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Gevrey Regularity of the Global Attractor for Damped Forced KdV Equation on the Real Line

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Abstract. We consider here a weakly damped KdV equation on the real line with forcing term that belongs to some Gevrey space. We prove that the global attractor is also contained into such a space of analytic functions.

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1 Introduction

1.1 Global attractor for damped forced KdV equation on the real line

This article is concerned with regularity issues for the global attractor of a weakly damped, forced Korteweg-de Vries (KdV) equation that reads

$$u_t+\gamma u+u_{xxx}+uu_x=f; \tag{1.1}$$

here $\gamma>0$ is the positive damping parameter and the forcing term f does not depends on t. Throughout this article we will assume that f is a real analytic function that belongs to some Gevrey space G^{τ_0} (see the precise definition of Gevrey spaces below). The unknown u is a function from $\mathbb{R}_t\times\mathbb{R}_x$ into $\mathbb{R}$. This article partakes of the infinite-dimensional dynamical system theory (see [10,14,18,24]). For weakly damped dispersive equations as (1.1) a first issue concerns the existence of a global attractor. Indeed, supplemented with an initial data in $H^\rho(\mathbb{R})$, the solution map $S_\rho(t)u_0=u(t)$ defines a semigroup in $H^\rho(\mathbb{R})$. When it exists, the global attractor $\mathcal{A}_\rho$ is a compact invariant attracting set in $H^\rho(\mathbb{R})$ (see [24]). For KdV equations on the whole line, P. Laurençot [13] proved the existence of a global attractor $\mathcal{A}_2$ for the semigroup in $H^2(\mathbb{R})$, while R. Rosa [19] proved the existence of a global attractor $\mathcal{A}_1$ for the semigroup in $H^1(\mathbb{R})$ by the so-called *energy*

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method (see [16] and the references therein). Eventually, in [8] the authors proved that the semigroup $S_0(t)$ in $L^2(\mathbb{R})$ has a compact global attractor $\mathcal{A}_0$ that is smooth, i.e. that if $f\in L^2(\mathbb{R})$ then $\mathcal{A}_0\subset H^3(\mathbb{R})$. As a consequence of this result we have that $\mathcal{A}_0=\mathcal{A}_1=\mathcal{A}_2$. We set $\mathcal{A}=\mathcal{A}_0$ for this global attractor in the sequel. Besides we know that if f is assumed smooth then $\mathcal{A}$ is a subset of $\cap_m H^m(\mathbb{R})$. We also refer to [25] for the study weakly damped forced KdV equations below $L^2(\mathbb{R})$ by the I-method of J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao (see [22] and the references therein).

We address in this article the following issue: is the global attractor $\mathcal{A}$ included in a set of analytic functions if the forcing term belongs to some Gevrey space? For weakly damped dispersive equations the answer is positive for periodic $1D$ nonlinear Schrödinger equations. This result is due to M. Oliver and E. S. Titi [17] and has some important consequences. The first one is that the Faedo-Galerkin approximation of a solution included in the global attractor converges exponentially fast. The second one is in estimating the number of determining nodes for solutions in the global attractor. This issue for KdV equation was still open. Our main result is as follows

Theorem 1.1. *Assume that the forcing term f belongs to some Gevrey space G^{τ_0}. Then there exists a τ that depends on γ, f such that the global attractor $\mathcal{A}$ is a bounded subset of G^τ.*

For the periodic KdV equations, the regularity of the attractor was proved in [15], [6]. For the Gevrey regularity issue in this periodic setting we refer to [7]. Besides, the study of the initial value problem for KdV equation, from the pioneering works [2,20,23] to more recent methods by C. Kenig, G. Ponce and L. Vega [11], has been boosted by the work of J. Bourgain [3] (see [5,12,22] and the references therein). Moreover the so-called Bourgain method was used in the last decade in [1,9,21] to tackle the initial value problem for KdV equations in Gevrey spaces and other issues related to the analyticity in space of solutions in the conservative case $\gamma=0$ and $f=0$. Therefore our strategy here is to combine the splitting method used in [8] with the use of Bourgain-Gevrey spaces. We consider a complete trajectory that is included in the global attractor, i.e. that is defined for all (positive and negative) times and bounded. We split the solution into a low frequency and a high frequency part, choosing the cut-off N large enough; we then introduce an auxiliary problem whose solution approximates the high frequency part of the trajectory in $L^2(\mathbb{R})$ for large times, and whose initial data belongs to some Gevrey space; we then prove the persistence of the Gevrey regularity for the solution of the auxiliary problem. We conclude shifting backward in time the solution of the auxiliary problem.

The outline of the article is the following. In the remaining of Section 1 we introduce the required mathematical framework for Bourgain-Gevrey spaces and then prove some useful bilinear estimates in these spaces. Section 2 is devoted to the proof of the Theorem 1.1 as described in the previous paragraph. In a last Section we discuss the size of the global attractor in Gevrey spaces G^τ for small τ.

1.2 Mathematical framework

We consider here

$$L^2(\mathbb{R})=\left\{u:\mathbb{R}\to\mathbb{R};\int_{\mathbb{R}}|\hat{u}(\xi)|^2d\xi<+\infty\right\},$$

equipped with the scalar product

$$<u,v>=\int_{\mathbb{R}}u(x)v(x)dx=\frac{1}{2\pi}\mathrm{Re}\int_{\mathbb{R}}\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi.$$

In the sequel we denote by $\hat{u}$ or $\mathcal{F}(u)$ the Fourier transform of u while u is a function of one variable x or two variables t,x. The Schwartz space is denoted by $\mathcal{S}(\mathbb{R}^2)$. We set D for the operator whose symbol is $|\xi|$, i.e. $D=\mathcal{H}\partial_x$ where $\mathcal{H}$ is the Hilbert transform. In the sequel when dealing with $H^\rho(\mathbb{R})$-based norms we will use equivalently either ∂_x or D. Introduce now for any given $\tau\geq 0$ the unbounded self-adjoint operator $e^{\tau D}$ whose symbol is $e^{\tau|\xi|}$, i.e. such that $\mathcal{F}(u)(\xi)=e^{\tau|\xi|}\hat{u}(\xi)$. The Gevrey space G^τ is then the space of (analytic) functions u such that $e^{\tau D}u$ belongs to $L^2(\mathbb{R})$. Following [4] we also define $G^{\tau,\rho}$ as the space of functions u such that u and $D^\rho e^{\tau D}u$ belongs to $L^2(\mathbb{R})$. The associated norm is, setting $<x>=\sqrt{1+x^2}$,

$$||u||_{G^{\tau,\rho}}=\left(\int_{\mathbb{R}}<\xi>^{2\rho}e^{2\tau|\xi|}|\hat{u}(\xi)|^2d\xi\right)^{\frac{1}{2}}.$$

