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Attractors for a Caginalp Phase-field Model with Singular Potential

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Abstract. We consider a phase field model based on a generalization of the Maxwell Cattaneo heat conduction law, with a logarithmic nonlinearity, associated with Neumann boundary conditions. The originality here, compared with previous works, is that we obtain global in time and dissipative estimates, so that, in particular, we prove, in one and two space dimensions, the existence of a unique solution which is strictly separated from the singularities of the nonlinear term, as well as the existence of the finite-dimensional global attractor and of exponential attractors. In three space dimensions, we prove the existence of a solution.

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1 Introduction

The Caginalp phase-field model

$$\frac{\partial u}{\partial t} - \Delta u + g(u) = \theta, \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \Delta \theta = -\frac{\partial u}{\partial t}, \tag{1.2}$$

has been proposed to model phase transition phenomena, for example melting-solidification phenomena, in certain classes of materials. Caginalp considered the Ginzburg-Landau free energy and the classical Fourier law to derive his system, see, e.g., [1,2].

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Here, u denotes the order parameter and θ the (relative) temperature. Furthermore, all physical constants have been set equal to one. For more details and references we refer the reader to [2–4]. This model has been extensively studied (see, e.g., [5] and the references therein). Now, a drawback of the Fourier law is the so-called "paradox of heat conduction", namely, it predicts that thermal signals propagate with infinite speed, which, in particular, violates causality (see, e.g., [5]). One possible modification, in order to correct this unrealistic feature, is the Maxwell-Cattaneo law. We refer the reader to [3,5,6] for more discussions on the subject.

In this paper, we consider the following model

$$\frac{\partial u}{\partial t}-\Delta u+g(u)=\frac{\partial \alpha}{\partial t}, \tag{1.3}$$

$$\frac{\partial^2 \alpha}{\partial t^2}+\frac{\partial \alpha}{\partial t}-\Delta \alpha=-\frac{\partial u}{\partial t}-u, \tag{1.4}$$

which is a generalization of the original Caginalp system (see [2]). In this context α is the thermal displacement variable, defined by

$$\alpha=\int_0^t \theta d\tau+\alpha_0. \tag{1.5}$$

As mentioned above the Caginalp system can be obtained by considering the Ginzburg-Landau free energy

$$\Psi(u,\theta)=\int_\Omega \left(\frac{1}{2}|\nabla u|^2+G(u)-\theta u\right)dx, \tag{1.6}$$

the enthalpy $H=u+\theta$ and by writing

$$\frac{1}{d}\frac{\partial u}{\partial t}=-\partial_u \Psi, \tag{1.7}$$

$$\frac{\partial H}{\partial t}=-\mathrm{div} q, \tag{1.8}$$

where $d>0$ is a relaxation parameter, ∂_u denotes a variational derivative and q is the thermal flux vector. Setting $d=1$ and taking the usual Fourier law

$$q=-\nabla\theta, \tag{1.9}$$

we find (1.1)-(1.2).

The Maxwell-Cattaneo law reads

$$(1+\eta\frac{\partial}{\partial t})q=-\nabla\theta, \tag{1.10}$$

where η is a relaxation parameter; when $\eta=0$, one recovers the Fourier law. Taking for simplicity $\eta=1$, it follows from (1.8) that

$$\left(1+\frac{\partial}{\partial t}\right)\frac{\partial H}{\partial t}-\Delta\theta=0,$$