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A Dimensional Splitting Method for 3D Elastic Shell with Mixed Tensor Analysis on a 2D Manifold Embedded into a Higher Dimensional Riemannian Space

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Abstract. In this paper, a mixed tensor analysis for a two-dimensional (2D) manifold embedded into a three-dimensional (3D) Riemannian space is conducted and its applications to construct a dimensional splitting method for linear and nonlinear 3D elastic shells are provided. We establish a semi-geodesic coordinate system based on this 2D manifold, providing the relations between metrics tensors, Christoffel symbols, covariant derivatives and differential operators on the 2D manifold and 3D space, and establish the Gateaux derivatives of metric tensor, curvature tensor and normal vector and so on, with respect to the surface $\vec{\Theta}$ along any direction $\vec{\eta}$ when the deformation of the surface occurs. Under the assumption that the solution of 3D elastic equations can be expressed in a Taylor expansion with respect to transverse variable, the boundary value problems satisfied by the coefficients of the Taylor expansion are given.

AMS subject classifications: O175

Key words: Dimensional splitting method, linear elastic shell, mixed tensor analysis, nonlinear elastic shell.

1 Introduction

A shell is a three-dimensional (3D) elastic body that is geometrically characterized by its middle surface and its small thickness. The middle surface $\Im$ is a compact surface in $\Re^3$ that is not a plane (otherwise the shell is a plate), and it may or may not have a boundary. For instance, the middle surface of a sail has a boundary, whereas that of a basketball has no boundary.

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At each point $s\in\Im$, let $n(s)$ denote a unit vector normal to $\Im$. Then the reference configuration of the shell, (that is, the subset of $\Re^3$ that occupies before forces are applied to it), is a set of the form $\{(\vec{\theta}+\xi n(s))\in\Re^3:\vec{\theta}\in\Im,|\xi|\le e(\vec{\theta})\}$, where the function e: $\Im\to\Re$ is sufficiently smooth and satisfies $0\le e(\vec{\theta})\le\varepsilon$ for all $s\in\Im$. Additionally, $\varepsilon>0$ is thought of as being 'small' compared with the 'characteristic' length of $\Im$ (its diameter, for instance). If $e(s)=\varepsilon,\forall s\in\Im$, the shell is said to have a constant thickness of 2ε. If e is not a constant function, the shell is said to have a variable thickness.

The theory of elastic plates and shells is one of the most important theories of elasticity. Thin shells and plates are widely used in civil engineering projects as well as engineering projects. Examples include aircraft, cars, missiles, orbital launch systems, rockets, and trains.

Considerable work on the subject was conducted by the Russian scholars A.I. Lurje (1937), V.Z. Vlasov (1944) and V.V. Novozhilov (1951) after the pioneer idea of Love. However, now it appears necessary to improve the mathematical understanding of the classical plate and shell models pioneered by these scholars. The reason for this is that the precision required in aircraft and spacecraft projects has intensified with the advent of powerful electronic computers. The goal therein is, on the one hand, to develop better finite approximations on elements and, on the other hand, to refine theoretical models when necessary.

A.L. Gol'denveizer (1953) first put forward the ideas of conducting an asymptotic analysis based on the thickness of a shell or plate. New formulations for shells and plates were obtained by relaxing the constitutive relations and reinforcing the equilibrium that must be satisfied. Even in presenting a much more detailed analysis than that offered by his predecessors, no mathematical justification was given by Gol'denveizer. As such, a large number of difficulties were still to be overcome. At the same time, the weak formulations of J.L. Lions and mixed formulations of Hellinger-Reissner and Hu-Washizu (1968) appeared to provide alleviation. Of particular significance was the work of J.L. Lions (1973) on singular perturbation, which provided the tools and explanation for what happens when the asymptotic method is applied to plates, shells, and beams. The mathematical foundation of elasticity can be found in [4,6]. The asymptotic method was revisited in a functional framework proposed by Li, Zhang and Huang [1], P.G. Ciarlet [3,4] and M. Bernadou [5]. Their work made convergence and error analysis possible therein for the first time.

The 3D models are derived directly from the principles of equilibrium in classical mechanics, and are viewed as singular perturbation problems dependent upon the small parameter ε (the half-thickness of the shell). 2D models are obtained by making some additional hypotheses that are not justified by physical law. Our aim here is to justify mathematically the assumptions that formulate the basis of a 2D model of elasticity, and whose solution approaches 3D displacement better than the solution of classical models.

This paper is organized as follows: in Section 2, we present a mixed tensor analysis on 2D manifolds embedded into 3D Euclidian space; in Section 3, we provide the exchange of tensors and curvature after the deformation of curvatures; in Section 4, we give differential operators in 3D Riemannian space under semi geodesic coordinate system; in Section 5, asymptotic forms of 3D linear and nonlinear elastic operators with respect to transverse variable ξ are derived; in Section 6, a specific shell as an example,is provided.

2 Mixed tensor analysis on a 2D manifold embedded into a 3D Riemannian space

Tensor analysis in Riemannian space can be found in [2]. Here it presents for mixed tensor analysis on a 2D manifold embedded into a 3D Riemannian space, in which we provide basic theorem and formula and gives proof in details.

Let us consider elastic shells, which is assumed to be a St Venant-Kirchhoff material and homogeneous and isotropic. Hence this material is characterized by its two Lamé constants $\lambda > 0$ and $\mu > 0$ which are thus independent of its thickness.

An elastic shell whose reference configuration $\{\hat{\Omega}_\varepsilon\} \subset E^3$ (3D-Euclidean space) consists of all points within a distance $\leq \varepsilon$ from a given surface $\Im \subset E^3$ and $\varepsilon > 0$ which is thought of being small. The 2D manifold $\Im \subset E^3$ is called the middle surface of the shell, and the parameter ε is called the semi-thickness of the shell. The surface $\Im$ can be defined as the image $\vec{\theta}$ of the closure of a domain $\omega \subset R^2$, where $\vec{\theta}: \bar{\omega} \to E^3$ is a smooth injective mapping. Let $\vec{n}$ denote an unit normal vector along $\Im$ and let

$$\Omega_\varepsilon = \omega \times (-\varepsilon, \varepsilon).$$

Hence the set $\{\hat{\Omega}_\varepsilon\}$ is given by $\{\hat{\Omega}_\varepsilon\} = \vec{\Theta}(\bar{\Omega}_\varepsilon)$ where the mapping $\vec{\Theta}: \bar{\Omega}_\varepsilon \subset R^3 \to E^3$ is defined by

$$\vec{\Theta}(x^1, x^2, \xi) = \vec{\theta}(x^1, x^2) + \xi \vec{n}, \quad \forall (x^1, x^2, \xi) \in \bar{\Omega}_\varepsilon, \quad (x^1, x^2) \in \omega. \tag{2.1}$$

The pair (x^1, x^2) is usually called Gaussian coordinate on $\Im$, and (x^1, x^2, ξ) is called semi-geodesic coordinate system (S-Coordinate System) (if E^3 is a Riemmannian space and $\Im$ is a 2D manifold). The boundary of shell $\{\hat{\Omega}_\varepsilon\}$ consists as follows: • top surface $\Gamma_t = \Im \times \{+\varepsilon\}$,

• bottom surface $\Gamma_b = \Im \times \{-\varepsilon\}$,

• lateral surface $\Gamma_l = \Gamma_0 \cap \Gamma_1 : \Gamma_0 = \gamma_0 \times \{-\varepsilon, +\varepsilon\}, \Gamma_1 = \gamma_1 \times \{-\varepsilon, +\varepsilon\}$, where $\gamma = \gamma_0 \cup \gamma_1$ is the boundary of ω:$\gamma = \partial\omega$.

In what follows, Latin indices and exponent: $(i,j,k,\cdots)$ take their values in the set $\{1,2,3\}$ whereas Greek indices and exponents $(\alpha,\beta,\gamma,\cdots,)$ take their values in the set $\{1,2\}$. In addition, Einstein's summation convention with respect to repeated indices and exponent is used.

It is well known that the covariant and contravariant component of the metric tensors on the surface $\Im$ are given by

$$a_{\alpha\beta} = \vec{\theta}_\alpha \vec{\theta}_\beta \quad \text{and} \quad a^{\alpha\beta} a_{\beta\lambda} = \delta^\alpha_\lambda, \tag{2.2}$$

where $\vec{\theta}_\alpha = \frac{\partial \vec{\theta}}{\partial x^\alpha}$. The second and third fundamental forms are given by

$$b_{\alpha\beta} = \vec{n}\vec{\theta}_{\alpha\beta} = -\vec{n}_\alpha \vec{\theta}_\beta, \;\; c_{\alpha\beta} = \vec{n}_\alpha \vec{n}_\beta, \;\; c_{\alpha\beta} = a^{\lambda\sigma} b_{\alpha\lambda} b_{\beta\sigma}. \tag{2.3}$$

Furthermore, it is well known that the contravariant components of $b_{\alpha\beta}, c_{\alpha\beta}$ are given by

$$b^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\sigma} b_{\lambda\sigma}, \quad c^{\alpha\beta} = a^{\alpha\lambda} a^{\beta\sigma} c_{\lambda\sigma},$$

while the inverse matrices $\hat{b}^{\alpha\beta}, \hat{c}^{\alpha\beta}$ of $b_{\alpha\beta}, c_{\alpha\beta}$ can be expressed by

$$\hat{b}^{\alpha\beta} b_{\beta\lambda} = \delta^\alpha_\lambda, \quad \hat{c}^{\alpha\beta} c_{\beta\lambda} = \delta^\alpha_\lambda, \tag{2.4}$$

which will play important role in the followings. In the same season we have to introduce permutation tensors in $\Re^3$ and on $\Im$ which are given by

$$\varepsilon_{ijk} = \begin{cases} \sqrt{g}, \\ -\sqrt{g}, \\ 0, \end{cases} \;\; \varepsilon^{ijk} = \begin{cases} \frac{1}{\sqrt{g}}, (i,j,k): & \text{even permutation of (1,2,3),} \\ -\frac{1}{\sqrt{g}}, (i,j,k): & \text{odd permutation of (1,2,3),} \\ 0, \text{otherwise,} \end{cases} \tag{2.5}$$

where $g = \det(g_{ij})$ and g_{ij} is metric tensor of $\Re^3$.

Similarly

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \;\; \varepsilon^{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}}, (\alpha,\beta): & \text{even permutation of (1,2),} \\ -\frac{1}{\sqrt{a}}, (\alpha,\beta): & \text{odd permutation of (1,2),} \\ 0, \text{otherwise.} \end{cases} \tag{2.6}$$

Let H and K denote mean curvature and Gaussian curvature respectively:

$$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}, \quad K = \frac{\det(b_{\alpha\beta})}{\det(a_{\alpha\beta})} = \frac{b}{a}.$$

Then the determinant of third fundamental tensor $c_{\alpha\beta}$ is

$$c = \det(c_{\alpha\beta}) = \det(a^{\lambda\sigma} b_{\alpha\lambda} b_{\beta\sigma}) = a^{-1} b^2 = bK.$$

Using permutation tensors in $\Re^3$ and on $\Im$ the following relationships are held

$$\begin{cases} a^{\alpha\beta} a_{\beta\lambda} = \delta^\alpha_\lambda, \quad \widehat{b}^{\alpha\beta} b_{\beta\lambda} = \delta^\alpha_\lambda, \quad \widehat{c}^{\alpha\beta} c_{\beta\lambda} = \delta^\alpha_\lambda, \\ a^{\alpha\beta} = \varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} a_{\lambda\sigma}, \quad a = \det(a_{\alpha\beta}), \\ b\widehat{b}^{\alpha\beta} = a\varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} b_{\lambda\sigma}, \quad K\widehat{b}^{\alpha\beta} = \varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} b_{\lambda\sigma}, \quad b = \det(b_{\alpha\beta}), \\ c\widehat{c}^{\alpha\beta} = a\varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} c_{\lambda\sigma}, \quad K^2 \widehat{c}^{\alpha\beta} = \varepsilon^{\alpha\lambda} \varepsilon^{\beta\sigma} c_{\lambda\sigma}, \quad c = \det(c_{\alpha\beta}), \end{cases} \tag{2.7}$$

and

$$\begin{cases} \frac{b}{a} = K, \quad \frac{c}{a} = K^2, \\ c = \det(c_{\alpha\beta}) = \det(a^{\lambda\sigma} b_{\alpha\lambda} b_{\beta\sigma}) = a^{-1} b^2 = bK = aK^2. \end{cases} \tag{2.8}$$

The following lemma present fundamental formula concerning many basic tensors of order two on the surface.

