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Energy Stable Finite Element/Spectral Method for Modified Higher-Order Generalized Cahn-Hilliard Equations

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Abstract. Our aim in this paper is to study a fully discrete scheme for modified higher-order (in space) anisotropic generalized Cahn-Hilliard models which have extensive applications in biology, image processing, etc. In particular, the scheme is a combination of finite element or spectral method in space and a second-order stable scheme in time. We obtain energy stability results, as well as the existence and uniqueness of the numerical solution, both for the space semi-discrete and fully discrete cases. We also give several numerical simulations which illustrate the theoretical results and, especially, the effects of the higher-order terms on the anisotropy.

AMS subject classifications: 35K55, 35J60

Key words: Modified Cahn-Hilliard equation, higher-order models, energy stability, anisotropy.

1 Introduction

The Cahn-Hilliard equation

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$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0 \tag{1.1}$$

plays an essential role in materials science and describes important qualitative features of two-phase systems related with phase separation processes, assuming isotropy and a constant temperature (see, e.g., [6,7,15,40,41]).

Here, u is the order parameter (e.g., a density of atoms) and f is the derivative of a double-well potential F. A thermodynamically relevant potential F is the following logarithmic function which follows from a mean-field model:

$$F(s)=\frac{\theta_c}{2}(1-s^2)+\frac{\theta}{2}\left((1-s)\ln(\frac{1-s}{2})+(1+s)\ln(\frac{1+s}{2})\right),$$
$$s\in(-1,1),\ \ 0<\theta<\theta_c, \tag{1.2}$$

i.e.,

$$f(s)=-\theta_c s+\frac{\theta}{2}\ln\frac{1+s}{1-s}, \tag{1.3}$$

although such a function is very often approximated by regular ones, typically,

$$F(s)=\frac{1}{4}(s^2-1)^2, \tag{1.4}$$

i.e.,

$$f(s)=s^3-s. \tag{1.5}$$

Now, it is interesting to note that the Cahn-Hilliard equation and some of its variants are also relevant in other phenomena than phase separation. We can mention, for instance, population dynamics (see [17]), tumor growth (see [1,31]), bacterial films (see [32]), thin films (see [43,45]), image processing (see [3,4,8,10,18]) and even the rings of Saturn (see [46]) and the clustering of mussels (see [34]).

In particular, several such phenomena can be modeled by the following generalized Cahn-Hilliard equation:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)+g(x,u)=0. \tag{1.6}$$

We studied in [36,39] (see also [1,10,16,20]) this equation.

The Cahn-Hilliard equation is based on the so-called Ginzburg-Landau free energy,

$$\Psi_{\mathrm{GL}}=\int_\Omega\left(\frac{1}{2}|\nabla u|^2+F(u)\right)dx, \tag{1.7}$$

where Ω is the domain occupied by the system (we assume here that it is a bounded and regular domain of $\mathbb{R}^d$, $d=1$, 2 or 3, with boundary Γ). In particular, in (1.7), the