1) and (5.2), we can find $(c(t), \bar{c}(t))$ with small $\|(c,\bar{c})\|_{C^2[0,T]\times C^1[0,T]}$ norm, being the value of $(u^{\bar{\bar{k}}}, u_x^{\bar{\bar{k}}})$ on the whole time interval $[0,T]$ at $x = x_0$, and

$$(c(t), \bar{c}(t)) = \begin{cases} (a(t), \bar{a}(t)), & 0 \le t \le T_0, \\ (b(t), \bar{b}(t)), & T - T_0 \le t \le T. \end{cases} \tag{5.10}$$

Noting (2.2), we now change the status of t and x, and consider the leftward mixed initial-boundary value problem (see Figure 6(a)) for system (2.1) with the initial condition

$$x = x_0: \quad u^{\bar{\bar{k}}} = c(t), \quad u_x^{\bar{\bar{k}}} = \bar{c}(t), \qquad 0 \le t \le T \tag{5.11}$$

and Dirichlet boundary conditions

$$t=0: \quad u^{\bar{\bar{k}}}=\phi_{\bar{\bar{k}}}(x), \qquad d_{\bar{\bar{k}}0}\leq x\leq x_0, \tag{5.12}$$

$$t=T: \quad u^{\bar{\bar{k}}}=\Phi_{\bar{\bar{k}}}(x), \qquad d_{\bar{\bar{k}}0}\leq x\leq x_0, \tag{5.13}$$

where $\phi_{\bar{\bar{k}}}(x)$ and $\Phi_{\bar{\bar{k}}}(x)$ are given by (2.5) and (2.15), respectively. Obviously, the conditions of C^2 compatibility at the points $(t,x)=(0,x_0)$ and (T,x_0) are satisfied. Then there exists a unique C^2 solution $u_l^{\bar{\bar{k}}}=u_l^{\bar{\bar{k}}}(t,x)$ with small C^2 norm on the domain $\mathcal{R}_{\bar{\bar{k}}l}(T)=\{(t,x)|0\leq t\leq T, d_{\bar{\bar{k}}0}\leq x\leq x_0\}$ and

$$|u_l^{\bar{\bar{k}}}(t,x)|+\left|\frac{\partial u_l^{\bar{\bar{k}}}}{\partial x}(t,x)\right|\leq\varepsilon_0, \qquad \forall (t,x)\in\mathcal{R}_{\bar{\bar{k}}l}(T). \tag{5.14}$$

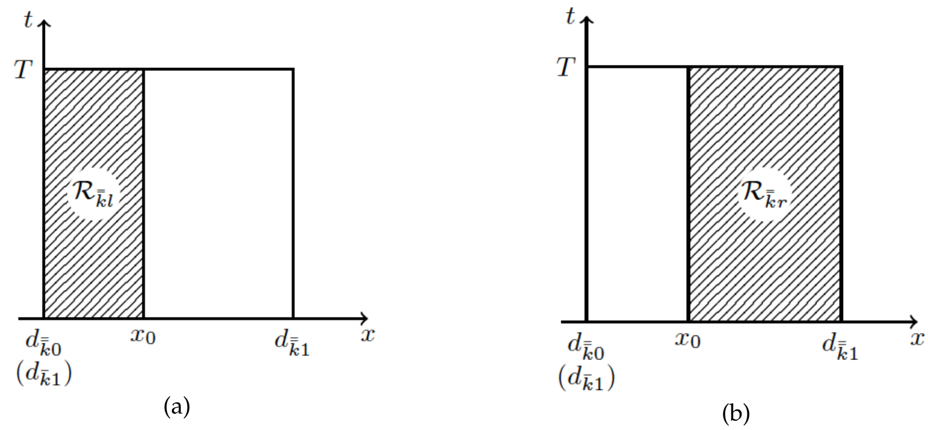

Figure 6: (a) Leftward mixed problem; (b) rightward mixed problem.