If $\tau=0$ then $G^{0,\rho}$ is the usual Sobolev space $H^\rho(\mathbb{R})$. Consider $W(t)u_0$ the solution of the Airy equation

$$u_t+u_{xxx}=0,\quad u(0)=u_0. \tag{1.2}$$

Following [9] we introduce the Bourgain-Gevrey norms of a function $u(t,x)$ as

$$||u||^2_{\mathbb{X}^{\tau,\rho,b}}=||W(-t)u||^2_{H^b_tG^{\tau,\rho}}=\int_{\mathbb{R}^2}<\theta-\xi^3>^{2b}e^{2\tau|\xi|}|\xi|^{2\rho}|\hat{u}(\theta,\xi)|^2d\xi d\theta; \tag{1.3}$$

here H^b_t is the usual Sobolev space $H^b(\mathbb{R})$. The notation $H^b_tG^{\tau,\rho}$ is a short hand notation for $H^b(\mathbb{R}_t;G^{\tau,\rho}_x)$. The Bourgain-Gevrey space $\mathbb{X}^{\tau,\rho,b}$ is then the closure of the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ for this norm. When $\rho=0$ we set $\mathbb{X}^{\tau,0,b}=\mathbb{X}^{\tau,b}$ for the sake of simplicity. When $\tau=0$ we use the usual notation $\mathbb{X}^{0,\rho,b}=X^{\rho,b}$. We recall that for $b>\frac{1}{2}$ we have the continuous embedding $\mathbb{X}^{\tau,b}\subset C(\mathbb{R},G^\tau)\cap L^\infty(\mathbb{R},G^\tau)$. For each of this space we consider their Fourier restriction norm as

$$||u||_{\mathbb{X}^{\tau,\rho,b}_{\mathcal{I}}}=\inf\left\{||v||_{\mathbb{X}^{\tau,\rho,b}};\, v=u\ \text{ for }\ t\in\mathcal{I}\right\}, \tag{1.4}$$

for any given interval $\mathcal{I}\subset\mathbb{R}_t$. If $\mathcal{I}=[-T,T]$ we set for simplicity $\mathbb{X}^{\tau,\rho,b}_{\mathcal{I}}=\mathbb{X}^{\tau,\rho,b}_T$. The following result (N. Tzvetkov, graduate course, Lille 2005, unpublished) will be used in the sequel

Lemma 1.1. *For u in* $\mathbb{X}_T^{\tau,\rho,b}$ *there exists* $\tilde{u}$ *in* $\mathbb{X}^{\tau,\rho,b}$ *such that* $||u||_{\mathbb{X}_T^{\tau,\rho,b}} = ||\tilde{u}||_{\mathbb{X}^{\tau,\rho,b}}$. *Moreover for any* $0 \le v \le \tau$, $0 \le \varrho \le \rho$ *we also have* $||u||_{\mathbb{X}_T^{v,\varrho,b}} = ||\tilde{u}||_{\mathbb{X}^{v,\varrho,b}}$.

Proof. Let us introduce the closed subspace of $\mathbb{X}^{\rho,\tau,b}$ defined as

$$E = \{u \in \mathbb{X}^{\rho,\tau,b}; v(t,x) = 0 \text{ for } t \in (-T,T)\}.$$

Set $\tilde{E}$ for the orthogonal complement of E in $\mathbb{X}^{\rho,\tau,b}$. Then the orthogonal projection $\tilde{u}$ of u in $\tilde{E}$ achieves $\inf\{||v||_{\mathbb{X}^{\tau,\rho,b}}; v = u \in (-T,T)\}$. Observing that $\tilde{u}$ is also the orthogonal projection in $\mathbb{X}^{v,\varrho,b}$ completes the proof of the Lemma. □

In the sequel we use a smooth cut-off function $\psi(t)$ that is 1 if $|t| \le 1$ and 0 if $|t| \ge 2$. Consider $T > 0$. We set $\psi_T(t) = \psi(\frac{t}{T})$. We also set $\psi_\gamma(t) = e^{-\gamma t}\psi(t)$. We now state

Lemma 1.2. *Consider* $b > \frac{1}{2}$. *There exists a constant* c_γ *that depends on* γ, ψ, b *such that for any u in* $\mathbb{X}^{\rho,\tau,b}$

$$||\psi_\gamma u||_{\mathbb{X}^{\rho,\tau,b}} \le c_\gamma ||u||_{\mathbb{X}^{\rho,\tau,b}}.$$

Proof. The proof of the Lemma amounts to prove the one-dimensional inequality

$$||\psi_\gamma a||_{H_t^b} \le c_\gamma ||a||_{H_t^b}, \tag{1.5}$$

for a smooth function $a(t)$. Since $H^b(\mathbb{R})$ is an algebra if $b > \frac{1}{2}$ then the results follows. □

Besides throughout this article we denote by c a numerical constant, and K a constant that depends on γ, f. These constants c and K may vary from one line to one another. For a function in $L^2(\mathbb{R})$ and for a level N we introduce the orthogonal projector P_N as $P_N u(x) = \frac{1}{2\pi}\int_{|\xi| \le N} \hat{u}(\xi)\exp(i\xi x)d\xi$ and $Q_N = Id - P_N$ the projector onto the orthogonal complement. Setting $y = P_N u$ and $z = Q_N u$ we recall the following specialized inverse and Poincaré inequalities (for ρ, τ nonnegative)

$$c\max(e^{-\tau N}||y||_{G^\tau}, N^{-\rho}||y||_{H^\rho}) \le ||y||_{L^2} \le ||u||_{L^2}, \tag{1.6}$$

$$||z||_{L^2} \le c\min(N^{-\rho}||z||_{H^\rho}, e^{-\tau N}||z||_{G^\tau}). \tag{1.7}$$