Lemma 2.1. *The third fundamental tensor is not independent of the first and second fundamental tensors $a_{\alpha\beta}, b_{\alpha\beta}$. There are following relationships*

$$\begin{cases} Ka_{\alpha\beta}-2Hb_{\alpha\beta}+c_{\alpha\beta}=0, \quad Ka^{\alpha\beta}-2Hb^{\alpha\beta}+c^{\alpha\beta}=0,\\ a^{\alpha\beta}-2H\hat{b^{\alpha\beta}}+K\hat{c^{\alpha\beta}}=0,\\ K\hat{b^{\alpha\beta}}+b^{\alpha\beta}-2Ha^{\alpha\beta}=0, \quad K^2\hat{c^{\alpha\beta}}+2Hb^{\alpha\beta}-(4H^2-K)a^{\alpha\beta}=0. \end{cases} \tag{2.9}$$

Besides, there are relationships between matrices $b_{\alpha\beta}, c_{\alpha\beta}$ and its inverse matrices $\widehat{b}^{\alpha\beta}, \widehat{c}^{\alpha\beta}$ as follows

$$\begin{cases} K\hat{b}^{\alpha\beta}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\lambda\sigma}, \quad K^2\hat{c}^{\alpha\beta}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}c_{\lambda\sigma},\\ \widehat{b}^{\alpha\beta}=\widehat{c}^{\alpha\lambda}b^{\beta}_{\lambda}, \quad \widehat{c}^{\alpha\beta}=\widehat{b}^{\alpha\lambda}\widehat{b}^{\beta}_{\lambda}, \quad b^{\alpha}_{\beta}=\widehat{b}^{\alpha\lambda}c_{\beta\lambda}. \end{cases} \tag{2.10}$$

Furthermore,

$$\begin{cases} b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=\varepsilon_{\mu\nu}\varepsilon_{\lambda\beta}b^{\nu}_{\alpha}b^{\mu}_{\sigma},\\ b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=K\varepsilon_{\alpha\sigma}\varepsilon_{\beta\lambda},\\ \varepsilon_{\alpha\beta}b^{\alpha}_{\lambda}b^{\beta}_{\sigma}=K\varepsilon_{\lambda\sigma}, \quad \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma}=2K, \quad \varepsilon_{\nu\mu}\varepsilon^{\beta\sigma}b^{\nu}_{\beta}b^{\mu}_{\sigma}=2K, \end{cases} \tag{2.11}$$

$$\begin{cases} c_{\alpha\lambda}b^{\lambda}_{\beta}=-2HKa_{\alpha\beta}+(4H^2-K)b_{\alpha\beta}=2Hc_{\alpha\beta}-Kb_{\alpha\beta},\\ c_{\alpha\lambda}c^{\lambda}_{\beta}=-K(4H^2-K)a_{\alpha\beta}+2H(4H^2-2K)b_{\alpha\beta},\\ c^{\alpha}_{\alpha}=a^{\alpha\beta}c_{\alpha\beta}=b^{\alpha\beta}b_{\alpha\beta}=4H^2-2K;\\ b^{\alpha\beta}c_{\alpha\beta}=8H^3-6HK; \quad c^{\alpha\beta}c_{\alpha\beta}=16H^4-16H^2K+2K^2. \end{cases} \tag{2.12}$$

Proof. First, we prove (2.11). For the simplicity, let $\mathbf{r}=\vec{\theta}$ and $\mathbf{n}$ denote the unite normal vector to surface $\Im$. (2.3) shows $b_{\alpha\beta}=-\mathbf{n}_{\alpha}\mathbf{r}_{\beta}$, therefore

$$b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=(-(\mathbf{n}_{\alpha}\cdot\mathbf{r}_{\beta})\mathbf{r}_{\lambda}+(\mathbf{n}_{\alpha}\cdot\mathbf{r}_{\lambda})\mathbf{r}_{\beta})\mathbf{n}_{\sigma}.$$

In terms of formula in vector analysis

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C},$$

it yields that

$$b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=(\mathbf{n}_{\alpha}\times(\mathbf{r}_{\lambda}\times\mathbf{r}_{\beta}))\mathbf{n}_{\sigma}.$$

On the other hand, applying Weingarten formula

$$\mathbf{n}_{\beta}=-b^{\alpha}_{\beta}\mathbf{r}_{\alpha}=-b_{\alpha\beta}\mathbf{r}^{\alpha}$$

and permutation tensor

$$\varepsilon_{\alpha\beta}\mathbf{n}=\mathbf{r}_{\alpha}\times\mathbf{r}_{\beta}, \quad \varepsilon_{\alpha\beta}=\mathbf{n}(\mathbf{r}_{\alpha}\times\mathbf{r}_{\beta}), \tag{2.13}$$

we can obtain

$$\begin{aligned} &b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=\varepsilon_{\lambda\beta}(\mathbf{n}_{\alpha}\times\mathbf{n})\mathbf{n}_{\sigma}\\ =&\varepsilon_{\lambda\beta}b^{\nu}_{\alpha}b^{\mu}_{\sigma}(\mathbf{r}_{\nu}\times\mathbf{n})\mathbf{r}_{\mu}=\varepsilon_{\lambda\beta}b^{\nu}_{\alpha}b^{\mu}_{\sigma}(\mathbf{r}_{\mu}\times\mathbf{r}_{\nu})\mathbf{n}=\varepsilon_{\mu\nu}\varepsilon_{\lambda\beta}b^{\nu}_{\alpha}b^{\mu}_{\sigma}. \end{aligned}$$

This is the first part of (2.11).

Next we prove last two formula in (2.11). Remember that

$$b=\frac{1}{2}\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma}=a\frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma}.$$

Therefore,

$$\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma}=2\frac{b}{a}=2K.$$

This is the forth part of (2.11). Using $\varepsilon^{\alpha\lambda}a_{\alpha\nu}a_{\lambda\mu}=\varepsilon_{\nu\mu}$ we derive

$$\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}a_{\alpha\nu}b^{\nu}_{\beta}a_{\lambda\mu}b^{\mu}_{\sigma}=\varepsilon_{\nu\mu}\varepsilon^{\beta\sigma}b^{\nu}_{\beta}b^{\mu}_{\sigma}=2K.$$

This is the fifth part of (2.11).

Applying $\varepsilon^{\beta\lambda}$ to contraction of indices of tensor for both sides of above equality

$$\varepsilon_{\beta\lambda}\varepsilon_{\nu\mu}\varepsilon^{\beta\sigma}b^{\nu}_{\beta}b^{\mu}_{\sigma}=2K\varepsilon_{\beta\lambda}$$

and using $\varepsilon_{\beta\lambda}\varepsilon^{\beta\sigma}=\delta^{\sigma}_{\lambda}$ we lead to

$$\varepsilon_{\nu\mu}b^{\nu}_{\beta}b^{\mu}_{\lambda}=2K\varepsilon_{\beta\lambda}.$$

This is the third part of (2.11).

Next, we prove second of (2.11). To do that, combing the first and third part of (2.11), we have

$$b_{\alpha\beta}b_{\lambda\sigma}-b_{\alpha\lambda}b_{\beta\sigma}=\varepsilon_{\mu\nu}\varepsilon_{\lambda\beta}b^{\nu}_{\alpha}b^{\mu}_{\sigma}=\varepsilon_{\lambda\beta}K\varepsilon_{\alpha\sigma}.$$

From this it yields the second part of (2.11).

Next we prove (2.9). To do that by contraction of tensor indices for the second part of (2.11) with $a^{\lambda\sigma}$ we have

$$a^{\lambda\sigma}b_{\alpha\beta}b_{\lambda\sigma}-a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma}=Ka^{\lambda\sigma}\varepsilon_{\alpha\sigma}\varepsilon_{\beta\lambda}.$$

Because of

$$a^{\lambda\sigma}\varepsilon_{\alpha\sigma}\varepsilon_{\beta\lambda}=a_{\alpha\beta},\quad a^{\lambda\sigma}b_{\alpha\beta}b_{\lambda\sigma}=2Hb_{\alpha\beta},\quad a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma}=c_{\alpha\beta},$$

we can obtain the first part of (2.9).

Using the trick of tensor indices left leads to the second part of (2.9).

In order to prove the third part of (2.9), multiplying both sides of tensor index of the first part of (2.9) by $\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}$, then using (2.7), we derive

$$Ka^{\lambda\sigma}-2HK\widehat{b}^{\lambda\sigma}+K^2\widehat{c}^{\lambda\sigma}=0.$$

This is the third part of (2.9).

Applying trick of tensor index left, the second part of (2.11) can be rewritten as

$$b^{\alpha\beta}b^{\lambda\sigma}-b^{\alpha\lambda}b^{\beta\sigma}=K\varepsilon^{\alpha\sigma}\varepsilon^{\beta\lambda},$$

multiplying both sides of above equality, by $b_{\lambda\sigma}$ and using $\widehat{b}^{\alpha\beta}=\varepsilon^{\alpha\sigma}\varepsilon^{\beta\lambda}b_{\lambda\sigma}$, it is easy to yield

$$b^{\alpha\beta}b^{\lambda\sigma}b_{\lambda\sigma}-b^{\alpha\lambda}b^{\beta\sigma}b_{\lambda\sigma}=K^2\widehat{b}^{\alpha\beta}. \tag{2.14}$$

On the other hand, the first part of (2.9) can be rewritten in mixed tensor formulae

$$K\delta^\alpha_\beta-2Hb^\alpha_\beta+c^\beta_\beta=0.$$

Using this formula and $2H=b^\alpha_\alpha$, we claim that

$$\begin{aligned}&b^{\lambda\sigma}b_{\lambda\sigma}=c^\lambda_\lambda=2Hb^\lambda_\lambda-K\delta^\lambda_\lambda=4H^2-2K,\\&b^{\alpha\lambda}b^{\beta\sigma}b_{\lambda\sigma}=b^{\alpha\lambda}c^\beta_\lambda=b^{\alpha\lambda}(2Hb^\beta_\lambda-K\delta^\beta_\lambda)=2Hc^{\alpha\beta}-Kb^{\alpha\beta}\\&\qquad=2H(2Hb^{\alpha\beta}-Ka^{\alpha\beta})-Kb^{\alpha\beta}=(4H^2-K)b^{\alpha\beta}-2HKa^{\alpha\beta}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}&b^{\alpha\beta}b^{\lambda\sigma}b_{\lambda\sigma}-b^{\alpha\lambda}b^{\beta\sigma}b_{\lambda\sigma}\\&=(4H^2-2K)b^{\alpha\beta}-[(4H^2-K)b^{\alpha\beta}-2HKa^{\alpha\beta}]=K(-b^{\alpha\beta}+2Ha^{\alpha\beta}).\end{aligned} \tag{2.15}$$

Combing (2.14) and (2.15), we prove the fourth part of (2.9).

Because of the third and fourth part of (2.9)

$$K^2\widehat{c}^{\alpha\beta}=K(2H\widehat{b}^{\alpha\beta}-a^{\alpha\beta})=2H(2Ha^{\alpha\beta}-b^{\alpha\beta})-Ka^{\alpha\beta}=(4H^2-K)a^{\alpha\beta}-2Hb^{\alpha\beta},$$

it is easy to derive the fifth part of (2.9). Now we prove (2.10). With

$$\begin{aligned}&\widehat{b}^{\alpha\beta}b_{\beta\lambda}=b^{\alpha\beta}\widehat{b}_{\beta\lambda}=\delta^\alpha_\lambda,\quad \widehat{b}^{\alpha\lambda}b^\beta_\lambda=a^{\beta\sigma}\widehat{b}^{\alpha\lambda}b_{\sigma\lambda}=a^{\beta\sigma}\delta^\alpha_\sigma=a^{\alpha\beta},\\&K\widehat{c}^{\alpha\lambda}b^\beta_\lambda=(2H\widehat{b}^{\alpha\lambda}-a^{\alpha\lambda})b^\beta_\lambda=2Ha^{\alpha\beta}-b^{\alpha\beta}=K\widehat{b}^{\alpha\beta},\end{aligned}$$

it yields the first part of (2.10):

$$\widehat{b}^{\alpha\beta}=\widehat{c}^{\alpha\lambda}b^\beta_\lambda.$$

Similarly,

$$K\widehat{b}^{\alpha\lambda}\widehat{b}^\beta_\lambda=(2Ha^{\alpha\lambda}-b^{\alpha\lambda})\widehat{b}^\beta_\lambda=(2H\widehat{b}^{\alpha\beta}-a^{\alpha\beta})=K\widehat{c}^{\alpha\beta}.$$

This is the second part of (2.10).

On the other hand, with (2.10) we derive

$$\begin{aligned}K\widehat{b}^{\alpha\lambda}c_{\beta\lambda}&=(2Ha^{\alpha\lambda}-b^{\alpha\lambda})(2Hb_{\beta\lambda}-Ka_{\beta\lambda})=4H^2b^\alpha_\beta-2HK\delta^\alpha_\beta-2Hc^\alpha_\beta+Kb^\alpha_\beta\\&=((4H^2+K)b^\alpha_\beta-2HK\delta^\alpha_\beta-2H(2Hb^\alpha_\beta-K\delta^\alpha_\beta)=Kb^\alpha_\beta.\end{aligned}$$

This is the third part of (2.10).