Similarly, for the rightward mixed initial-boundary value problem (see Figure 6(b)) for system (2.1) with the initial condition (5.11) and Dirichlet boundary conditions

$$t=0: \quad u^{\bar{\bar{k}}}=\phi_{\bar{\bar{k}}}(x), \qquad x_0\leq x\leq d_{\bar{\bar{k}}1}, \tag{5.15}$$

$$t=T: \quad u^{\bar{\bar{k}}}=\Phi_{\bar{\bar{k}}}(x), \qquad x_0\leq x\leq d_{\bar{\bar{k}}1}, \tag{5.16}$$

there exists a unique C^2 solution $u_r^{\bar{\bar{k}}}=u_r^{\bar{\bar{k}}}(t,x)$ with small C^2 norm on the domain$\mathcal{R}_{\bar{\bar{k}}r}(T)=\{(t,x)|0\leq t\leq T, x_0\leq x\leq d_{\bar{\bar{k}}1}\}$ and

$$|u_r^{\bar{\bar{k}}}(t,x)|+\left|\frac{\partial u_r^{\bar{\bar{k}}}}{\partial x}(t,x)\right|\leq\varepsilon_0, \qquad \forall (t,x)\in\mathcal{R}_{\bar{\bar{k}}r}(T). \tag{5.17}$$

Let

$$u^{\bar{\bar{k}}}(t,x)=\begin{cases} u_l^{\bar{\bar{k}}}(t,x), & (t,x)\in\mathcal{R}_{\bar{k}l}(T),\\ u_r^{\bar{\bar{k}}}(t,x), & (t,x)\in\mathcal{R}_{\bar{k}r}(T).\end{cases} \tag{5.18}$$

Obviously, $u^{\bar{\bar{k}}}(t,x)$ is a C^2 solution on the domain $\mathcal{R}_{\bar{k}}(T)=\{(t,x)|0\le t\le T,d_{\bar{k}0}\le x\le d_{\bar{k}1}\}$.

The C^2 solutions $u_l^{\bar{\bar{k}}}$(resp. $u_r^{\bar{\bar{k}}}$) and $u_f^{\bar{\bar{k}}}$ satisfy simultaneously the same system (2.1) (in which we take $i=\bar{\bar{k}}$), the same initial condition (5.11) and the same boundary condition(5.12)(resp. (5.15)), by uniqueness of C^2 solutions to the one-sided mixed initial boundary value problem (see [7]), $u^{\bar{\bar{k}}}\equiv u_f^{\bar{\bar{k}}}$ on the trapezoidal area (see Figure 7)

$$\{(t,x)|0\le t\le T_0-A_{\bar{k}}(x_0-x),d_{\bar{k}0}\le x\le x_0\} \tag{5.19a}$$

$$(\text{resp.}\{(t,x)|0\le t\le T_0-A_{\bar{k}}(x-x_0),x_0\le x\le d_{\bar{k}1}\}), \tag{5.19b}$$

where

$$A_{\bar{k}}=\sup_{|u^{\bar{\bar{k}}}|+|v^{\bar{\bar{k}}}|\le\varepsilon_0}\frac{1}{\sqrt{K^{\bar{\bar{k}}}_{v^{\bar{k}}}(u^{\bar{\bar{k}}},v^{\bar{\bar{k}}})}}.$$

In particular, on the interval $[d_{\bar{k}0},d_{\bar{k}1}]$ on the x-axis, $u^{\bar{\bar{k}}}$ satisfies the initial condition (2.5) (in which we take $i=\bar{\bar{k}}$), and on two ends of string $S_{\bar{k}}$ we have

$$x=d_{\bar{k}0}:\ (u^{\bar{\bar{k}}},u_x^{\bar{\bar{k}}})=(a_{\bar{k}}(t),\bar{a}_{\bar{k}}(t)),\qquad 0\le t\le T_0-A_{\bar{k}}(x_0-d_{\bar{k}0}), \tag{5.20}$$

$$x=d_{\bar{k}1}:\ (u^{\bar{\bar{k}}},u_x^{\bar{\bar{k}}})=(\alpha_{\bar{k}}(t),\bar{\alpha}_{\bar{k}}(t)),\qquad 0\le t\le T_0-A_{\bar{k}}(d_{\bar{k}1}-x_0), \tag{5.21}$$

where $(a_{\bar{k}}(t),\bar{a}_{\bar{k}}(t))$ and $(\alpha_{\bar{k}}(t),\bar{\alpha}_{\bar{k}}(t))$ are given by the forward solution $(u_f^{\bar{\bar{k}}},u_{fx}^{\bar{\bar{k}}})$ obtained in (i).