1.3 Bilinear and other inequalities

To handle the initial value problem we begin with an estimate for the inhomogeneous linear problem. For a proof see [5] and the references therein

Proposition 1.1. *Assume* $0 < b' < 1 - b < \frac{1}{2}$ *and* $T \le 1$. *There exists a constant c that is independent of* T, τ, ρ *such that if g belongs to* $\mathbb{X}^{\tau,\rho,-b'}$, *then*

$$||\psi_T(t)\int_0^t W(t-s)g(s)ds||_{\mathbb{X}^{\tau,\rho,b}} \le cT^{1-b-b'}||g||_{\mathbb{X}^{\tau,\rho,-b'}}. \tag{1.8}$$

We now state bilinear estimate required to handle the local well-posedness issue; these bilinear estimates are slight modifications of classical bilinear Bourgain estimates (see [12] and the references therein) for KdV equations.

Proposition 1.2. *Assume $b' \geq \frac{1}{4}$ and $b > \frac{1}{2}$. Assume $\tau,\rho \geq 0$. There exists a constant c that is independent of τ such that for any function u in $\mathbb{X}^{\tau,\rho,b}$*

$$||D(uv)||_{\mathbb{X}^{\tau,\rho,-b'}} \leq c||u||_{\mathbb{X}^{\tau,\rho,b}}||v||_{\mathbb{X}^{\tau,\rho,b}}. \tag{1.9}$$

Proof. We proceed by a duality argument. Consider u,v,G in $\mathcal{S}(\mathbb{R}^2)$ such that $||G||_{L^2(\mathbb{R}^2)} = ||u||_{L^2(\mathbb{R}^2)} = ||v||_{L^2(\mathbb{R}^2)} = 1$. To prove (1.9) amounts to prove the estimate,

$$q = \left| \int_{\mathbb{R}^4} \frac{|\xi| <\xi>^{\rho} e^{\tau(|\xi|-|\xi_1|-|\xi_2|)}}{<\xi_1>^{\rho}<\xi_2>^{\rho}><\theta-\xi^3>^{b'}} \frac{\hat{G}\hat{u}_1\hat{u}_2}{<\theta_1-\xi^3>^{b}<\theta_2-\xi^3>^{b}} d\xi d\xi_1 d\theta d\theta_1 \right| \leq C, \tag{1.10}$$

setting $\hat{G}=\hat{G}(\theta,\xi)$, $\hat{u}_1=\hat{u}(\theta_1,\xi_1)$, $\hat{v}_2=\hat{v}(\theta_2,\xi_2)$ and being understood here and in the sequel that $\xi=\xi_1+\xi_2$ and $\theta=\theta_1+\theta_2$. We set $\sigma=\theta-\xi^3$ and $\sigma_j=\theta_j-\xi_j^3$ (for $j\in 1,2$). We then use $e^{\tau|\xi|} \leq e^{\tau|\xi_1|}e^{\tau|\xi_2|}$ and $<\xi>^{\rho} \leq c<\xi_1>^{\rho}<\xi_2>^{\rho}$. Then we just have to prove

$$\tilde{q} = \int_{\mathbb{R}^4} \frac{|\xi|}{<\sigma>^{b'}<\sigma_1>^{b}<\sigma_2>^{b}} |\hat{G}||\hat{u}_1||\hat{v}_2| d\xi d\xi_1 d\theta d\theta_1 \leq C. \tag{1.11}$$

Introduce

$$I(\theta,\xi) = \frac{|\xi|}{<\sigma>^{b'}} \left(\int_{\mathbb{R}^2} \frac{d\theta_1 d\xi_1}{<\sigma_1>^{2b}<\sigma_2>^{2b}} \right)^{\frac{1}{2}}.$$

Appealing Lemma 2.4 in [12] we know that if $b > \frac{1}{2}$ and $b' \geq \frac{1}{4}$ then I belongs to $L^\infty(\mathbb{R}^2)$. Then applying twice Cauchy-Schwarz inequality gives

$$\tilde{q} \leq c||I||_{L^\infty(\mathbb{R}^2)}||u||_{L^2(\mathbb{R}^2)}||v||_{L^2(\mathbb{R}^2)}||G||_{L^2(\mathbb{R}^2)} \leq C, \tag{1.12}$$

that completes the proof of the Proposition. □

We now state another specific bilinear estimate to handle Gevrey spaces. Consider two smooth functions u,v in $\mathcal{S}(\mathbb{R}^2)$ and introduce for a given $\tau \geq 0$ the bilinear commutator $B(u,v) = (e^{\tau D}u)(e^{\tau D}v) - e^{\tau D}(uv)$. We now state (see Lemma 16 in [21])

Proposition 1.3. *Assume* $\max(\frac{7}{16},\frac{b}{2}) \leq b' < \frac{1}{2} < b$. *Assume $\tau \geq 0$. There exists a constant c that is independent of τ such that*

$$||D(B(u,v))||_{X^{0,-b'}} \leq c\sqrt{\tau}||u||_{\mathbb{X}^{\tau,b}}||v||_{\mathbb{X}^{\tau,b}}. \tag{1.13}$$