Next we prove (2.12). Repeatedly using (2.9)

$$\begin{aligned}b^{\alpha\lambda}c^\beta_\lambda&=b^{\alpha\lambda}(-K\delta^\beta_\lambda+2Hb^\beta_\lambda)=-Kb^{\alpha\beta}+2Hc^{\alpha\beta}=-Kb^{\alpha\beta}+2H(-Ka^{\alpha\beta}+2Hb^{\alpha\beta})\\&=-2HKa^{\alpha\beta}+(4H^2-K)b^{\alpha\beta}=2Hc_{\alpha\beta}-Kb_{\alpha\beta}.\end{aligned}$$

Then, we have

$$\begin{aligned}c_{\alpha\lambda}c^{\lambda}_{\beta}&=c_{\alpha\lambda}(-K\delta^{\lambda}_{\beta}+2Hb^{\lambda}_{\beta})=-Kc_{\alpha\beta}+2Hb^{\lambda}_{\beta}c_{\alpha\lambda}=-Kc_{\alpha\beta}+2H(-2HKa_{\alpha\beta}+(4H^2-K)b_{\alpha\beta})\\&=-Kc_{\alpha\beta}-4H^2Ka_{\alpha\beta}+2H(4H^2-K)b_{\alpha\beta}=K(-2Hb_{\alpha\beta}+Ka_{\alpha\beta})-4H^2Ka_{\alpha\beta}\\&\quad+2H(4H^2-K)b_{\alpha\beta}=(K^2-4H^2K)a_{\alpha\beta}+2H(4H^2-2K)b_{\alpha\beta},\\b^{\alpha\beta}c_{\alpha\beta}&=b^{\alpha\beta}(2Hb_{\alpha\beta}-Ka_{\alpha\beta})=2H(4H^2-2K)-K2H=8H^3-6HK,\\c^{\alpha\beta}c_{\alpha\beta}&=c^{\alpha\beta}(2Hb_{\alpha\beta}-Ka_{\alpha\beta})=2H(8H^3-6HK)-K(4H^2-2K)=16H^4-16H^2K+2K^2.\end{aligned}$$

Those are (2.12), thus, we complete our proof. □

In the following sections, we consider the metric tensor of 3D Euclidean space under semi-geodesic coordinate (x^1,x^2,ξ).

Lemma 2.2. *Under semi-geodesic coordinate system, the covariant components $g_{ij}=\vec{\Theta}_i\vec{\Theta}_j$ of metric tensor of 3D Euclidian space E^3 are the polynomials of two degree with respect transversal variable ξ and can be expressed by means of the first, second and third fundamental form of surface $\Im$:*

$$\begin{cases} g_{\alpha\beta}(x,\xi)=a_{\alpha\beta}(x)-2\xi b_{\alpha\beta}(x)+\xi^2c_{\alpha\beta}(x)=(1-K\xi^2)a_{\alpha\beta}(x)+2\xi(H\xi-1)b_{\alpha\beta}(x);\\ g_{\alpha3}(x,\xi)=g_{3\alpha}(x,\xi)=0,\quad g_{33}(x,\xi)=1,\quad g(x,\xi)=\det(g_{ij})=\theta(\xi)^2a(x),\end{cases}\tag{2.16}$$

where

$$\theta(\xi)=1-2H\xi+K\xi^2=(1-\kappa_1\xi)(1-\kappa_2\xi),\tag{2.17}$$

$\kappa_\lambda,\lambda=1,2$ are the principle curvatures of $\Im$.

Its contravariant components g^{ij}, $g_{ij}g^{jk}=\delta^k_i$ are the rational functions of transverse variable ξ and can be expressed by inverse matrices $\widehat{b}^{\alpha\beta}$, $\widehat{c}^{\alpha\beta}$:

$$\begin{cases} g^{\alpha\beta}(x,\xi)=\theta^{-2}(a^{\alpha\beta}(x)-2Kb^{\hat{\alpha\beta}}(x)\xi+K^2\xi^2c^{\hat{\alpha\beta}}(x))=\theta^{-2}(p(\xi)a^{\alpha\beta}(x)+q(\xi)b^{\alpha\beta}(x));\\ g^{3\alpha}(x,\xi)=g^{\alpha3}(x,\xi)=0,\quad g^{33}(x,\xi)=1,\end{cases}\tag{2.18}$$

where $x=(x^1,x^2)$ and

$$p(\xi)=1-4H\xi+(4H^2-K)\xi^2,\quad q(\xi)=2\xi(1-H\xi).\tag{2.19}$$

In particular, $g^{\alpha\beta}$ admits a Taylor expansion

$$g^{\alpha\beta}=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^3+\cdots.\tag{2.20}$$

Proof. Let $\forall(x,\xi)\in E^3$

$$\mathbf{R}(x,\xi)=\mathbf{r}(x)+\xi\mathbf{n}$$

and its derivatives

$$\mathbf{R}_\alpha(x,\xi)=\frac{\partial R}{\partial x^\alpha}=\mathbf{r}_\alpha(x)+\xi\mathbf{n}_\alpha,\quad \mathbf{R}_3=\mathbf{n}.$$

It is obvious that they are independent on Ω. Hence

$$\begin{aligned}g_{\alpha\beta}=&\mathbf{R}_\alpha(x,\xi)\cdot\mathbf{R}_\beta(x,\xi)=\mathbf{r}_\alpha(x,\xi)\cdot\mathbf{r}_\beta(x,\xi)+\xi\Big\{\mathbf{r}_\alpha(x,\xi)\cdot\mathbf{n}_\beta(x,\xi)+\mathbf{n}_\alpha(x,\xi)\cdot\mathbf{r}_\beta(x,\xi)\Big\}\\&+\xi^2\mathbf{n}_\alpha(x,\xi)\cdot\mathbf{n}_\beta(x,\xi)=a_{\alpha\beta}-2\xi b_{\alpha\beta}+\xi^2c_{\alpha\beta}.\end{aligned}$$

Combing the first part of (2.9), we derive

$$g_{\alpha\beta}=p_0(\xi)a_{\alpha\beta}+q_0(\xi)b_{\alpha\beta}.$$

Since $|\mathbf{n}|^2=1$, $\mathbf{n}_\alpha(x,\xi)\cdot\mathbf{n}(x,\xi)=0$, we claim

$$g_{3\alpha}=(\mathbf{r}_\alpha(x,\xi)+\xi\mathbf{n}_\beta(x,\xi))\cdot\mathbf{n}=0,\quad g_{33}=\mathbf{n}(x,\xi)\cdot\mathbf{n}(x,\xi)=0.$$

Consider determinant

$$\begin{aligned}\det(g_{ij})=\det(g_{\alpha\beta})=&\tfrac{a}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}g_{\alpha\beta}g_{\lambda\sigma}=\tfrac{a}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}[a_{\alpha\beta}a_{\lambda\sigma}-2\xi(a_{\alpha\beta}b_{\lambda\sigma}+a_{\lambda\sigma}b_{\alpha\beta})\\&+\xi^2(a_{\alpha\beta}c_{\lambda\sigma}+a_{\lambda\sigma}c_{\alpha\beta}+4b_{\alpha\beta}b_{\lambda\sigma})-2\xi^3(b_{\alpha\beta}c_{\lambda\sigma}+b_{\lambda\sigma}c_{\alpha\beta})+\xi^4c_{\alpha\beta}c_{\lambda\sigma}).\end{aligned}$$

With (2.8)

$$\begin{cases}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}a_{\lambda\sigma}=a^{\alpha\beta},\quad \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}a_{\alpha\beta}a_{\lambda\sigma}=a^{\alpha\beta}a_{\alpha\beta}=2,\\ K\widehat{b}^{\alpha\beta}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\lambda\sigma},\quad K^2\widehat{c}^{\alpha\beta}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}c_{\lambda\sigma},\end{cases}\tag{2.21}$$

it infers

$$\begin{aligned}g:=\det(g_{ij})=&\tfrac{a}{2}[a_{\alpha\beta}a^{\alpha\beta}-4\xi(a^{\lambda\sigma}b_{\lambda\sigma})+2\xi^2(a^{\lambda\sigma}c_{\lambda\sigma}+2K\widehat{b}^{\lambda\sigma}b_{\lambda\sigma})\\&-4\xi^3(K\widehat{b}^{\lambda\sigma}c_{\lambda\sigma})+\xi^4K^2\widehat{c}^{\lambda\sigma}c_{\lambda\sigma}).\end{aligned}$$

From (2.8) and (2.9)

$$\begin{aligned}&a_{\alpha\beta}a^{\alpha\beta}=2,\quad a^{\lambda\sigma}b_{\lambda\sigma}=2H,\quad a^{\lambda\sigma}c_{\lambda\sigma}=4H^2-2K,\quad \widehat{b}^{\lambda\sigma}b_{\lambda\sigma}=2,\quad \widehat{c}^{\lambda\sigma}c_{\lambda\sigma}=2,\\&K\widehat{b}^{\lambda\sigma}c_{\lambda\sigma}=(2Ha^{\lambda\sigma}-b^{\lambda\sigma})c_{\lambda\sigma}=2H(4H^2-2K)-(8H^3-6HK)=2HK,\end{aligned}\tag{2.22}$$

we obtain

$$g=\tfrac{a}{2}\{2-8H\xi+2(4H^2+2K)\xi^2-4HK\xi^3+2K\xi^4\}=a\theta^2.$$

We complete the proof of (2.16).

In order to prove (2.18), we observe $[g^{ij}]=[g_{ij}]^{-1}$,

$$\begin{cases}g^{ij}=\frac{1}{2}\varepsilon^{ikl}\varepsilon^{jmn}g_{km}g_{ln},\\ g^{\alpha\beta}=\frac{1}{2}\varepsilon^{\alpha kl}\varepsilon^{\beta mn}g_{km}g_{ln}\\ \quad=\frac{1}{2}[\varepsilon^{\alpha3\lambda}\varepsilon^{\beta mn}g_{3m}g_{\lambda n}+\varepsilon^{\alpha\lambda3}\varepsilon^{\beta mn}g_{\lambda m}g_{3n}]\\ \quad=\frac{1}{2}[\varepsilon^{\alpha3\lambda}\varepsilon^{\beta3\sigma}g_{33}g_{\lambda\sigma}+\varepsilon^{\alpha3\lambda}\varepsilon^{\beta\sigma3}g_{3\sigma}g_{\lambda3}+\varepsilon^{\alpha\lambda3}g^{\beta3\sigma}g_{\lambda3}g_{3\sigma}+\varepsilon^{\alpha\lambda3}\varepsilon^{\beta\sigma3}g_{33}g_{\lambda\sigma}]\\ \quad=(by(2.16))\frac{1}{2}[\varepsilon^{\alpha3\lambda}\varepsilon^{\beta3\sigma}g_{33}g_{\lambda\sigma}+\varepsilon^{\alpha\lambda3}\varepsilon^{\beta\sigma3}g_{33}g_{\lambda\sigma}],\end{cases}$$

and

$$\varepsilon^{3\alpha\beta}=\frac{\sqrt{a}}{\sqrt{g}}\varepsilon^{\alpha\beta}=\theta^{-1}\varepsilon^{\alpha\beta}.$$

Then

$$g^{\alpha\beta}=\frac{1}{2}[2\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}g_{\lambda\sigma}]=\theta^{-2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}(a_{\lambda\sigma}-2\xi b_{\lambda\sigma}+\xi^2c_{\lambda\sigma})=a^{\alpha\beta}-2\xi K\widehat{b}^{\alpha\beta}+K^2\widehat{c}^{\alpha\beta}. \tag{2.23}$$

Similarly

$$\begin{aligned}
g^{3\alpha}&=\frac{1}{2}\varepsilon^{3kl}g^{\alpha mn}g_{km}g_{ln}=\frac{1}{2}\varepsilon^{3\lambda\sigma}\varepsilon^{\alpha mn}g_{\lambda m}g_{\sigma n}=\frac{1}{2}\varepsilon^{3\lambda\sigma}\varepsilon^{\alpha 3\beta}g_{\lambda 3}g_{\sigma\beta}+\frac{1}{2}\varepsilon^{3\lambda\sigma}\varepsilon^{\alpha\beta 3}g_{\lambda\beta}g_{\sigma 3}=0,\\
g^{33}&=\frac{1}{2}\varepsilon^{3kl}\varepsilon^{3mn}g_{km}g_{ln}=\frac{1}{2}\varepsilon^{3\lambda\sigma}\varepsilon^{3\alpha\beta}g_{\lambda\alpha}g_{\sigma\beta}=\frac{1}{2}\theta^{-2}\varepsilon^{\lambda\sigma}\varepsilon^{\alpha\beta}g_{\lambda\alpha}g_{\sigma\beta}\\
&=\frac{1}{2}\theta^{-2}\varepsilon^{\lambda\sigma}\varepsilon^{\alpha\beta}(a_{\alpha\lambda}-2\xi b_{\alpha\lambda}+\xi^2c_{\alpha\lambda})(a_{\beta\sigma}-2\xi b_{\beta\sigma}+\xi^2c_{\beta\sigma})\\
&=\frac{1}{2}\theta^{-2}\varepsilon^{\lambda\sigma}\varepsilon^{\alpha\beta}[a_{\alpha\lambda}a_{\beta\sigma}-2\xi(a_{\alpha\lambda}b_{\beta\sigma}+a_{\beta\sigma}b_{\alpha\lambda})+\xi^2(a_{\alpha\lambda}c_{\beta\sigma}+a_{\beta\sigma}c_{\alpha\lambda}+4b_{\alpha\lambda}b_{\beta\sigma})\\
&\qquad-2\xi^3(b_{\alpha\lambda}c_{\beta\sigma}+b_{\beta\sigma}c_{\alpha\lambda})+\xi^4c_{\alpha\lambda}c_{\beta\sigma})].
\end{aligned}$$