Similarly, we obtain that $u^{\bar{\bar{k}}}$ also satisfies the final condition (2.15) (in which we take $i=\bar{\bar{k}}$), and on two ends of string $S_{\bar{k}}$ we have

$$x=d_{\bar{k}0}:\ (u^{\bar{\bar{k}}},u_x^{\bar{\bar{k}}})=(b_{\bar{k}}(t),\bar{b}_{\bar{k}}(t)),\qquad T-T_0+A_{\bar{k}}(x_0-d_{\bar{k}0})\le t\le T; \tag{5.22}$$

$$x=d_{\bar{k}1}:\ (u^{\bar{\bar{k}}},u_x^{\bar{\bar{k}}})=(\beta_{\bar{k}}(t),\bar{\beta}_{\bar{k}}(t)),\qquad T-T_0+A_{\bar{k}}(d_{\bar{k}1}-x_0)\le t\le T, \tag{5.23}$$

where$(b_{\bar{k}}(t),\bar{b}_{\bar{k}}(t))$ and $(\beta_{\bar{k}}(t),\bar{\beta}_{\bar{k}}(t))$ are given by the backward solution $(u_b^{\bar{\bar{k}}},u_{bx}^{\bar{\bar{k}}})$ obtained in (ii).

(v) If $d_{\bar{k}1}$ is a simple node, the rightward constructive procedure is over. Otherwise, on the multiple node $d_{\bar{k}1}$, in a similar way as (v) in the proof of Lemma 4.1, we can find $(c_l(t),\bar{c}_l(t))\in C^3[0,T]\times C^1[0,T]$ $(l=\bar{\bar{k}}$ and $l\in\mathcal{J}_{\bar{k}})$ with small $C^3[0,T]\times C^1[0,T]$ norm, being the value of (u^l,u_x^l) $(l=\bar{\bar{k}}$ and $l\in\mathcal{J}_{\bar{k}})$, and for $l=\bar{\bar{k}}$ and $l\in\mathcal{J}_{\bar{k}}$,

$$(c_l(t),\bar{c}_l(t))=\begin{cases}(\alpha_l(t),\bar{\alpha}_l(t)), & 0\le t\le T_0-A_{\bar{k}}(d_{\bar{k}1}-x_0),\\ (\beta_l(t),\bar{\beta}_l(t)), & T-T_0+A_{\bar{k}}(d_{\bar{k}1}-x_0)\le t\le T,\end{cases} \tag{5.24}$$

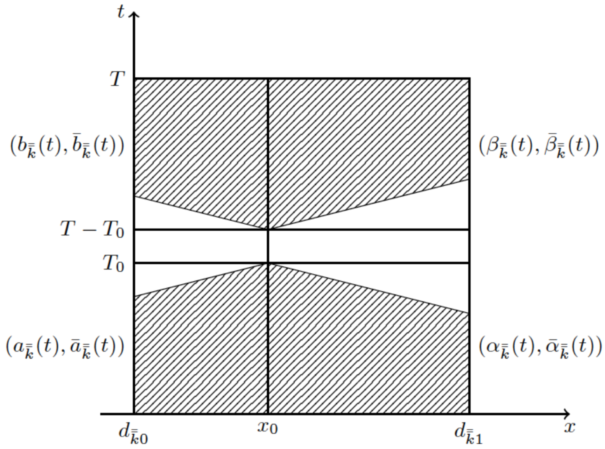

Figure 7: Uniqueness of solutions on the shaded domain.

where $(\alpha_l(t),\bar{\alpha}_l(t))(l\in\mathcal{J}_{\bar{\bar{k}}})$ and $(\beta_l(t),\bar{\beta}_l(t))$ $(l\in\mathcal{J}_{\bar{\bar{k}}})$ are determined by the forward solution (u^l_f,u^l_{fx}) and the backward solution (u^l_b,u^l_{bx}) obtained in (i) and (ii), respectively. Thus, $(u^l,u^l_t,u^l_{tt},u^l_x)=(c_l(t),c_l'(t),c_l''(t),\bar{c}_l(t))$ $(l=\bar{\bar{k}}$ and $l\in\mathcal{J}_{\bar{\bar{k}}})$ satisfies the interface conditions (2.9) on the whole interval $[0,T]$.

For $l\in\mathcal{J}_{\bar{\bar{k}}}$, we consider d_{l0} as a starting node and solve the rightward mixed initial-boundary value problem for system (2.1) with the initial condition

$$x=d_{l0}:\quad u^l=c_l(t),\quad u^l_x=\bar{c}_l(t),\qquad 0\le t\le T \tag{5.25}$$

and Dirichlet boundary conditions

$$t=0:\quad u^l=\phi_l(x),\qquad d_{l0}\le x\le d_{l1}, \tag{5.26}$$

$$t=T:\quad u^l=\Phi_l(x),\qquad d_{l0}\le x\le d_{l1}. \tag{5.27}$$

There exists a unique C^2 solution $u^l=u^l(t,x)$ with small C^2 norm on the domain $\mathcal{R}_l(T)=\{(t,x)|0\le t\le T,d_{l0}\le x\le d_{l1}\}$ and

$$|u^l(t,x)|+\left|\frac{\partial u^l}{\partial x}(t,x)\right|\le\varepsilon_0,\qquad \forall(t,x)\in\mathcal{R}_l(T). \tag{5.28}$$