Proof. We proceed by a duality argument. Consider u,v,G in $\mathcal{S}(\mathbb{R}^2)$ such that $||G||_{L^2(\mathbb{R}^2)} = ||u||_{L^2(\mathbb{R}^2)} = ||v||_{L^2(\mathbb{R}^2)} = 1$. To prove (1.13) amounts to prove

$$q = \left| \int_{\mathbb{R}^4} |\xi| \frac{e^{\tau|\xi_1|+\tau|\xi_2|} - e^{\tau|\xi|}}{e^{\tau|\xi_1|+\tau|\xi_2|}} \frac{\hat{G}\hat{u}_1\hat{v}_2}{<\sigma>^{b'}<\sigma_1>^{b}<\sigma_1>^{b}} d\xi d\xi_1 d\theta d\theta_1 \right| \leq c\sqrt{\tau}, \tag{1.14}$$

setting as above $\hat{G}=\hat{G}(\theta,\xi)$, $\hat{w}_j=\hat{w}(\theta_j,\xi_j)$ for $w=u$ or v, $j=1$ or 2, and being understood that $\xi=\xi_1+\xi_2$ and $\theta=\theta_1+\theta_2$. We have set as above $\sigma=\theta-\xi^3$ and $\sigma_j=\theta_j-\xi_j^3$. We then use (see Lemma 12 in [21])

$$e^{\tau|\xi_1|+\tau|\xi_2|}-e^{\tau|\xi|}\le\sqrt{2\tau}\min(|\xi_1|,|\xi_2|)^{\frac{1}{2}}e^{\tau|\xi_1|+\tau|\xi_2|}. \tag{1.15}$$

This is valid since the left hand side vanishes if $\xi_1\xi_2>0$ and in the remaining case, assuming $|\xi_1|\le|\xi_2|$, then

$$e^{\tau|\xi_1|+\tau|\xi_2|}-e^{\tau|\xi|}=2\sinh(\tau|\xi_1|)e^{\tau|\xi_2|}\le\sqrt{2\tau|\xi_1|}e^{\tau|\xi_1|+\tau|\xi_2|}.$$

Therefore

$$q\le\sqrt{2\tau}\int_{\mathbb{R}^4}\frac{\min(|\xi_1|,|\xi_2|)^{\frac{1}{2}}|\xi|}{<\sigma>^{b'}<\sigma_1>^{b}<\sigma_1>^{b}}|\hat{G}||\hat{u}_1||\hat{u}_2|d\xi d\xi_1 d\theta d\theta_1. \tag{1.16}$$

We then use that $\xi^3-\xi_1^3-\xi_2^3=3\xi\xi_1\xi_2$ and then

$$\min(|\xi_1|,|\xi_2|)|\xi|^2\le2|\xi_1||\xi_2||\xi|\le2\max(|\sigma|,|\sigma_1|,|\sigma_2|). \tag{1.17}$$

We now divide q into $q_0+q_1+q_2$ where respectively the domain of integration of q_0, q_1, q_2 are the regions where respectively $|\sigma|=\max(|\sigma|,|\sigma_1|,|\sigma_2|)$, $|\sigma_j|=\max(|\sigma|,|\sigma_1|,|\sigma_2|)$ for $j=1,2$. We first handle q_0. Assume without loss of generality that $|\xi_1|\le|\xi_2|$. In this case

$$\frac{\min(|\xi_1|,|\xi_2|)^{\frac{1}{2}}|\xi|}{<\sigma>^{b'}}\le c|\xi_1|^{\frac{1}{2}-b'}|\xi_2|^{1-2b'},$$

and then

$$q_0\le c\sqrt{\tau}\int_{\mathbb{R}^4}\frac{|\hat{G}||\xi_1|^{\frac{1}{2}-b'}|\hat{u}_1||\xi_2|^{1-2b'}|\hat{v}_2|}{<\sigma_1>^{b}<\sigma_1>^{b}}d\xi_1 d\xi_2 d\theta d\theta_1. \tag{1.18}$$

Therefore

$$q_0\le c\sqrt{\tau}||G||_{L^2(\mathbb{R}^2)}\left\|\mathcal{F}^{-1}\left(\frac{|\xi_1|^{\frac{1}{2}-b'}|\hat{u}_1|}{<\sigma_1>^{b}}\right)||_{L^4(\mathbb{R}^2)}||\mathcal{F}^{-1}\left(\frac{|\xi_2|^{1-2b'}|\hat{v}_2|}{<\sigma_2>^{b}}\right)\right\|_{L^4(\mathbb{R}^2)}. \tag{1.19}$$

Appealing the Bourgain's embedding theorems (see [8] and the references therein) we know that for $b>\frac{1}{2}$ then $X^{0,\frac{b}{2}}\subset L^4(\mathbb{R}^2)$ and that

$$||D^{\frac{1}{8}}v||_{L^4(\mathbb{R}^2)}\le c||v||_{X^{0,\frac{3b}{4}}}.$$

Therefore for $1-2b'\le\frac{1}{8}$, i.e. $b'\ge\frac{7}{16}$ then by interpolation

$$\left\|\mathcal{F}^{-1}\left(\frac{|\xi_2|^{1-2b'}|\hat{v}_2|}{<\sigma_2>^{b}}\right)\right\|_{L^4(\mathbb{R}^2)}\le c||v||_{L^2(\mathbb{R}^2)}\le c.$$

The term involving u can be bounded by above by the same argument. We now tackle q_1 as follows, using that since $b>\frac{1}{2}$ then $\min(|\xi_1|,|\xi_2|)^{\frac{1}{2}}|\xi|<\sigma_1>^{-b}\leq c$, and this gives

$$q_1 \leq c\sqrt{\tau}\int_{\mathbb{R}^4}\frac{|\hat{G}||\hat{u}_1||\hat{v}_2|}{<\sigma>^{b'}<\sigma_2>^{b}}d\xi_1 d\xi_2 d\theta d\theta_1. \tag{1.20}$$

Then, proceeding as above using $X^{0,\frac{b}{2}}\subset L^4(\mathbb{R}^2)$ and $b'\geq\frac{b}{2}$

$$q_1 \leq c\sqrt{\tau}||u||_{L^2(\mathbb{R}^2)}||G||_{X^{0,\frac{b}{2}-b'}}||v||_{X^{0,-\frac{b}{2}}}\leq c\sqrt{\tau}. \tag{1.21}$$

The upper bound for q_2 is very similar and then omitted. □

We need one more bilinear estimate that is a slight modification of a bilinear estimate in [8].