With (2.21), we have

$$\begin{aligned}
g^{33}=\frac{1}{2}\theta^{-2}[&a^{\beta\sigma}a_{\beta\sigma}-2\xi(a^{\beta\sigma}b_{\beta\sigma}+a^{\alpha\lambda}b_{\alpha\lambda})+\xi^2(a^{\beta\sigma}c_{\beta\sigma}+a^{\alpha\lambda}c_{\alpha\lambda}+4K\widehat{b}^{\beta\sigma}b_{\beta\sigma})\\
&-2\xi^3(K\widehat{b}^{\beta\sigma}c_{\beta\sigma}+K\widehat{b}^{\alpha\lambda}c_{\alpha\lambda})+\xi^4K^2c^{\beta\sigma}c_{\beta\sigma})].
\end{aligned}$$

Using (2.12), we obtain

$$g^{33}=\frac{1}{2}\theta^{-2}[2-8H\xi+2\xi^2(4H^2+2K)-8HK\xi^3+2\xi^4K^2]=\theta^{-2}\theta^2=1.$$

Making Taylor expansion

$$\begin{aligned}
&\theta^{-2}=1+4H\xi+(12H^2-2K)\xi^2\cdots,\\
&g^{\alpha\beta}=(1+4H\xi+(12H^2-2K)\xi^2+\cdots)(p(\xi)a^{\alpha\beta}+q(\xi)b^{\alpha\beta})\\
&\quad=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3(-Ka^{\alpha\beta}+2Hb^{\alpha\gamma})\xi^2+\cdots(\text{appying second of}(2.9))\\
&\quad=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^2+\cdots=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^2+\cdots,
\end{aligned} \tag{2.24}$$

gives (2.20). The proof is completed. □

Since $\Im$ is as a 2D manifold embedded in 3D Euclidian space E^3, we need consider the mixed tensor, in particular, mixed covariant derivative for the mixed tensor. Lemmas 2.2 and 2.3 provide the relations between the tensors in E^3 and those on $\Im$. What follows, we consider others relations, for example, let Γ^i_{jk},∇_i, and $\overset{*}{\Gamma}{}^{\alpha}{}_{\beta\gamma},\overset{*}{\nabla}_\alpha$ denote Christoffel symbols and covariant derivative in E^3 and on $\Im$ respectively,

$$\left\{\begin{aligned}
&\Gamma_{ij,k}=\frac{1}{2}\Big(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\Big),\quad \Gamma^m_{ij}=g^{mk}\Gamma_{ij,k},\\
&\overset{*}{\Gamma}_{\alpha\beta,\lambda}=\frac{1}{2}\Big(\frac{\partial a_{\alpha\lambda}}{\partial x^\beta}+\frac{\partial a_{\beta\lambda}}{\partial x^\alpha}-\frac{\partial a_{\alpha\beta}}{\partial x^\lambda}\Big),\quad \overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}=a^{\lambda\sigma}\overset{*}{\Gamma}_{\alpha\beta,\sigma},\\
&\nabla_iu^j=\frac{\partial u^j}{\partial x^i}+\Gamma^j_{ik}u^k,\quad \overset{*}{\nabla}_\alpha u^\beta=\frac{\partial u^\beta}{\partial x^\alpha}+\overset{*}{\Gamma}{}^{\beta}{}_{\alpha\lambda}u^\lambda,\\
&\mathrm{div}u=\nabla_iu^i,\quad \overset{*}{\mathrm{div}}\,u=\overset{*}{\nabla}_\alpha u^\alpha.
\end{aligned}\right. \tag{2.25}$$

Then we have

Lemma 2.3. *Under S-coordinate system, Christoffel symbols* $(\Gamma^k_{ij},\Gamma_{ij,k})$ *in* E^3 *can be expressed by means of Christoffel symbols* $(\overset{*}{\Gamma}{}^{\alpha}{}_{\beta\lambda},\overset{*}{\Gamma}_{\alpha\beta,\lambda})$ *of* $\Im$

$$\begin{cases} \Gamma_{\alpha\beta,\lambda}=g_{\lambda\sigma}\overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}+\xi(H\xi-1)\overset{*}{\nabla}_{\lambda}b_{\alpha\beta}+2\xi(H\xi-1)(\overset{*}{\Gamma}{}^{\sigma}{}_{\beta\lambda}b_{\sigma\alpha}-b_{\lambda\sigma}\overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}), \\ \Gamma_{\alpha\beta,3}=-J_{\alpha\beta}(\xi), \quad \Gamma_{\alpha3,\beta}=\Gamma_{3\alpha,\beta}=J_{\alpha\beta}(\xi), \quad \Gamma_{ij,k}=0, \quad \textit{other case}, \end{cases} \tag{2.26}$$

$$\begin{cases} \Gamma^{\lambda}_{\alpha\beta}=\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}+\Phi^{\lambda}_{\alpha\beta}, \quad \Gamma^{\alpha}_{\beta3}=\Gamma^{\alpha}_{3\beta}=\theta^{-1}I^{\alpha}_{\beta}, \quad \Gamma^{3}_{\alpha\beta}=J_{\alpha\beta}, \\ \Gamma^3_{33}=\Gamma^3_{3\beta}=\Gamma^3_{\beta3}=\Gamma^{\alpha}_{33}=0, \end{cases} \tag{2.27}$$

where

$$\begin{array}{l} \Phi^{\alpha}_{\beta\lambda}=\theta^{-1}R^{\alpha}_{\beta\lambda}, \quad R^{\alpha}_{\beta\lambda}=(-\delta^{\alpha}_{\mu}\xi+(2H\delta^{\alpha}_{\mu}-b^{\alpha}_{\mu})\xi^2)\overset{*}{\nabla}_{\lambda}b^{\mu}_{\beta}; \\ I^{\alpha}_{\beta}=-b^{\alpha}_{\beta}+K\xi\delta^{\alpha}_{\beta}, \quad J_{\alpha\beta}=b_{\alpha\beta}-\xi c_{\alpha\beta}, \quad g^{\alpha\beta}J_{\beta\sigma}=-\theta^{-1}I^{\alpha}_{\sigma}. \end{array} \tag{2.28}$$

Proof. With Weingarten formula and Gaussian formulae

$$\begin{cases} \mathbf{r}_{\alpha\beta}=\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}\mathbf{r}_{\lambda}+b_{\alpha\beta}\mathbf{n}, \quad \partial_{\beta}\mathbf{r}^{\alpha}=-\overset{*}{\Gamma}{}^{\alpha}{}_{\beta\lambda}\mathbf{r}^{\lambda}+b^{\alpha}_{\beta}\mathbf{n}, \\ \mathbf{n}_{\beta}=-b^{\alpha}_{\beta}\mathbf{r}_{\alpha}=-b_{\alpha\beta}\mathbf{r}^{\alpha}. \end{cases} \tag{2.29}$$

we have

$$\begin{cases} \mathbf{R}_{\alpha}=\mathbf{r}_{\alpha}+\xi\mathbf{n}_{\alpha}=(\delta^{\lambda}_{\alpha}-\xi b^{\lambda}_{\alpha})\mathbf{r}_{\lambda}, \\ \mathbf{R}_3=\mathbf{n}, \end{cases} \tag{2.30}$$

$$\begin{aligned} \mathbf{R}_{\alpha\beta}&=(\delta^{\lambda}_{\alpha}-\xi b^{\lambda}_{\alpha})\mathbf{r}_{\lambda\beta}-\xi\partial_{\beta}b^{\lambda}_{\alpha}r_{\lambda} \\ &=\overset{*}{\Gamma}{}^{\nu}{}_{\lambda\beta}(\delta^{\lambda}_{\alpha}-\xi b^{\lambda}_{\alpha})\mathbf{r}_{\nu}+b_{\lambda\beta}(\delta^{\lambda}_{\alpha}-\xi b^{\lambda}_{\alpha})\mathbf{n}-\xi\partial_{\beta}b^{\lambda}_{\alpha}r_{\lambda} \\ &=[\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\Gamma}{}^{\nu}{}_{\lambda\beta}b^{\lambda}_{\alpha}+\partial_{\beta}b^{\nu}_{\alpha})]\mathbf{r}_{\nu}+J_{\alpha\beta}\mathbf{n} \\ &=[\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}+\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}b^{\nu}_{\mu})]\mathbf{r}_{\nu}+J_{\alpha\beta}\mathbf{n}. \end{aligned}$$

Therefore,

$$\begin{cases} \mathbf{R}_{\alpha\beta}=[\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}+\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}b^{\nu}_{\mu})]\mathbf{r}_{\nu}+J_{\alpha\beta}\mathbf{n}, \\ \mathbf{R}_{\alpha3}=\mathbf{R}_{3\alpha}=\mathbf{n}_{\alpha}=-b^{\lambda}_{\alpha}\mathbf{r}_{\lambda}, \\ \mathbf{R}_{33}=0. \end{cases} \tag{2.31}$$

Here we used Gadazzi formula and the covariant derivative of $b_{\alpha\beta}$

$$\frac{\partial b_{\alpha\lambda}}{\partial x^{\beta}}=\overset{*}{\nabla}_{\beta}b_{\alpha\lambda}+\overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}b_{\sigma\lambda}+\overset{*}{\Gamma}{}^{\sigma}{}_{\beta\lambda}b_{\alpha\sigma}, \quad \overset{*}{\nabla}_{\alpha}b_{\beta\lambda}=\overset{*}{\nabla}_{\beta}b_{\alpha\lambda}. \tag{2.32}$$

Since $\mathbf{n}\mathbf{r}_{\lambda}=0$, $c_{\lambda\mu}=b^{\nu}_{\mu}b_{\lambda\nu}$, $g_{\lambda\mu}=a_{\lambda\mu}-2\xi b_{\lambda\mu}+\xi^2c_{\lambda\mu}$, we have

$$\begin{aligned} \Gamma_{\alpha\beta,\lambda}&=\mathbf{R}_{\alpha\beta}\mathbf{R}_{\lambda}=[\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}+\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}b^{\nu}_{\mu})]\mathbf{r}_{\nu}\cdot(\delta^{\sigma}_{\lambda}-\xi\delta^{\sigma}_{\lambda})\mathbf{r}_{\sigma} \\ &=(a_{\lambda\nu}-\xi b_{\lambda\nu})[\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}+\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}b^{\nu}_{\mu})] \\ &=a_{\lambda\nu}\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}-\xi(\overset{*}{\nabla}_{\beta}b_{\alpha\lambda}+2\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}b^{\nu}_{\mu})+\xi^2(b_{\lambda\nu}\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}+c_{\lambda\mu}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\beta}) \\ &=\overset{*}{\Gamma}_{\alpha\beta,\lambda}+Q_{\alpha\beta,\lambda}(\xi)=g_{\lambda\mu}\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\beta}+R_{\alpha\beta,\lambda}(\xi), \end{aligned}$$

where

$$\begin{cases} R_{\alpha\beta,\lambda}(\xi) := -\xi \overset{*}{\nabla}_\beta b_{\alpha\lambda} + \xi^2 b_{\lambda\mu} \overset{*}{\nabla}_\beta b^\mu_\alpha, \\ Q_{\alpha\beta,\lambda} := -\xi(\overset{*}{\nabla}_\beta b_{\alpha\lambda} + 2b_{\lambda\mu} \overset{*}{\Gamma}{}^\mu{}_{\alpha\beta}) + \xi^2(b_{\lambda\mu} \overset{*}{\nabla}_\beta b^\mu_\alpha + c_{\lambda\mu} \overset{*}{\Gamma}{}^\mu{}_{\alpha\beta}). \end{cases}$$

We complete the proof of first part of (2.26).