By uniqueness of C^2 solutions to the one-sided mixed initial-boundary value problem (see [7]), it is easy to prove that $u^l=u^l(t,x)$ satisfies the initial condition (2.5) (in which we take $i=l$) and the final condition (in which we take $i=l$), and on node d_{l1} we have

$$(u^l,u^l_x)=\begin{cases}(a_l(t),\bar{a}_l(t)), & 0\le t\le T_0-A_{\bar{\bar{k}}}(d_{\bar{\bar{k}}1}-x_0)-A_lL_l,\\ (b_l(t),\bar{b}_l(t)), & T-T_0+A_{\bar{\bar{k}}}(d_{\bar{\bar{k}}0}-x_0)+A_lL_l\le t\le T,\end{cases} \tag{5.29}$$

where $(a_l(t),\bar{a}_l(t))$ and $(b_l(t),\bar{b}_l(t))$ are given by the forward solution (u^l_f,u^l_{fx}) and the backward solution (u^l_b,u^l_{bx}) obtained in (i) and (ii), respectively.

(vi) The rightward constructive procedure is over when d_{l1} is a simple node. Otherwise, considering d_{l1} as a starting node and do the constructive process above again, it is easy to see that we can continue this rightward constructive procedure on the right half tree-like network.

When d_{j1} is a node on the right half of the tree-like network, there is a unique chain-like subnetwork connecting point $x=x_0$ and node d_{j1}. Let $\mathcal{D}_{x_0,j}$ stand for the set of indices k of all nodes d_{k0} contained in this subnetwork. Noting (5.1),(5.2) and the definition of x_0 given by (5.8), we have

$$T_0-A_{\bar{k}}(d_{\bar{k}1}-x_0)-\sum_{k\in\mathcal{D}_{x_0,j}}A_kL_k\geq 0, \tag{5.30a}$$

$$T-T_0+A_{\bar{k}}(d_{\bar{k}1}-x_0)+\sum_{k\in\mathcal{D}_{x_0,j}}A_kL_k\geq 0. \tag{5.30b}$$

Thus, for any given $k_1,k_2\in\mathcal{D}_{x_0,j}$ satisfying $d_{k_11}=d_{k_20}$, we have $u^{k_2}\equiv u^{k_2}_f$ (resp. $u^{k_2}\equiv u^{k_2}_b$)

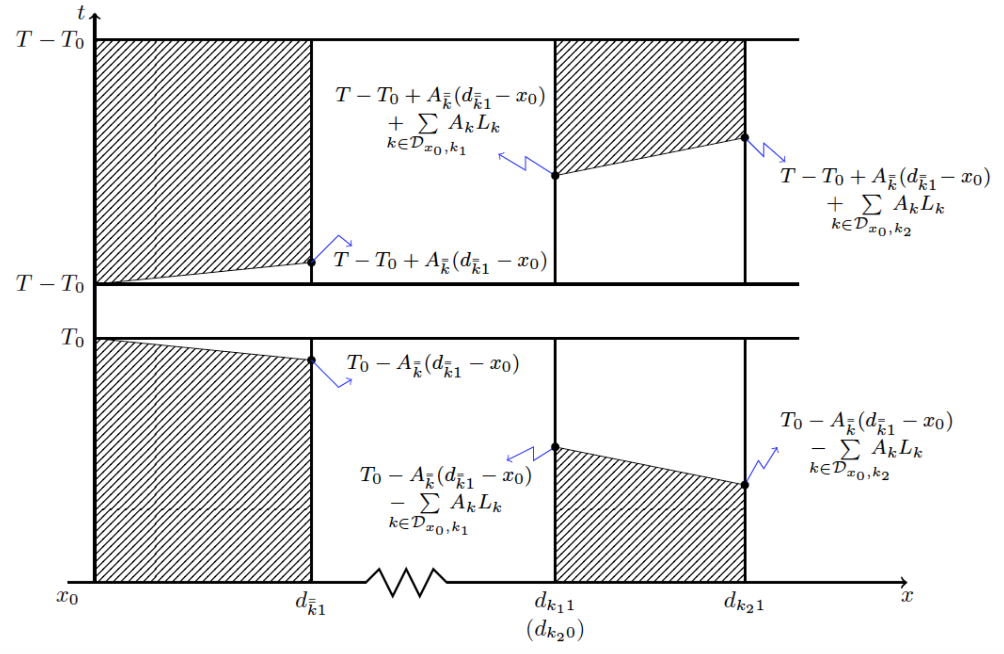

Figure 8: The rightward procedure and uniqueness of solutions on the shaded domain.