Proposition 1.4. *Assume* $\max(\frac{11}{24},\frac{b}{2})\leq b'<\frac{1}{2}<b$. *Consider a smooth function* $Z=Q_N Z$ *for a given* N. *Then there exists a constant* c *such that*

$$||D(Z^2)||_{X^{0,-b'}}\leq c\frac{||Z||^2_{X^{0,b}}}{\sqrt{N}}. \tag{1.22}$$

Proof. We proceed by a duality argument. Introduce a smooth function G such that $||G||_{L^2(\mathbb{R}^2)}=1$. To prove (1.13) amounts to prove that for $Z=Q_N Z$ in the Schwartz space that satisfy $||Z||_{L^2(\mathbb{R}^2)}=1$ the following estimate is valid

$$q=|\int_{\mathbb{R}^4}|\xi|\frac{\hat{G}\hat{Z}_1\hat{Z}_2}{<\sigma>^{b'}<\sigma_1>^{b}<\sigma_1>^{b}}d\xi d\xi_1 d\theta d\theta_1|\leq\frac{c}{\sqrt{N}}, \tag{1.23}$$

setting as above $\hat{G}=\hat{G}(\theta,\xi)$, $\hat{Z}_j=\hat{Z}(\theta_j,\xi_j)$, $\sigma=\theta-\xi^3$, $\sigma_j=\theta_j-\xi_j^3$, $j=1$ or 2, and being understood that $\xi=\xi_1+\xi_2$ and $\theta=\theta_1+\theta_2$. We now divide q into $q_0+q_1+q_2$ where respectively the domain of integration of q_0, q_1, q_2 are the regions where respectively $|\sigma|=\max(|\sigma|,|\sigma_1|,|\sigma_2|)$, $|\sigma_j|=\max(|\sigma|,|\sigma_1|,|\sigma_2|)$ for $j=1,2$. We first handle q_0. Assume without loss of generality that $N\leq|\xi_1|\leq|\xi_2|$. In this case

$$\frac{|\xi|}{<\sigma>^{b'}}\leq c\frac{|\xi_2|^{\frac{3}{2}-3b'}}{|\xi_1|^{b'}|\xi_2|^{\frac{1}{2}-b'}}\leq c\frac{|\xi_2|^{\frac{3}{2}-3b'}}{N^{\frac{1}{2}}},$$

and then

$$q_0\leq\frac{c}{\sqrt{N}}\int_{\mathbb{R}^4}\frac{|\hat{G}||\hat{u}_1||\xi_2|^{3(\frac{1}{2}-b')}||\hat{v}_2|}{<\sigma_1>^{b}<\sigma_1>^{b}}d\xi_1 d\xi_2 d\theta d\theta_1. \tag{1.24}$$

Therefore proceeding as in (1.19) we have the result for $\frac{1}{2}-b'\leq\frac{1}{24}$ that is $b'\geq\frac{11}{24}$. We now tackle q_1,q_2 as we did for (1.20)-(1.21) using that since $b>\frac{1}{2}$ and $Z=Q_N Z$ then $N^{\frac{1}{2}}|\xi|<\sigma_1>^{-b}\leq c$. □

2 Proof of the main theorem

Since we plan to use the material introduced in Section 1, we assume in the following that we have a fixed pair (b,b') that satisfies the estimates of the various propositions. For instance we fix $b'=\frac{11}{24}$, $b=\frac{25}{48}$. We then set $\beta=\frac{1}{48}=1-b-b'>0$.

2.1 The auxiliary problem

Consider a complete trajectory $u(t)$ that belongs to $\mathcal{A}$. Set $y(t)=P_N u(t)$ and $z(t)=u(t)-y(t)=Q_N u(t)$. We set, for ρ an integer

$$\sup_{u\in\mathcal{A}} ||u||_{H^\rho(\mathbb{R})}=M_\rho, \tag{2.1}$$

where M_ρ depends on ρ,γ,f. We now follow [8]. Consider for a given N and a given τ the solution of the following auxiliary problem

$$Z_t+\gamma Z+Z_{xxx}+\frac{1}{2}Q_N(\partial_x(y+Z)^2)=Q_N f, \tag{2.2a}$$

$$Z(0)=Z_0\in G^\tau. \tag{2.2b}$$

We make precise the choice of $Z(0)$ in the sequel. We will choose $Z(0)$ small and smooth; we then may assume that $||Z(0)||_{L^2}\le M_0$ without loss of generality. In [8] (see Theorem 3.2 and Section 4.1) we have solved this auxiliary problem in $X_T^{0,b}$ and proved the following result

Proposition 2.1. *There exists K,N_0 that depends on γ,f such that for any fixed $N\ge N_0$, then for any positive t*

$$||Q_N u(t)-Z(t)||_{L^2}\le K\exp(-\gamma t/2). \tag{2.3}$$

We now prove estimates for $Z(t)$ in Gevrey spaces.

2.2 Local in time estimate

Consider T_0 the local time existence in $X^{0,b}$ for the original problem (1.1) and for the auxiliary problem (2.2). We will seek a local in time solution in Bourgain-Gevrey spaces $\mathbb{X}_T^{\tau,b}$ assuming without loss of generality that $T\le\min(T_0,1)$. Consider a smooth cut-off function $\psi(t)$ as above. Seek a mild local solution for (2.2) as a fixed point to, setting $v(t)=y(t)+Z(t)$,

$$\begin{aligned}\psi_T(t)Z(t)=&\psi(t)W(t)Z(0)+\psi(t)Q_N\left(\int_0^t W(t-s)ds\right)f\\&-\psi_T(t)\gamma\int_0^t W(t-s)Z(s)ds-\psi_T(t)Q_N\int_0^t W(t-s)vv_x ds.\end{aligned} \tag{2.4}$$

This problem was solved in $X^{0,b}_{T_0}$ in [8]. For $|t|\le T$ the $Z(t)$ solution of the fixed point above is a solution to (2.2). We prove here an estimate in $\mathbb{X}^{\tau,b}_T$ for the solution Z of this equation.

Proposition 2.2. *Fix N large enough depending on γ,f as in Proposition 2.1. Fix $\tau=N^{-1}$ (and then $\exp(\tau N)=e$). There exists numerical constant c_1, and another constant $\tilde{M}$ large enough depending on γ,f, such that for any initial data $Z(0)$ satisfying $||Z(0)||_{G^\tau}\le\tilde{M}$ then for T small enough depending on $\tilde{M}$ the solution of (2.4) satisfies*

$$||Z||_{\mathbb{X}^{\tau,b}_T}\le c_1\tilde{M}.$$

Moreover we also have that for any $t\in[0,T]$ then $||Z(t)||_{G^\tau}\le c\tilde{M}$.