Similarly,

$$\begin{aligned} &\Gamma_{\alpha\beta,3} = \mathbf{R}_{\alpha\beta}\mathbf{n} = J_{\alpha\beta}(\xi), \\ &\Gamma_{\alpha3,\beta} = \Gamma_{3\alpha,\beta} = \mathbf{R}_{3\alpha}\mathbf{R}_\beta = -b^\lambda_\alpha \mathbf{r}_\lambda \mathbf{R}_\beta = -J_{\alpha\beta}(\xi), \\ &\Gamma_{33,\alpha} = \Gamma_{3\beta,3} = 0. \end{aligned}$$

Next we prove (2.27). In deed, from the first part of (2.26), we derive

$$\Gamma^\lambda_{\alpha\beta} \quad = g^{\lambda\sigma}\Gamma_{\alpha\beta,\sigma} = g^{\lambda\sigma} g_{\sigma\nu} \overset{*}{\Gamma}{}^\nu{}_{\alpha\beta} + g^{\lambda\sigma} R_{\alpha\beta,\sigma} = \overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta} + \Phi^\lambda_{\alpha\beta}(\xi).$$

We use $g^{\lambda\sigma} g_{\sigma\nu} = \delta^\lambda_\nu$ and get

$$\begin{aligned} \Phi^\lambda_{\alpha\beta}(\xi) &:= g^{\lambda\sigma} R_{\alpha\beta,\sigma} \\ &= \theta^{-2}(p(\xi)a^{\lambda\sigma} + q(\xi)b^{\lambda\sigma})(-\xi \overset{*}{\nabla}_\beta b_{\alpha\sigma} + \xi^2 b_{\sigma\mu} \overset{*}{\nabla}_\beta b^\mu_\alpha)) \\ &= \theta^{-2}[p(\xi)(-\xi \overset{*}{\nabla}_\beta b^\lambda_\alpha + \xi^2 b^\lambda_\mu \overset{*}{\nabla}_\beta b^\mu_\alpha) + q(\xi)(-\xi b^{\lambda\sigma} \overset{*}{\nabla}_\beta b_{\alpha\sigma} + \xi^2 c^\lambda_\mu \overset{*}{\nabla}_\beta b^\mu_\alpha)]. \end{aligned}$$

As the covariant derivative of metric tensor is vanished, from Godazzi formula, we have

$$\begin{aligned} &c^\lambda_\mu = -K\delta^\lambda_\mu + 2Hb^\lambda_\mu, \\ &b^\lambda_\mu \overset{*}{\nabla}_\beta b^\mu_\alpha = b^{\lambda\sigma} \overset{*}{\nabla}_\beta b_{\alpha\sigma} = b^{\lambda\sigma} \overset{*}{\nabla}_\sigma b_{\alpha\beta}. \end{aligned}$$

It can also be expressed by

$$\Phi^\lambda_{\alpha\beta} = \theta^{-2}[\varphi_1(\xi) \overset{*}{\nabla}_\beta b^\lambda_\alpha + \varphi_2(\xi) b^{\lambda\sigma} \overset{*}{\nabla}_\beta b_{\alpha\sigma}],$$

$$\varphi_1(\xi) := -\xi(p(\xi) + K\xi), \quad \varphi_2(\xi) := \xi(p\xi - q + 2H\xi q).$$

Taking (2.19) into account, simply computation shows

$$\varphi_1(\xi) = \xi(2H\xi - 1)\theta, \quad \varphi_2(\xi) = -\xi^2\theta.$$

Therefore,

$$\begin{aligned} \Phi^\lambda_{\alpha\beta} &= \theta^{-1}(\xi)(\xi(2H\xi - 1) \overset{*}{\nabla}_\beta b^\lambda_\alpha - \xi^2 b^{\lambda\sigma} \overset{*}{\nabla}_\sigma b_{\alpha\beta}) \\ &= \theta^{-1}(\xi)[(-\delta^\alpha_\mu \xi + (2H\delta^\alpha_\mu - b^\alpha_\mu)\xi^2) \overset{*}{\nabla}_\lambda b^\mu_\beta]. \end{aligned}$$

This is the first part of (2.27). In addition,

$$\Gamma^\alpha_{3\beta} = g^{\alpha\lambda}\Gamma_{3\beta,\lambda} = -g^{\alpha\lambda} J_{\beta\lambda} = \theta^{-1} I^\alpha_\beta.$$

Other conclusions can be proved easily. □

As we all know, the covariant derivatives of tensor in E^3 and $\Im$ are defined by

$$\nabla_j u^i = \frac{\partial u^j}{\partial x^j} + \Gamma^i_{jk} u^k, \quad \overset{*}{\nabla}_\beta u^\alpha = \frac{\partial u^\alpha}{\partial x^\beta} + \overset{*}{\Gamma}{}^\alpha{}_{\beta\gamma} u^\gamma, \quad \overset{*}{\nabla}_\beta u^3 = \frac{\partial u^\alpha}{\partial \xi}. \tag{2.33}$$

Since $\Im$ is embedded into E^3, there are relations between covariant derivatives of tensor at different level.

Lemma 2.4. *Under S-coordinate system covariant derivative of a vector $\vec{u}$ in E^3 can be expressed by derivatives of its components on the tangent space at $\Im$. Furthermore it is a rational function of transversal variable ξ*

$$\begin{cases} \nabla_\alpha u^\beta = \overset{*}{\nabla}_\alpha u^\beta + \theta^{-1} I^\beta_\alpha u^3 + \Phi^\beta_{\alpha\lambda} u^\lambda, \quad \nabla_3 u^3 = \frac{\partial u^3}{\partial \xi}; \\ \nabla_3 u^\beta = \frac{\partial u^\beta}{\partial \xi} + \theta^{-1} I^\beta_\lambda u^\lambda; \quad \nabla_\alpha u^3 = \overset{*}{\nabla}_\alpha u^3 + J_{\alpha\lambda} u^\lambda; \\ divu = \overset{*}{div}\, u + \frac{\partial u^3}{\partial \xi} + d_k(\xi) u^k, \\ d_\alpha(\xi) = \theta^{-1}[-2\overset{*}{\nabla}_\alpha H\xi + \overset{*}{\nabla}_\alpha K\xi^2], \quad d_3(\xi) = \theta^{-1}(-2H + 2K\xi), \\ \theta = 1 - 2H\xi + K\xi^2, \end{cases} \tag{2.34}$$

which admit to make Taylor expansion with respect to transversal variable ξ

$$\begin{cases} \nabla_i u^j = \overset{0}{\nabla}_i u^j + \overset{1}{\nabla}_i u^j \xi + \overset{2}{\nabla}_i u^j \xi^2 + \cdots, \\ divu = \frac{\partial u^3}{\partial \xi} + \overset{0}{div}\, u + \overset{1}{div}\, u\xi + \overset{2}{div}\, u\xi^2 + \cdots \end{cases} \tag{2.35}$$

where

$$\begin{cases} \overset{0}{\nabla}_\alpha u^\beta := \overset{*}{\nabla}_\alpha u^\beta - b^\beta_\alpha u^3, \quad \overset{1}{\nabla}_\alpha u^\beta := -(c^\beta_\alpha u^3 + \overset{*}{\nabla}_\lambda b^\beta_\alpha u^\lambda), \\ \overset{2}{\nabla}_\alpha u^\beta := (K b^\beta_\alpha - 2H c^\beta_\alpha) u^3 - b^\beta_\lambda \overset{*}{\nabla}_\sigma b^\lambda_\alpha u^\sigma, \\ \overset{0}{\nabla}_\alpha u^3 := \overset{*}{\nabla}_\alpha u^3 + b_{\beta\alpha} u^\beta, \quad \overset{1}{\nabla}_\alpha u^3 := -c_{\beta\alpha} u^\beta, \quad \overset{2}{\nabla}_\alpha u^3 := 0, \\ \overset{0}{\nabla}_3 u^\beta := \frac{\partial u^\beta}{\partial \xi} - b^\beta_\lambda u^\lambda, \quad \overset{1}{\nabla}_3 u^\beta := -c^\beta_\lambda u^\lambda, \quad \overset{2}{\nabla}_3 u^\beta := (K b^\beta_\lambda - 2H c^\beta_\lambda) u^\lambda, \\ \overset{0}{\nabla}_3 u^3 := \frac{\partial u^3}{\partial \xi}, \quad \overset{1}{\nabla}_3 u^3 := 0, \quad \overset{2}{\nabla}_3 u^3 := 0. \end{cases} \tag{2.36}$$

$$\begin{cases} \overset{0}{div}\, u := \overset{*}{div}\, u - 2H u^3, \quad \overset{1}{div}\, u := -[(4H^2 - 2K) u^3 + 2u^\alpha \overset{*}{\nabla}_\alpha H], \\ \overset{2}{div}\, u := -[(8H^3 - 6HK) u^3 + u^\alpha \overset{*}{\nabla}_\alpha (2H^2 - K)]. \end{cases} \tag{2.37}$$

Proof. Note that (2.34) can be easly derived from (2.33) and (2.27). In order to consider Taylor expansion, it is value to mention that when ξ is small enough, function θ^{-1} can be made Taylor expansion

$$\theta^{-1} = 1 + 2H\xi + (4H^2 - K)\xi^2 + \cdots. \tag{2.38}$$

Since

$$\begin{aligned}\theta^{-1}I^\beta_\alpha u^3+\Phi^\beta_{\alpha\lambda}u^\lambda&=\theta^{-1}[-b^\beta_\alpha u^3-(\overset{*}{\nabla}_\alpha b^\beta_\lambda u^\lambda-K\delta^\beta_\lambda u^3)\xi+(2H\overset{*}{\nabla}_\alpha b^\beta_\lambda-b^{\beta\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\sigma})u^\lambda\xi^2]\\&=-b^\beta_\alpha u^3-(c^\beta_\alpha u^3+\overset{*}{\nabla}_\alpha b^\beta_\lambda u^\lambda)\xi+((-2Hc^\beta_\alpha+Kb^\beta_\alpha)u^3-b^{\beta\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\sigma}u^\lambda)\xi^2+o(|\xi|^3),\\\theta^{-1}I^\beta_\alpha&=-b^\beta_\alpha-c^\beta_\alpha\xi+(-2Hc^\beta_\alpha+Kb^\beta_\alpha)\xi^2+o(|\xi|^3),\end{aligned}$$

it immediately yields (2.35) and (2.36). Remind that we have to compute divergence. In fact,

$$\mathrm{div}u=\overset{*}{\mathrm{div}}\,u+\frac{\partial u^3}{\partial\xi}+\Phi^\alpha_{\alpha\lambda}(\xi)u^\lambda+\theta^{-1}I^\alpha_\alpha(\xi)+o(|\xi|^3).$$

Applying Godazzi formula and Lemma 2.1,

$$\begin{aligned}&\overset{*}{\nabla}_\alpha b_{\lambda\sigma}=\overset{*}{\nabla}_\lambda b_{\alpha\sigma},\quad b^{\alpha\beta}b_{\alpha\beta}=4H^2-2K,\\&2b^{\lambda\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\sigma}=\overset{*}{\nabla}_\alpha(b^{\lambda\sigma}b_{\lambda\sigma})=\overset{*}{\nabla}_\alpha(4H^2-2K),\end{aligned}\tag{2.39}$$

we have

$$I^\alpha_\alpha=-b^\alpha_\alpha+K\xi\delta^\alpha_\alpha=-2H+2K\xi,\quad\Phi^\alpha_{\alpha\lambda}(\xi)=\theta^{-1}(-2\overset{*}{\nabla}_\lambda H\xi+\xi^2\overset{*}{\nabla}_\lambda K).\tag{2.40}$$

Combining above results, it is easy to obtain (2.35)–(2.37). The proof is completed. □

Following lemma is very useful throughout this paper which indicates the relations between $J_{\alpha\beta},I^\alpha_\beta,R^\alpha_{\beta\lambda}$ and so on.

Lemma 2.5. *The following formulae are valid*

$$\begin{cases}g_{\alpha\beta}I^\beta_\sigma=-\theta J_{\alpha\sigma};\quad J_{\alpha\beta}I^\beta_\sigma=-\theta c_{\alpha\sigma};\quad g^{\alpha\beta}J_{\alpha\lambda}=-\theta^{-1}I^\beta_\lambda;\\g_{\alpha\beta}\Phi^\beta_{\sigma\lambda}=-\xi\overset{*}{\nabla}_\alpha b_{\sigma\lambda}+\xi^2b_{\alpha\gamma}\overset{*}{\nabla}_\sigma b^\gamma_\lambda;\quad J_{\alpha\beta}\Phi^\beta_{\sigma\lambda}=-\xi b_{\alpha\gamma}\overset{*}{\nabla}_\sigma b^\gamma_\lambda;\\g_{\alpha\beta}\Phi^\alpha_{\sigma\lambda}\Phi^\beta_{\nu\mu}=\xi^2a_{\gamma\eta}\overset{*}{\nabla}_\sigma b^\gamma_\lambda\overset{*}{\nabla}_\nu b^\eta_\mu;\\g_{\lambda\sigma}I^\lambda_\alpha I^\sigma_\beta=c_{\alpha\beta}\theta^2;\quad g_{\lambda\sigma}I^\lambda_\nu\Phi^\sigma_{\alpha\beta}=\xi\theta b_{\nu\mu}\overset{*}{\nabla}_\alpha b^\mu_\beta.\\g^{\beta\sigma}\Phi^\lambda_{\beta\sigma}=\theta^{-3}\{((2H\xi^2-\xi)a^{\lambda\mu}-\xi^2b^{\lambda\mu})(2p(\xi)+4Hq(\xi)\overset{*}{\nabla}_\mu H-q(\xi)\overset{*}{\nabla}_\mu(K))\}.\end{cases}\tag{2.41}$$