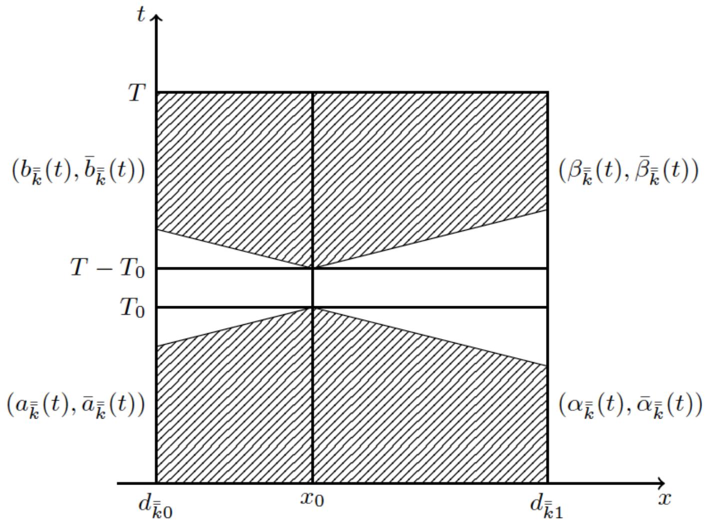

Figure 9: The case that x_0 is exactly a multiple node.

on the the trapezoidal area (see Figure 8)

$$\Big\{(t,x)|0\le t\le T_0-A_{\bar{\bar{k}}}(d_{\bar{\bar{k}}1}-x_0)-\sum_{k\in\mathcal{D}_{x_0,k_1}}A_kL_k-A_{k_2}(x-d_{k_20}),d_{k_20}\le x\le d_{k_21}\Big\}$$
$$\left(\text{resp. }\Big\{\Big((t,x)|T-T_0+A_{\bar{\bar{k}}}(d_{\bar{\bar{k}}1}-x_0)+\sum_{k\in\mathcal{D}_{x_0,k_1}}A_kL_k+A_{k_2}(x-d_{k_20})\le t\le T,d_{k_20}\le x\le d_{k_21}\Big\}\right). \tag{5.31}$$

Thus, the piecewise C^2 solution $u^k(k\in\mathcal{D}_{x_0,j})$ constructed above satisfies simultaneously the initial condition (2.5) and the final condition (2.15).

(vii) Similarly to the rightward constructive procedure shown in step (v) and (vi), starting from x_0, we can finish the leftward constructive procedure on the left half of the tree-like network by steps.

We now discuss the case that $x_0=d_{\bar{k}0}(x_0=d_{\bar{k}1})$, namely, x_0 is exactly the multiple node $d_{\bar{k}1}$ on the network (see Figure 9).

In this case, by steps (i) and (ii), we can determine the value of $(u_f^{\bar{k}},u_{fx}^{\bar{k}})$, $(u_b^{\bar{k}},u_{bx}^{\bar{k}})$ and

$(u_f^l, u_{fx}^l)(l \in \mathcal{J}_{\bar{k}})$, $(u_b^l, u_{bx}^l)(l \in \mathcal{J}_{\bar{k}})$ at $x = x_0$ as follows:

$$x = x_0: \quad \begin{array}{lll} (u_f^{\bar{k}}, u_{fx}^{\bar{k}}) = (a_{\bar{k}}(t), \bar{a}_{\bar{k}}(t)), & 0 \le t \le T_0, & \\ (u_b^{\bar{k}}, u_{bx}^{\bar{k}}) = (b_{\bar{k}}(t), \bar{b}_{\bar{k}}(t)), & T - T_0 \le t \le T, & \\ (u_f^l, u_{fx}^l) = (\alpha_l(t), \bar{\alpha}_l(t)), & 0 \le t \le T_0, & l \in \mathcal{J}_{\bar{k}}, \\ (u_b^l, u_{bx}^l) = (\beta_l(t), \bar{\beta}_l(t)), & T - T_0 \le t \le T, & l \in \mathcal{J}_{\bar{k}}, \end{array} \tag{5.32}$$