Remark 2.1. We may assume without loss of generality that $\tilde{M}\ge M_1\ge M_0$.

Proof of Proposition 2.2. The last inequality in the statement of Proposition 2.2 is a mere consequence of the embedding $\mathbb{X}^{\tau,b}_T\subset L^\infty_t G^\tau$. We now proceed to bound $||\psi_T Z||_{\mathbb{X}^{\tau,b}_T}$ by above. We first handle the fourth term in the right hand side of (2.4) as follows, appealing (1.8),

$$||\psi_T(t)Q_N\int_0^t W(t-s)D(\psi_T v)^2ds||_{\mathbb{X}^{\tau,b}}\le cT^\beta||D(\psi_T v)^2||_{\mathbb{X}^{\tau,-b'}}. \tag{2.5}$$

Due to inverse inequality (1.6) and to (1.2), we also have,

$$\begin{aligned}||D(\psi_T v)^2||_{\mathbb{X}^{\tau,-b'}}\le& cT^\beta(||\psi_T y||^2_{\mathbb{X}^{\tau,b}}+||\psi_T Z||^2_{\mathbb{X}^{\tau,b}})\\ \le& cT^\beta(e^{2\tau N}||\psi_T y||^2_{X^{0,b}}+||\psi_T Z||^2_{\mathbb{X}^{\tau,b}}).\end{aligned} \tag{2.6}$$

We observe that $||\psi_T y||^2_{X^{0,b}}\le cM_0^2$ where M_0 is the L^2 bound for the attractor; in fact $||\psi_T u||_{X^{0,b}}$ remains bounded by $c||u_0||_{L^2}$, when solving the initial value problem in $X^{0,b}_{T_0}$ as in [8]. Hence, since $\exp(\tau N)=e$, then the right hand side of (2.6) is bounded by above by $cT^\beta(M_0^2+||\psi_T Z||^2_{\mathbb{X}^{\tau,b}})$. We now handle the third term as follows

$$\gamma||\psi_T\int_0^t W(t-s)Zds||_{\mathbb{X}^{\tau,b}}\le cT^\beta\gamma||\psi_T Z||_{\mathbb{X}^{\tau,-b'}}\le cT^\beta\gamma||\psi_T Z||_{\mathbb{X}^{\tau,b}}. \tag{2.7}$$

For the first term we have

$$||\psi(t)W(t)Z(0)||_{\mathbb{X}^{\tau,b}}\le c(\psi,b)||Z(0)||_{G^\tau}. \tag{2.8}$$

The second term reads $\psi(t)e^{\tau D}(W(t)-Id)D^{-3}(Q_N f)$ that is bounded by above in $\mathbb{X}^{\tau,b}$ by $c||f||_{G^\tau}$. Gathering these inequalities we have that there exists a constant K_0 that depends on γ,f such that

$$||\psi_T Z||_{\mathbb{X}^{\tau,b}}\le K_0+c\tilde{M}+cT^\beta||\psi_T Z||^2_{\mathbb{X}^{\tau,b}}. \tag{2.9}$$

Chose now $\tilde{M}$ large enough depending on γ, f such that $c\tilde{M} \geq K_0$. Then for T small enough depending on $\tilde{M}$, i.e. for instance

$$16cT^{\beta}\tilde{M} = 1, \tag{2.10}$$

we have the result since $||Z||_{\mathbb{X}_T^{\tau,b}} \leq ||\psi_T Z||_{\mathbb{X}^{\tau,b}}$. □

We now improve the previous estimate.

Proposition 2.3. *Assume $\tilde{M}, T$ being fixed as in Proposition 2.2. Then there exist N large enough then (and $\tau = \frac{1}{N}$ small enough) such that Proposition 2.1 remains valid and such that*

$$||Z(T)||_{G^\tau} \leq \tilde{M}. \tag{2.11}$$

Proof. Consider the scalar product of (2.2) with $\exp(2\tau D)Z$ to obtain

$$\frac{1}{2}\frac{d}{dt}||Z||_{G^\tau}^2 + \gamma||Z||_{G^\tau}^2 = -\frac{1}{2} < \partial_x e^{\tau D}(v^2), e^{\tau D}Z > + < e^{\tau D}f, e^{\tau D}Z >. \tag{2.12}$$

Let us observe that

$$| < e^{\tau D}f, e^{\tau D}Z > | \leq ||Z(t)||_{G^\tau}||Q_N f||_{G^\tau} \leq \frac{\gamma}{2}||Z(t)||_{G^\tau}^2 + \frac{||Q_N f||_{G^\tau}^2}{2\gamma}. \tag{2.13}$$

Therefore, integrating in time for $t \leq T$ we then have

$$||Z(t)||_{G^\tau}^2 e^{\gamma t} \leq ||Z(0)||_{G^\tau}^2 + |\int_0^t e^{\gamma s} < \partial_x e^{\tau D}(v^2), e^{\tau D}Z > ds| + \frac{||Q_N f||_{G^\tau}^2}{\gamma^2} e^{\gamma t}. \tag{2.14}$$

We now use $e^{\tau D}(v^2) = (e^{\tau D}v)^2 - B(v,v)$, where B is the commutator defined above. Besides, we also have, setting $\underline{y} = e^{\tau D}y$, $\underline{Z} = e^{\tau D}Z$ and $\underline{v} = e^{\tau D}v$, since $< \underline{Z}^2, \underline{Z}_x > = 0$,

$$< \underline{v}^2, \underline{Z}_x > = < \underline{y}^2, \underline{Z}_x > + < \underline{y}, \partial_x(\underline{Z})^2 >. \tag{2.15}$$

Therefore the second term in the right hand side of (2.14) is bounded by above by, for $t \leq T$, setting $\psi_\gamma(s) = \psi(s)e^{-\gamma s}$,

$$\left|\int_0^t \psi_\gamma < DB(v,v), \underline{Z} > ds\right| + \left|\int_0^t \psi_\gamma < D(\underline{y})^2, \underline{Z} > ds\right| + \left|\int_0^t \psi_\gamma < D(\underline{Z})^2, \underline{y} > ds\right|. \tag{2.16}$$