Proof. Repeatedly using Lemma 2.1, we have

$$\begin{aligned}
g_{\alpha\beta}I^{\beta}_{\sigma} &= (a_{\alpha\beta}-2\xi b_{\alpha\beta}+\xi^2 c_{\alpha\beta})(-b^{\beta}_{\sigma}+K\xi\delta^{\beta}_{\sigma})\\
&= -b_{\alpha\sigma}+2\xi c_{\alpha\sigma}-\xi^2 b^{\beta}_{\sigma}c_{\alpha\beta}+K\xi a_{\alpha\sigma}-2\xi^2 K b_{\alpha\sigma}+\xi^3 K c_{\alpha\sigma}\\
&= -b_{\alpha\sigma}+\xi(c_{\alpha\sigma}+Ka_{\alpha\beta})+\xi c_{\alpha\sigma}-\xi^2(-2HKa_{\alpha\sigma}+(4H^2-K)b_{\alpha\sigma})-2\xi^2 K b_{\alpha\sigma}+\xi^3 K c_{\alpha\sigma}\\
&= -b_{\alpha\sigma}+2H\xi b_{\alpha\sigma}-K\xi^2 b_{\alpha\sigma}+\xi c_{\alpha\sigma}-\xi^2(-2HKa_{\alpha\sigma}+(4H^2)b_{\alpha\sigma})+\xi^3 K c_{\alpha\sigma}\\
&= -\theta b_{\alpha\sigma}+\xi c_{\alpha\sigma}-\xi^2 2Hc_{\alpha\sigma}+\xi^3 K c_{\alpha\sigma} = -\theta b_{\alpha\sigma}+\xi\theta c_{\alpha\sigma} = -\theta J_{\alpha\sigma},\\
J_{\alpha\beta}I^{\beta}_{\sigma} &= (b_{\alpha\beta}-\xi c_{\alpha\beta})(-b^{\alpha}_{\sigma}+K\xi\delta^{\alpha}_{\sigma}) = -c_{\alpha\beta}+\xi(c_{\alpha\beta}b^{\beta}_{\sigma}+Kb_{\alpha\sigma})-Kc_{\alpha\sigma}\xi^2\\
&= -c_{\alpha\sigma}+\xi(-2HKa_{\alpha\sigma}+(4H^2-K)b_{\alpha\sigma}+Kb_{\alpha\sigma})-Kc_{\alpha\sigma}\xi^2\\
&= -c_{\alpha\sigma}+2H\xi(2Hb_{\alpha\sigma}-Ka_{\alpha\sigma})-Kc_{\alpha\sigma}\xi^2 = -\theta c_{\alpha\sigma},\\
g^{\alpha\beta}J_{\alpha\lambda} &= \theta^{-2}(a^{\alpha\beta}-2K\widehat{b}^{\alpha\beta}\xi+K^2\widehat{c}^{\alpha\beta}\xi^2)(b_{\alpha\lambda}-\xi c_{\alpha\lambda})\\
&= \theta^{-2}(b^{\beta}_{\lambda}-\xi(c^{\beta}_{\lambda}+2K\delta^{\beta}_{\lambda})+\xi^2(2K\widehat{b}^{\alpha\beta}c_{\alpha\lambda}+K^2\widehat{c}^{\alpha\beta}b_{\alpha\lambda})-K^2\delta^{\beta}_{\lambda}\xi^3).
\end{aligned}$$

Since

$$\begin{aligned}
&2K\widehat{b}^{\alpha\beta}c_{\alpha\lambda}+K^2\widehat{c}^{\alpha\beta}b_{\alpha\lambda} = K^2\widehat{b}^{\beta}_{\lambda}+2Kb^{\beta}_{\lambda} = 2HK\delta^{\beta}_{\lambda}+Kb^{\beta}_{\lambda},\\
&c^{\beta}_{\lambda}+2K\delta^{\beta}_{\lambda} = 2Hb^{\beta}_{\lambda}+K\delta^{\beta}_{\lambda},
\end{aligned}$$

we have,

$$\begin{aligned}
g^{\alpha\beta}J_{\alpha\lambda} &= \theta^{-2}(b^{\beta}_{\lambda}-\xi(2Hb^{\beta}_{\lambda}+K\delta^{\beta}_{\lambda})+\xi^2(2Hk\delta^{\beta}_{\lambda}+Kb^{\beta}_{\lambda})-K^2\delta^{\beta}_{\lambda}\xi^3)\\
&= \theta^{-1}(Hb^{\beta}_{\lambda}-K\xi\delta^{\beta}_{\lambda}) = -\theta^{-1}I^{\beta}_{\lambda}.
\end{aligned}$$

Next, we compute

$$g_{\lambda\sigma}\Phi^{\sigma}_{\alpha\beta} = \theta^{-1}[(2H\xi^2-\xi)g_{\lambda\nu}-\xi^2 g_{\lambda\sigma}b^{\sigma}_{\nu}]\overset{*}{\nabla}_{\beta}b^{\nu}_{\alpha}.$$

Since

$$g_{\lambda\sigma}b^{\sigma}_{\nu} = b^{\sigma}_{\nu}(a_{\lambda\sigma}-2\xi b_{\lambda\sigma}+\xi^2 c_{\lambda\sigma}) = b_{\lambda\sigma}-2\xi c_{\lambda\sigma}+\xi^2 b^{\sigma}_{\nu}c_{\lambda\sigma},$$

this follows from the first part of (2.12) that

$$b^{\sigma}_{\nu}c_{\lambda\sigma} = -2HKa_{\lambda\nu}+(4H^2-K)b_{\lambda\nu},$$

we obtain

$$\begin{aligned}
&g_{\lambda\sigma}b^{\sigma}_{\nu} = -2HK\xi^2 a_{\lambda\nu}+(1+(4H^2-K)\xi^2)b_{\lambda\nu}-2\xi c_{\lambda\nu},\\
&(2H\xi^2-\xi)g_{\lambda\nu}-\xi^2 g_{\lambda\sigma}b^{\sigma}_{\nu}\\
&\quad = (-\xi+2H\xi^2+2HK\xi^4)a_{\lambda\nu}+(\xi^2-2H\xi^3+(K-4H^2)\xi^4)b_{\lambda\nu}+(2H\xi^4+\xi^3)c_{\lambda\nu}\\
&\quad = \theta(\xi^2 b_{\lambda\nu}-\xi a_{\lambda\nu}+(1+2H\xi)\xi^3(Ka_{\lambda\nu}-2Hb_{\lambda\nu}+c_{\lambda\nu}) = \theta(\xi^2 b_{\lambda\nu}-\xi a_{\lambda\nu}).
\end{aligned}$$

Combining above results we get the forth part of (2.41).

Taking the first and second part of (2.41) into account,

$$g_{\lambda\sigma}I^\lambda_\alpha I^\sigma_\beta=(-\theta J_{\alpha\sigma})I^\sigma_\beta=-(\theta)(-\theta)c_{\alpha\beta}=\theta^2 c_{\alpha\beta}.$$

It yields seventh of (2.41)

$$g_{\lambda\sigma}I^\lambda_\alpha I^\sigma_\beta=\theta^2 c_{\alpha\beta}.$$

Next we prove the fifth part of (2.41). Note that

$$\begin{aligned}J_{\lambda\sigma}\Phi^\sigma_{\alpha\beta}(\xi)&=\theta^{-1}\{(\xi(2H\xi-1)J_{\lambda\sigma}\delta^\sigma_\nu-\xi^2 J_{\lambda\sigma}b^\sigma_\nu\}\overset{*}{\nabla}_\beta b^\nu_\alpha\\&=\theta^{-1}\{(\xi(2H\xi-1)(b_{\lambda\nu}-\xi c_{\lambda\nu})-\xi^2(b^\sigma_\nu b_{\lambda\sigma}-\xi b^\sigma_\nu c_{\lambda\nu})\}\overset{*}{\nabla}_\beta b^\nu_\alpha.\end{aligned}$$

By (2.12)

$$b^\sigma_\nu c_{\lambda\nu}=-2HKa_{\lambda\nu}+(4H^2-k)b_{\lambda\nu},$$

and $\theta=1-2H\xi+K\xi^2$ and (2.9)

$$Ka_{\lambda\nu}-2Hb_{\lambda\nu}+c_{\lambda\nu}=0,$$

simple calculation shows that

$$J_{\lambda\sigma}\Phi^\sigma_{\alpha\beta}(\xi)=\theta^{-1}\{-\xi\theta b_{\lambda\nu}-2H\xi^3(Ka_{\lambda\nu}-2Hb_{\lambda\nu}+c_{\lambda\nu})\}\overset{*}{\nabla}_\beta b^\nu_\alpha=-\xi b_{\lambda\nu}\overset{*}{\nabla}_\beta b^\nu_\alpha.$$

This is the fifth part of (2.41).

In order to prove the eighth part of (2.41), by $g_{\lambda\sigma}I^\lambda_\nu=-\theta J_{\nu\sigma}$, we obtain

$$g_{\lambda\sigma}I^\lambda_\nu\Phi^\sigma_{\alpha\beta}=-\theta J_{\sigma\nu}\Phi^\sigma_{\alpha\beta}(\text{using fifth parts of (2.41)})=\theta\xi b_{\nu\mu}\overset{*}{\nabla}_\alpha b^\mu_\beta.$$

Next, we prove sixth of (2.41). Indeed, from fourth of (2.41) it gives that

$$\begin{aligned}g_{\alpha\beta}\Phi^\alpha_{\sigma\lambda}\Phi^\beta_{\nu\mu}=&(-\xi a_{\beta\gamma}+\xi^2 b_{\beta\gamma})\overset{*}{\nabla}_\lambda b^\gamma_\sigma\theta^{-1}(-\xi\delta^\beta_\eta+(2H\delta^\beta_\eta-b^\beta_\eta)\xi^2)\overset{*}{\nabla}_\nu b^\eta_\mu\\=&\theta^{-1}\{\xi^2 a_{\gamma\eta}-\xi^3(2Ha_{\gamma\eta}-b_{\gamma\eta})-\xi^3 b_{\gamma\eta}+\xi^4(2Hb_{\gamma\eta}-c_{\gamma\eta})\}\overset{*}{\nabla}_\lambda b^\gamma_\sigma\overset{*}{\nabla}_\nu b^\eta_\mu\\=&\theta^{-1}\xi^2((1-2H\xi)a_{\gamma\eta}+2Hb_{\gamma\eta}-c_{\gamma\eta}\xi^2)\overset{*}{\nabla}_\lambda b^\gamma_\sigma\overset{*}{\nabla}_\nu b^\eta_\mu\\&\quad(\text{owing to }-c_{\gamma\eta}=Ka_{\gamma\eta}-2Hb_{\gamma\eta})\\=&\theta^{-1}a_{\gamma\eta}(1-2H\xi+K\xi^2)\overset{*}{\nabla}_\lambda b^\gamma_\sigma\overset{*}{\nabla}_\nu b^\eta_\mu=\xi^2 a_{\gamma\eta}\overset{*}{\nabla}_\lambda b^\gamma_\sigma\overset{*}{\nabla}_\nu b^\eta_\mu.\end{aligned}$$

This is sixth of (2.41).