and, by the hidden regularity on the multiple nodes,

$$\begin{array}{l} \|(a_{\bar{k}}, \bar{a}_{\bar{k}})\|_{C^3[0,T_0] \times C^1[0,T_0]}, \|(\alpha_l, \bar{\alpha}_l)\|_{C^3[0,T_0] \times C^1[0,T_0]} \ (l \in \mathcal{J}_{\bar{k}}), \\ \|(b_{\bar{k}}, \bar{b}_{\bar{k}})\|_{C^3[T-T_0,T] \times C^1[T-T_0,T]}, \quad \|(\beta_l, \bar{\beta}_l)\|_{C^3[T-T_0,T] \times C^1[T-T_0,T]} (l \in \mathcal{J}_{\bar{k}}) \end{array}$$

are all small enough. Furthermore, by the interface conditions (2.9) (in which we take $i = \bar{k}$), we have

$$\begin{array}{lll} a_{\bar{k}}(t) = \alpha_l(t), & 0 \le t \le T_0, & l \in \mathcal{J}_{\bar{k}}, \\ b_{\bar{k}}(t) = \beta_l(t), & T - T_0 \le t \le T, & l \in \mathcal{J}_{\bar{k}}. \end{array} \tag{5.33}$$

Noting (5.1) and (5.2), we can find $(c(t), \bar{c}(t))$ with small $\|(c, \bar{c})\|_{C^3[0,T] \times C^1[0,T]}$ norm, being the value of $(u^{\bar{k}}, u_x^{\bar{k}})$ on the whole time interval $[0,T]$ at $x = x_0$, such that

$$(c_{\bar{k}}(t), \bar{c}_{\bar{k}}(t)) = \begin{cases} (a_{\bar{k}}(t), \bar{a}_{\bar{k}}(t)), & 0 \le t \le T_0, \\ (b_{\bar{k}}(t), \bar{b}_{\bar{k}}(t)), & T - T_0 \le t \le T. \end{cases} \tag{5.34}$$

Similar to the proof in (v), we can find $(c_l(t), \bar{c}_l(t))(l \in \mathcal{J}_{\bar{k}})$ with small $\|(c_l, \bar{c}_l)\|_{C^3[0,T] \times C^1[0,T]}$ $(l \in \mathcal{J}_{\bar{k}})$ norm, being the value of $(u^l, u_x^l)(l \in \mathcal{J}_{\bar{k}})$ on the whole time interval $[0,T]$ at $x = x_0$, such that

$$(c_l(t), \bar{c}_l(t)) = \begin{cases} (\alpha_l(t), \bar{\alpha}_l(t)), & 0 \le t \le T_0, \\ (\beta_l(t), \bar{\beta}_l(t)), & T - T_0 \le t \le T, \end{cases} \quad l \in \mathcal{J}_{\bar{k}}, \tag{5.35}$$

and $(c_{\bar{k}}(t), \bar{c}_{\bar{k}}(t))$, $(c_l(t), \bar{c}_l(t))$ $(l \in \mathcal{J}_{\bar{k}})$ satisfy the interface conditions (2.9) (in which we take $i = \bar{k}$) on the whole interval $[0,T]$ at $x = d_{\bar{k}1}$ $(x = x_0)$.

Starting from $x = x_0 (x = d_{\bar{k}1})$, similarly to (v) and (vi), we can finish the leftward and rightward constructive procedure on the corresponding left and right half of tree-like network. Thus, we get a piecewise C^2 solution $\mathbf{u} = (u_1, \ldots, u_N)$ with small piecewise C^2 norm.

(vii) Noting the setting of T_0 given by (4.2), for any given $i \in \mathcal{S}$, we have

$$T_0 - \sum_{j \in \mathcal{D}_i} T_j \ge 0,$$

which promises that the piecewise C^2 solutions $u^i=u^i(t,x)(i=1,...,N)$ obtained above satisfy

$$t=0:\ u^i\equiv u^i_f,\quad d_{i0}\le x\le d_{i1}\ \ (i=1,...,N),$$
$$t=T:\ u^i\equiv u^i_b,\quad d_{i0}\le x\le d_{i1}\ \ (i=1,...,N),$$

then, $\mathbf{u}=(u^1,...,u^N)=(u^1(t,x),...,u^N(t,x))$ satisfies the initial condition (2.5) and the final condition (2.12). Consequently, this solution meets with all the requirements given in Lemma 5.1. □

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