We now chose $\tilde{Z}$ the function that satisfies $||\tilde{Z}||_{\mathbb{X}^{\tau,b}} = ||Z||_{\mathbb{X}_T^{\tau,b}}$, and $\tilde{y}$ that satisfies $||\tilde{y}||_{\mathbb{X}^{\tau,b}} = ||y||_{\mathbb{X}_T^{\tau,b}}$ (see Lemma 1.1 above). Since y and Z are orthogonal we then have $\widetilde{y+Z} = \tilde{y} + \tilde{Z}$. By duality and by Lemma 1.2, the first term in (2.16) can be bounded as follows

$$||\psi_\gamma \tilde{Z}||_{\mathbb{X}^{\tau,b}}||DB(\tilde{v},\tilde{v})||_{X^{0,-b}} \leq c_\gamma ||\tilde{Z}||_{\mathbb{X}^{\tau,b}}||DB(\tilde{v},\tilde{v})||_{X^{0,-b}}. \tag{2.17}$$

Since $b'\le b$, by Proposition 1.3 we then have

$$\left|\int_0^t \psi_\gamma < DB(v,v),\underline{Z}> ds\right| \le c_\gamma\sqrt{\tau}||Z||_{\mathbb{X}_T^{\tau,b}}\left(||y||^2_{\mathbb{X}_T^{\tau,b}}+||Z||^2_{\mathbb{X}_T^{\tau,b}}\right). \tag{2.18}$$

Due to inverse inequality (1.6) we have $||y||^2_{\mathbb{X}_T^{\tau,b}}\le e^{2\tau N}||y||^2_{X_T^{0,b}}\le cM_0^2$. The second term in (2.16) can be bounded similarly as

$$\left|\int_0^t \psi_\gamma < D(\underline{y})^2,\underline{Z}> ds\right| \le c_\gamma||Z||_{\mathbb{X}_T^{\tau,b}}||Q_N D(\underline{\tilde{y}})^2||_{X^{0,-b}}. \tag{2.19}$$

Using specialized Poincaré inequality (1.7), bilinear inequality (1.9) (for $\tau=0$ and $\rho=1$) and the fact that $H^1(\mathbb{R})$ is an algebra we then have

$$||Q_N D(\underline{\tilde{y}})^2||_{X^{0,-b}}\le N^{-1}||D(\underline{\tilde{y}})^2||_{X^{1,-b'}}\le cN^{-1}||y||^2_{\mathbb{X}_T^{\tau,1,b}}\le cN^{-1}\exp(2\tau N)M_1^2.$$

The third term in (2.16) can be bounded as follows

$$\left|\int_0^t \psi_\gamma < D(\underline{Z})^2,\underline{y}> ds\right| \le c_\gamma||y||_{\mathbb{X}_T^{\tau,b}}||D(\underline{\tilde{Z}})^2||_{X^{0,-b'}}. \tag{2.20}$$

Appealing (1.22) this can be bounded by above by $c_\gamma M_0 N^{-\frac{1}{2}}||Z||^2_{\mathbb{X}_T^{\tau,b}}$. Gathering these inequalities with $\tau N=1$ we then have

$$||Z(T)||^2_{G^\tau}\le \tilde{M}^2 e^{-\gamma T}+\frac{c}{\sqrt{N}}\tilde{M}^3+\frac{||Q_N f||^2_{G^\tau}}{\gamma^2}. \tag{2.21}$$

The constants $\tilde{M},T$ that depend on γ,f being fixed we now chose N such that

$$\frac{c}{\sqrt{N}}\tilde{M}^3+\frac{||Q_N f||^2_{G^\tau}}{\gamma^2}\le (1-e^{-\gamma T})\tilde{M}^2.$$

Therefore we have that $||Z(T)||_{G^\tau}\le\tilde{M}$. □

2.3 Completing the proof of Theorem 1.1

We conclude as follows. Consider a complete trajectory that is in the global attractor. For any m in $\mathbb{N}$ solve the following time shifted auxiliary problem, for $t\ge -mT$

$$Z^m_t+\gamma Z^m+Z^m_{xxx}+\frac{1}{2}Q_N(\partial_x(y+Z^m)^2)=Q_N f, \tag{2.22a}$$

$$Z^m(-mT)=0. \tag{2.22b}$$

On one hand due to Proposition 2.3 the sequence $Z_m=Z^m(mT)$ remains bounded in G^τ. On the other hand due to Proposition 2.1 Z_m converges towards Qu_0 in L^2. Therefore $||Qu_0||_{G^\tau}\le\tilde{M}$. Since $||Pu_0||_{G^\tau}\le e^{\tau N}||u_0||_{L^2}\le cM_0$ then u_0 remains bounded in G^τ. This completes the proof of the main theorem. □

3 Miscelleanous results

3.1 Size of the global attractor in Gevrey space

The main result states that for a given $f \in G^{\tau_0}$ and for τ small enough then the global attractor $\mathcal{A}$ is a bounded subset of G^{τ}. We now introduce

Definition 3.1. *For a given $\tau \leq \tau_0$ such that $\mathcal{A} \subset G^{\tau}$ we set*

$$M(\tau) = \sup_{a \in \mathcal{A}} ||a||_{G^{\tau}}.$$

It is clear that the function $\tau \mapsto M(\tau)$ is a nondecreasing function. Moreover since we cannot expect that even stationary solutions are as regular as f is, then $M(\tau)$ may blow up for $\tau < \tau_0$. We then define

$$\tau_{\max} = \sup\{\tau < \tau_0; M(\tau) < +\infty\}.$$

We state and prove

Lemma 3.1. *The map $M : [0,\tau_{\max}) \to \mathbb{R}^+; \tau \mapsto M(\tau)$ is a continuous function.*

Proof. Fix $\tau < \tau_{\max}$. Consider here small parameters ε in $(0,\tau_{\max} - \tau)$. Since $\tau \mapsto M(\tau)$ is non decreasing then

$$\limsup_{\varepsilon \to 0} M(\tau - \varepsilon) \leq M(\tau) \leq \liminf_{\varepsilon \to 0} M(\tau + \varepsilon).$$