Finally, we have to prove the night part of (2.41). From (2.18), (2.28) and Godazzi formula $\overset{*}{\nabla}_\alpha b_{\beta\lambda}=\overset{*}{\nabla}_\lambda b_{\alpha\beta}$, it leads to

$$\begin{aligned}&g^{\alpha\beta}=\theta^{-2}(p(\xi)a^{\alpha\beta}+q(\xi)b^{\alpha\beta}),\\&\Phi^\lambda_{\alpha\beta}=\theta^{-1}\{-\xi a^{\lambda\sigma}\overset{*}{\nabla}_\alpha b_{\beta\sigma}+\xi^2(2H\delta^\lambda_\sigma-b^\lambda_\sigma)a^{\sigma\mu}\overset{*}{\nabla}_\alpha b_{\mu\beta}\}\\&\quad=(Godazzi)=\theta^{-1}\{-\xi a^{\lambda\sigma}\overset{*}{\nabla}_\sigma b_{\alpha\beta}+\xi^2(2H\delta^\lambda_\sigma-b^\lambda_\sigma)a^{\sigma\mu}\overset{*}{\nabla}_\mu b_{\alpha\beta}\}\\&\quad=\theta^{-1}((2H\xi^2-\xi)a^{\lambda\mu}-\xi^2 b^{\lambda\mu})\overset{*}{\nabla}_\mu b_{\alpha\beta}.\end{aligned}$$

Therefore,

$$g^{\beta\sigma}\Phi^{\lambda}_{\beta\sigma}=\theta^{-3}\{((2H\xi^2-\xi)a^{\lambda\mu}-\xi^2 b^{\lambda\mu})(p(\xi)a^{\beta\sigma}+q(\xi)b^{\beta\sigma})\overset{*}{\nabla}_\mu b_{\beta\sigma}\}.$$

Since covariant derivative of metric tensor is vanished, from Lemma 2.1, with $b^{\beta\sigma}b_{\beta\sigma}=c^{\beta}_{\beta}=4H^2-2K$, it is not difficult to obtain

$$a^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=2\overset{*}{\nabla}_\mu(H),\quad b^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=\overset{*}{\nabla}_\mu(2H^2-K).$$

Indeed,

$$a^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=\overset{*}{\nabla}_\mu(a^{\beta\sigma}b_{\beta\sigma})=\overset{*}{\nabla}_\mu(2H),\quad b^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=\overset{*}{\nabla}_\mu(b^{\beta\sigma}b_{\beta\sigma})-b_{\beta\sigma}\overset{*}{\nabla}_\mu b^{\beta\sigma},$$

but

$$b^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=b_{\beta\sigma}\overset{*}{\nabla}_\mu b^{\beta\sigma},$$

hence

$$b^{\beta\sigma}\overset{*}{\nabla}_\mu b_{\beta\sigma}=\frac{1}{2}\overset{*}{\nabla}_\mu(b^{\beta\sigma}b_{\beta\sigma}=\overset{*}{\nabla}_\mu(2H^2-K).$$

Finally we obtain

$$\begin{aligned}g^{\beta\sigma}\Phi^{\lambda}_{\beta\sigma}&=\theta^{-3}\{((2H\xi^2-\xi)a^{\lambda\mu}-\xi^2 b^{\lambda\mu})(2p(\xi)\overset{*}{\nabla}_\mu H+q(\xi)\overset{*}{\nabla}_\mu(2H^2-K))\}\\&=\theta^{-3}\{((2H\xi^2-\xi)a^{\lambda\mu}-\xi^2 b^{\lambda\mu})(2p(\xi)+4Hq(\xi)\overset{*}{\nabla}_\mu H-q(\xi)\overset{*}{\nabla}_\mu(K))\}.\end{aligned}$$

The proof is completed. □

In what follows, we consider strain tensor $e_{ij}(u)$ in E^3 associated with displacement vector u

$$e_{ij}(u)=\frac{1}{2}(\nabla_i u_j+\nabla_j u_i)=\frac{1}{2}(g_{jk}\nabla_i u^k+g_{ik}\nabla_j u^k).\tag{2.42}$$

We use derivative of metric tensor being vanished

$$\nabla_i g_{jk}=0,\quad \overset{*}{\nabla}_\alpha a_{\beta\sigma}=0,\tag{2.43}$$

which will be frequently used throughout this paper.

The contravariant component of vector u: $u^k=g^{kj}u_j$ instead of covariant component of vector. The strain tensor on $\Im$ associated with displacement vector u:

$$\overset{*}{e}_{\alpha\beta}(u)=\frac{1}{2}(\overset{*}{\nabla}_\alpha u_\beta+\overset{*}{\nabla}_\beta u_\alpha)=\frac{1}{2}(a_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\mu+a_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda).\tag{2.44}$$

The contravariant components of strain tensor are defined by

$$e^{ij}(u)=g^{ik}g^{jm}e_{km}(u),\quad \overset{*}{e^{\alpha\beta}}(u)=a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\lambda\sigma}(u).$$

We also consider Green St.Vennant strain tensor $E_{ij}(u)$ of the displacement u for non-linear elastic case

$$E_{ij}(u)=e_{ij}(u)+D_{ij}(u),\quad D_{ij}(u)=\tfrac{1}{2}g_{kl}\nabla_i u^k\nabla_j u^l,\quad E^{ij}(u)=g^{ik}g^{jl}E_{kl}(u).\tag{2.45}$$

Lemma 2.6. *Under S-coordinate system the strain tensor and Green St.Vennant strain tensor are the polynomials of degree two with respect to transverse variable* ξ

$$\begin{cases} e_{ij}(u)=\gamma_{ij}(u)+\overset{1}{\gamma}_{ij}(u)\xi+\overset{2}{\gamma}_{ij}(u)\xi^2, \\ E_{ij}(u)=e_{ij}(u)+D_{ij}(u)=\sum_{k=0}^{2}\overset{k}{E}_{ij}(u)\xi^k, \end{cases} \tag{2.46}$$

where

$$\begin{aligned}
&\gamma_{\alpha\beta}(u)=\overset{*}{e}_{\alpha\beta}(u)-b_{\alpha\beta}u^3=\frac{1}{2}[a_{\beta\lambda}\overset{0}{\nabla}_\alpha u^\lambda+a_{\alpha\lambda}\overset{0}{\nabla}_\beta u^\lambda],\\
&\overset{1}{\gamma}_{\alpha\beta}(u)=-(b_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+b_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)+c_{\alpha\beta}u^3-\overset{*}{\nabla}_\lambda b_{\alpha\beta}u^\lambda\\
&\qquad\quad=-[b_{\beta\lambda}\overset{0}{\nabla}_\alpha u^\lambda+b_{\alpha\lambda}\overset{0}{\nabla}_\beta u^\lambda]-c_{\alpha\beta}u^3-\overset{*}{\nabla}_\lambda b_{\alpha\beta}u^\lambda,\\
&\overset{2}{\gamma}_{\alpha\beta}(u)=\frac{1}{2}(c_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+c_{\beta\lambda}\overset{*}{\nabla}_\alpha u_\lambda+\overset{*}{\nabla}_\lambda c_{\alpha\beta}u^\lambda)=\frac{1}{2}[b_{\beta\lambda}\overset{*}{\nabla}_\alpha(b^\lambda_\sigma u^\sigma)+b_{\alpha\lambda}\overset{*}{\nabla}_\beta(b^\lambda_\sigma u^\sigma)],\\
&\gamma_{\alpha3}(u)=\frac{1}{2}(a_{\alpha\beta}\frac{\partial u^\beta}{\partial\xi}+\overset{*}{\nabla}_\alpha u^3),\quad \overset{1}{\gamma}_{\alpha3}(u)=-b_{\alpha\beta}\frac{\partial u^\beta}{\partial\xi},\quad \overset{2}{\gamma}_{\alpha3}(u)=\frac{1}{2}c_{\alpha\beta}\frac{\partial u^\beta}{\partial\xi},\\
&\gamma_{33}(u)=\frac{\partial u^3}{\partial\xi},\quad \overset{1}{\gamma}_{33}(u)=\overset{2}{\gamma}_{33}(u)=0,\quad \gamma^{\alpha\beta}(u)=a^{\alpha\nu}a^{\beta\sigma}\gamma_{\nu\sigma}(u),
\end{aligned} \tag{2.47}$$

$$\begin{aligned}
&\overset{*}{E}_{ij}(u,u):=\overset{0}{E}_{ij}(u,u)=\gamma_{ij}(u)+\varphi_{ij}(u,u),\\
&\overset{1}{E}_{ij}(u,u)=\overset{1}{\gamma}_{ij}(u)+\varphi^1_{ij}(u,u),\quad \overset{2}{E}_{ij}(u,u)=\overset{2}{\gamma}_{ij}(u)+\varphi^2_{ij}(u,u),\\
&D_{ij}(u,v):=\varphi_{ij}(u,v)+\varphi^1_{ij}(u,v)\xi+\varphi^2_{ij}(u,v)\xi^2,
\end{aligned} \tag{2.48}$$

where the strain tensors on the two-dimensional manifold S are given as:

$$\begin{cases} \overset{*}{e}_{\alpha\beta}(u)=\frac{1}{2}(a_{\alpha\lambda}\delta^\sigma_\beta+a_{\beta\lambda}\delta^\sigma_\alpha)\overset{*}{\nabla}_\sigma u^\lambda,\\ \overset{1}{e}_{\alpha\beta}(u)=-(b_{\alpha\lambda}\delta^\sigma_\beta+b_{\beta\lambda}\delta^\sigma_\alpha)\overset{*}{\nabla}_\sigma u^\lambda,\quad \overset{2}{e}_{\alpha\beta}(u)=\frac{1}{2}(c_{\alpha\sigma}\delta^\lambda_\beta+c_{\beta\sigma}\delta^\lambda_\sigma)\overset{*}{\nabla}_\lambda u^\sigma, \end{cases} \tag{2.49}$$

$$\begin{cases} \varphi_{\alpha\beta}(u)=\frac{1}{2}a_{ij}\overset{0}{\nabla}_\alpha u^i\overset{0}{\nabla}_\beta u^j,\\ \varphi^1_{\alpha\beta}(u)=-\frac{1}{2}[\overset{*}{\nabla}_\alpha(b_{\lambda\sigma}u^\lambda)\overset{0}{\nabla}_\beta u^\sigma+\overset{0}{\nabla}_\alpha u^\lambda\overset{*}{\nabla}_\beta(b_{\lambda\sigma}u^\sigma)+(c_{\alpha\nu}\overset{0}{\nabla}_\beta u^3+c_{\beta\nu}\overset{0}{\nabla}_\alpha u^3)u^\nu],\\ \varphi^2_{\alpha\beta}(u)=\frac{1}{2}[\overset{*}{\nabla}_\alpha(b_{\lambda\sigma}u^\lambda)\overset{*}{\nabla}_\beta(b^\sigma_\nu u^\nu)+c_{\alpha\lambda}c_{\beta\sigma}u^\lambda u^\sigma], \end{cases} \tag{2.50}$$

$$\begin{cases} \varphi^0_{3\alpha}(u)=\frac{1}{2}[(a_{\lambda\sigma}\frac{\partial u^\sigma}{\partial\xi}-b_{\lambda\sigma})\overset{0}{\nabla}_\alpha u^\lambda+\overset{0}{\nabla}_\alpha u^3\frac{\partial u^3}{\partial\xi}],\\ \varphi^1_{3\alpha}(u)=-\frac{1}{2}[b_{\lambda\sigma}\overset{0}{\nabla}_\alpha u^\lambda+\overset{*}{\nabla}_\alpha b_{\sigma\lambda}u^\lambda]\frac{\partial u^\sigma}{\partial\xi}+\frac{1}{2}c_{\lambda\sigma}u^\sigma(-\delta^\lambda_\alpha\frac{\partial u^3}{\partial\xi}+\overset{*}{\nabla}_\alpha u^\lambda),\\ \varphi^2_{3\alpha}(u)=\frac{1}{2}b_{\lambda\gamma}\overset{*}{\nabla}_\alpha b^\gamma_\sigma u^\lambda\frac{\partial u^\sigma}{\partial\xi}, \end{cases} \tag{2.51}$$

$$\begin{cases} \varphi_{33}(u)=\frac{1}{2}[(a_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}-2b_{\alpha\beta}u^\alpha)\frac{\partial u^\beta}{\partial\xi}+c_{\alpha\beta}u^\alpha u^\beta+\frac{\partial u^3}{\partial\xi}\frac{\partial u^3}{\partial\xi}],\\ \varphi^1_{33}(u)=[b_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}+c_{\alpha\beta}u^\alpha]\frac{\partial u^\beta}{\partial\xi}],\quad \varphi^2_{33}(u)=c_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi}, \end{cases} \tag{2.52}$$

where

$$a_{ij}=g_{ij}|_{\xi=0}=(a_{\alpha\beta},a_{3\alpha}=a_{\alpha 3}=0,a_{33}=1). \tag{2.53}$$

Remark 2.1. Since displacement vector $u^3=0$ and $\frac{\partial u}{\partial \xi}=0$ at tangent space of 2D manifold $\Im$, the strain tensor is given by $\overset{*}{e}_{\alpha\beta}(u)$ while displacement vector are in 3D space, the strain tensor of displacement vector restrict on the $\Im$ will be given by $\gamma_{\alpha\beta}(u)$.