Then we just have to prove that

$$\limsup_{\varepsilon \to 0} M(\tau + \varepsilon) \leq M(\tau) \leq \liminf_{\varepsilon \to 0} M(\tau - \varepsilon). \tag{3.1}$$

We focus on the first inequality. Pick a in $\mathcal{A} \subset G^{v}$ for a given v in $(\tau,\tau_{\max})$. Then for $\varepsilon < v - \tau$

$$||a||^2_{G^{\tau+\varepsilon}} - ||a||^2_{G^{\tau}} \leq 2\varepsilon \int_{\mathbb{R}} e^{2(\tau+\varepsilon)|\xi|} |\xi| |\hat{a}(\xi)|^2 d\xi.$$

We infer from this inequality that

$$M(\tau+\varepsilon)^2 \leq M(\tau)^2 + 2\varepsilon \sup_{a \in \mathcal{A}} ||a||^2_{G^{v,\frac{1}{2}}}.$$

Since $G^{\frac{\tau_{\max}+v}{2}} \subset G^{v,\frac{1}{2}}$ we can pass to the limit when $\varepsilon \to 0$. The proof of the second inequality in (3.1) is similar and then omitted. ☐

3.2 Upper bound for $M(\tau)$

This subsection is devoted to prove an estimate that generalizes the upper bound for M_0 the size of the attractor in $L^2(\mathbb{R})$.

Proposition 3.1. *There exists* $\tau_1 \le \tau_{\max}$ *such that for* $\tau \le \tau_1$ *small enough then*

$$M(\tau) \le \sqrt{2}\frac{||f||_{G^\tau}}{\gamma}.$$

Proof. Consider a complete trajectory $u(t)$ that is included in $\mathcal{A} \subset G^\tau$. For a given $t>0$ we consider the trajectory that starts from $u(-t)$ and we solve (1.1) forward in time until t. Consider then the scalar product of (1.1) with $e^{2\tau D}u$. Using that

$$2 < e^{\tau D}u, e^{\tau D} f > \le \frac{||f||^2_{G^\tau}}{\gamma} + \gamma ||u||^2_{G^\tau},$$

we then have, after time integration

$$||u(0)||^2_{G^\tau} \le ||u(-t)||^2_{G^\tau} e^{-\gamma t} + \left| \int_{-t}^0 e^{\gamma s} < e^{\tau D}u_x(s), e^{\tau D}(u(s)^2) > ds \right| + \frac{||f||^2_{G^\tau}}{\gamma^2}. \tag{3.2}$$

We then introduce $T(\tau)$ the local time existence in Bourgain-Gevrey space $\mathbb{X}^{\tau,b}$ associated to an initial data whose G^τ norm is bounded by above by $M(\tau)$. We introduce n such that $nT(\tau) \le t < (n+1)T(\tau)$. We then split the integral in (3.2) as follows

$$\left| \int_{-t}^0 e^{\gamma s} < e^{\tau D}u_x, e^{\tau D}(u^2) > ds \right| \le \sum_{k=0}^n e^{-\gamma k T(\tau)} \left| \int_{\mathcal{I}_k} e^{\gamma(s+kT(\tau))} < e^{\tau D}u_x, e^{\tau D}(u^2) > ds \right|, \tag{3.3}$$

setting $\mathcal{I}_k = [-(k+1)T(\tau), -kT(\tau)]$. We then use $< e^{\tau D}u_x, e^{\tau D}(u^2) > = - < e^{\tau D}u_x, B(u,u) >$. Appealing Lemma 1.1 we introduce $\tilde{u}$ that achieves $||u||_{\mathbb{X}^{\tau,b}_{\mathcal{I}_k}}$ to write,

$$\left| \int_{\mathcal{I}_k} e^{\gamma(s+kT(\tau))} < e^{\tau D}u_x, e^{\tau D}(u^2) > ds \right| \le ||\psi_\gamma(.+kT(\tau))\tilde{u}||_{\mathbb{X}^{\tau,\frac{1}{2}}} ||B(\tilde{u},\tilde{u})||_{X^{0,-\frac{1}{2}}}. \tag{3.4}$$

Since $b' < \frac{1}{2} < b$, then $X^{0,-\frac{1}{2}} \subset X^{0,-b'}$ and $\mathbb{X}^{0,b} \subset \mathbb{X}^{0,\frac{1}{2}}$. Therefore due to Lemma 1.2

$$||\psi_\gamma(.+kT(\tau))\tilde{u}||_{\mathbb{X}^{\tau,\frac{1}{2}}} \le c_\gamma ||u||_{\mathbb{X}^{\tau,b}_{\mathcal{I}_k}},$$

and due to bilinear estimate (1.13)

$$||B(\tilde{u},\tilde{u})||_{X^{0,-\frac{1}{2}}} \le c\sqrt{\tau} ||u||^2_{\mathbb{X}^{0,b}_{\mathcal{I}_k}}.$$

Then the right hand side of (3.4) is bounded by above by $cc_\gamma\sqrt{\tau}M(\tau)^3$. We then have, since $\gamma T(\tau)\sum_{k=0}^n e^{-\gamma kT(\tau)}\le c$,

$$||u(0)||^2_{G^\tau}\le ||u(-t)||^2_{G^\tau}e^{-\gamma\tau}+\frac{cc_\gamma\sqrt{\tau}M(\tau)^3}{\gamma T(\tau)}+\frac{||f||^2_{G^\tau}}{\gamma^2}. \tag{3.5}$$

We easily infer from (3.5) letting $t\to+\infty$ that

$$M(\tau)^2\le\frac{cc_\gamma\sqrt{\tau}M(\tau)^3}{\gamma T(\tau)}+\frac{||f||^2_{G^\tau}}{\gamma^2}. \tag{3.6}$$

Introducing

$$\tau_1=\inf_{\tau>0}\left\{\frac{2\sqrt{2}cc_\gamma\sqrt{\tau}||f||_{G^\tau}}{\gamma^2T(\tau)}>1\right\},$$

for $\tau\le\tau_1$ we have the result. □

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