Proof.

$$\begin{aligned}e_{\alpha\beta}(u)&=\tfrac{1}{2}(g_{\alpha\lambda}\nabla_\beta u^\lambda+g_{\beta\lambda}\nabla_\alpha u^\lambda)\\&=\tfrac{1}{2}[g_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+g_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda+\theta^{-1}(g_{\alpha\lambda}I^\lambda_\beta+g_{\beta\lambda}I^\lambda_\alpha)u^3+(g_{\alpha\lambda}\Phi^\lambda_{\beta\nu}+g_{\beta\lambda}\Phi^\lambda_{\alpha\nu})u^\nu]\\&=\tfrac{1}{2}(g_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+g_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)+\tfrac{1}{2}(-2J_{\alpha\beta}u^3)\\&\quad+\tfrac{1}{2}(-(\overset{*}{\nabla}_\alpha b_{\beta\nu}+\overset{*}{\nabla}_\beta b_{\alpha\nu})\xi+(b_{\alpha\mu}\overset{*}{\nabla}_\beta b^\mu_\nu+b_{\beta\mu}\overset{*}{\nabla}_\alpha b^\mu_\nu)u^\nu\xi^2).\end{aligned}$$

Since of Godazzi formula and vanishing of covariant derivatives of metric tensor, we get

$$b_{\alpha\mu}\overset{*}{\nabla}_\beta b^\mu_\nu+b_{\beta\mu}\overset{*}{\nabla}_\alpha b^\mu_\nu=b_{\alpha\mu}\overset{*}{\nabla}_\nu b^\mu_\beta+b_{\beta\mu}\overset{*}{\nabla}_\nu b^\mu_\alpha=\overset{*}{\nabla}_\nu(b^\mu_\beta b_{\alpha\mu})=\overset{*}{\nabla}_\nu c_{\alpha\beta},$$

$$\begin{aligned}e_{\alpha\beta}(u)&=\tfrac{1}{2}(g_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+g_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)-J_{\alpha\beta}u^3+(-\overset{*}{\nabla}_\nu b_{\alpha\beta}\xi+\xi^2\overset{*}{\nabla}_\nu c_{\alpha\beta})u^\nu\\&=\tfrac{1}{2}(a_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+a_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)-b_{\alpha\beta}u^3+\xi[-(b_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+b_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)+c_{\alpha\beta}u^3-\overset{*}{\nabla}_\nu b_{\alpha\beta}u^\nu]\\&\quad+\xi^2\tfrac{1}{2}[(c_{\alpha\lambda}\overset{*}{\nabla}_\beta u^\lambda+c_{\beta\lambda}\overset{*}{\nabla}_\alpha u^\lambda)+\overset{*}{\nabla}_\nu c_{\alpha\beta}u^\nu]=\gamma_{\alpha\beta}(u)+\overset{1}{\gamma}_{\alpha\beta}(u)\xi+\overset{2}{\gamma}_{\alpha\beta}(u)\xi^2.\end{aligned}$$

Similarly, using $J_{\alpha\lambda}+\theta^{-1}g_{\alpha\beta}I^\beta_\lambda=0$, we have

$$\begin{aligned}e_{3\alpha}(u)&=\tfrac{1}{2}(g_{\alpha\beta}\nabla_3 u^\beta+\nabla_\alpha u^3)\\&=\tfrac{1}{2}(g_{\alpha\beta}\tfrac{\partial u^\beta}{\partial\xi}+\theta^{-1}g_{\alpha\beta}I^\beta_\lambda u^\lambda+\overset{*}{\nabla}_\alpha u^3+J_{\alpha\lambda}u^\lambda)=\tfrac{1}{2}(g_{\alpha\beta}\tfrac{\partial u^\beta}{\partial\xi}+\overset{*}{\nabla}_\alpha u^3),\\e_{33}(u)&=\nabla_3 u^3=\tfrac{\partial u^3}{\partial\xi}.\end{aligned}$$

Next, we prove the second part of (2.46). With (2.41) and Godazzi formula, we have

$$\Psi_{\alpha\beta\lambda}(\xi)=g_{\lambda\sigma}\Phi^\sigma_{\alpha\beta}(\xi)=(-\xi\delta^\sigma_\beta+\xi^2 b^\sigma_\beta)\overset{*}{\nabla}_\alpha b_{\sigma\lambda}. \tag{2.54}$$

It infers that

$$\begin{aligned}D_{\alpha\beta}(u)&=\tfrac{1}{2}(g_{\lambda\sigma}\nabla_\alpha u^\lambda\nabla_\beta u^\sigma+\nabla_\alpha u^3\nabla_\beta u^3)\\&=\tfrac{1}{2}[g_{\lambda\sigma}(\overset{*}{\nabla}_\alpha u^\lambda+\theta^{-1}I^\lambda_\alpha u^3+\Phi^\lambda_{\alpha\nu}u^\nu)(\overset{*}{\nabla}_\beta u^\sigma+\theta^{-1}I^\sigma_\beta u^3+\Phi^\sigma_{\beta\nu}u^\nu)\\&\qquad+(\overset{*}{\nabla}_\alpha u^3+J_{\alpha\nu}u^\nu)(\overset{*}{\nabla}_\beta u^3+J_{\beta\nu}u^\nu)]\\&=\tfrac{1}{2}\{g_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda\overset{*}{\nabla}_\beta u^\sigma-(J_{\alpha\sigma}\overset{*}{\nabla}_\beta u^\sigma+J_{\beta\sigma}\overset{*}{\nabla}_\alpha u^\sigma)u^3+(\Psi_{\beta\nu\lambda}\overset{*}{\nabla}_\alpha u^\lambda\\&\qquad+\Psi_{\alpha\nu\sigma}\overset{*}{\nabla}_\beta u^\sigma)u^\nu+\Psi_{\alpha\nu\sigma}\Phi^\sigma_{\beta\mu}u^\nu u^\mu+\overset{*}{\nabla}_\alpha u^3\overset{*}{\nabla}_\beta u^3\\&\qquad+(J_{\alpha\nu}\delta^\sigma_\beta+J_{\beta\nu}\delta^\sigma_\alpha)\overset{*}{\nabla}_\sigma u^3 u^\nu+J_{\alpha\nu}J_{\beta\mu}u^\nu u^\mu\}=\varphi_{\alpha\beta}(u)+\xi\varphi^1_{\alpha\beta}(u)+\xi^2\varphi^2_{\alpha\beta}(u).\end{aligned}$$

In what follows, we prove

$$\Psi_{\alpha\nu\sigma}(\xi)\Phi^{\sigma}_{\beta\mu}(\xi)=\xi^2\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\lambda}_{\mu}.$$
$$(\Psi_{\beta\nu\lambda}\overset{*}{\nabla}_\alpha u^\lambda+\Psi_{\alpha\nu\sigma}\overset{*}{\nabla}_\beta u^\sigma)u^\nu=(-\xi(\delta^{\sigma}_{\beta}\delta^{\eta}_{\alpha}+\delta^{\sigma}_{\alpha}\delta^{\eta}_{\beta})+\xi^2(b^{\sigma}_{\beta}\delta^{\eta}_{\alpha}+b^{\sigma}_{\alpha}\delta^{\eta}_{\beta}))\overset{*}{\nabla}_\sigma b_{\lambda\nu}\overset{*}{\nabla}_\eta u^\lambda u^\nu.$$

Indeed, by (2.28) we derive

$$\begin{aligned}\Psi_{\alpha\nu\sigma}(\xi)\Phi^{\sigma}_{\beta\mu}(\xi)&=\theta^{-1}(-\xi\delta^{\lambda}_{\sigma}+\xi^2 b^{\lambda}_{\sigma})((2H\xi^2-\xi)\delta^{\sigma}_{\eta}-\xi^2 b^{\sigma}_{\eta})\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\eta}_{\mu}\\&=\theta^{-1}[\xi(\xi-2H\xi^2)\delta^{\lambda}_{\eta}+\xi^3 b^{\lambda}_{\eta}+\xi^2(2H\xi^2-\xi)b^{\lambda}_{\eta}-\xi^4 c^{\lambda}_{\eta}]\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\eta}_{\mu},\end{aligned}$$
$$\xi^3 b^{\lambda}_{\eta}+\xi^2(2H\xi^2-\xi)b^{\lambda}_{\eta}-\xi^4 c^{\lambda}_{\eta}=\xi^4(2Hb^{\lambda}_{\eta}-c^{\lambda}_{\eta})=\xi^4 K\delta^{\lambda}_{\eta}(\text{see}(2.9)),$$
$$\Psi_{\alpha\nu\sigma}(\xi)\Phi^{\sigma}_{\beta\mu}(\xi)=\theta^{-1}(\xi^2(1-2H\xi+K\xi^2))\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\eta}_{\mu}=\xi^2\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\eta}_{\mu}.$$

Taking into account of

$$(\Psi_{\beta\nu\lambda}\overset{*}{\nabla}_\alpha u^\lambda+\Psi_{\alpha\nu\sigma}\overset{*}{\nabla}_\beta u^\sigma)u^\nu=(\Psi_{\beta\nu\lambda}\delta^{\eta}_{\alpha}+\Psi_{\alpha\nu\lambda}\delta^{\eta}_{\beta})u^\nu\overset{*}{\nabla}_\eta u^\lambda,$$

and (2.54), we immediately obtain our conclusion. By above discussion, it infers that

$$\begin{aligned}D_{\alpha\beta}(u)=\tfrac12\{&g_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda\overset{*}{\nabla}_\beta u^\sigma+(-\xi(\delta^{\sigma}_{\beta}\delta^{\eta}_{\alpha}+\delta^{\sigma}_{\alpha}\delta^{\eta}_{\beta})+\xi^2(b^{\sigma}_{\beta}\delta^{\eta}_{\alpha}+b^{\sigma}_{\alpha}\delta^{\eta}_{\beta}))\overset{*}{\nabla}_\sigma b_{\lambda\nu}\overset{*}{\nabla}_\eta u^\lambda u^\nu\\&+(\xi^2\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\lambda}_{\mu}+J_{\alpha\nu}J_{\beta\mu})u^\nu u^\mu+\overset{*}{\nabla}_\alpha u^3\overset{*}{\nabla}_\beta u^3\\&+(J_{\alpha\nu}\delta^{\sigma}_{\beta}+J_{\beta\nu}\delta^{\sigma}_{\alpha})(\overset{*}{\nabla}_\sigma u^3 u^\nu-\overset{*}{\nabla}_\sigma u^\nu u^3)\}=\varphi_{\alpha\beta}(u)+\varphi^1_{\alpha\beta}(u)\xi+\varphi^2_{\alpha\beta}(u)\xi^2,\end{aligned}$$

where

$$\varphi_{\alpha\beta}(u)=\tfrac12 a_{ij}\overset{*}{\nabla}_\alpha u^i\overset{*}{\nabla}_\beta u^j+\varphi^0_{\alpha\beta\lambda\sigma}u^\lambda u^\sigma+\varphi^{0\sigma}_{\alpha\beta3\lambda}(\overset{*}{\nabla}_\sigma u^3 u^\lambda-\overset{*}{\nabla}_\sigma u^\lambda u^3),$$
$$\varphi^1_{\alpha\beta}(u)=-b_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda\overset{*}{\nabla}_\beta u^\sigma+\varphi^1_{\alpha\beta\lambda\sigma}u^\lambda u^\sigma+\varphi^{1\eta}_{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\eta u^\lambda u^\sigma+\varphi^{1\sigma}_{\alpha\beta3\lambda}(\overset{*}{\nabla}_\sigma u^3 u^\lambda-\overset{*}{\nabla}_\sigma u^\lambda u^3),$$
$$\varphi^2_{\alpha\beta}(u)=c_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda\overset{*}{\nabla}_\beta u^\sigma+\varphi^2_{\alpha\beta\lambda\sigma}u^\lambda u^\sigma+\varphi^{2\eta}_{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\eta u^\lambda u^\sigma,$$

with

$$\varphi^0_{\alpha\beta\lambda\sigma}=\tfrac12 b_{\alpha\lambda}b_{\beta\sigma},\quad \varphi^{0\sigma}_{\alpha\beta3\lambda}=\tfrac12(b_{\alpha\lambda}\delta^{\sigma}_{\beta}+b_{\beta\lambda}\delta^{\sigma}_{\alpha}),\quad \varphi^1_{\alpha\beta\lambda\sigma}=-\tfrac12(c_{\alpha\lambda}b_{\beta\sigma}+c_{\alpha\lambda}b_{\alpha\sigma}),$$
$$\varphi^{1\eta}_{\alpha\beta\lambda\sigma}=-\tfrac12(\delta^{\nu}_{\beta}\delta^{\eta}_{\alpha}+\delta^{\nu}_{\alpha}\delta^{\eta}_{\beta})\overset{*}{\nabla}_\sigma b_{\lambda\nu},\quad \varphi^{1\sigma}_{\alpha\beta3\lambda}=-\tfrac12(c_{\alpha\lambda}\delta^{\sigma}_{\beta}+c_{\beta\lambda}\delta^{\sigma}_{\alpha}),$$
$$\varphi^2_{\alpha\beta\lambda\sigma}=\tfrac12(c_{\alpha\lambda}c_{\beta\sigma}+\overset{*}{\nabla}_\alpha b_{\nu\lambda}\overset{*}{\nabla}_\beta b^{\nu}_{\sigma}),\quad \varphi^{2\eta}_{\alpha\beta\lambda\sigma}=\tfrac12(b^{\nu}_{\beta}\delta^{\eta}_{\alpha}+b^{\nu}_{\alpha}\delta^{\eta}_{\beta})\overset{*}{\nabla}_\nu b_{\lambda\sigma}.$$

Applying (2.36) immediately derive formula (2.50).

By similar manner, the Calculations show that

$$\begin{aligned}D_{3\alpha}(u)=D_{\alpha3}(u)&=\tfrac12 g_{km}\nabla_\alpha u^k\nabla_3 u^m=\tfrac12 g_{\lambda\sigma}\nabla_\alpha u^\lambda\nabla_3 u^\sigma+\tfrac12\nabla_\alpha u^3\nabla_3 u^3\\&=\tfrac12 g_{\lambda\sigma}(\overset{*}{\nabla}_\alpha u^\lambda+(\theta^{-1}I^{\lambda}_{\alpha}u^3+\Phi^{\lambda}_{\alpha\nu}u^\nu))(\tfrac{\partial u^\sigma}{\partial\xi}+\theta^{-1}I^{\sigma}_{\mu}u^\mu)\\&\quad+\tfrac12\tfrac{\partial u^3}{\partial\xi}(\overset{*}{\nabla}_\alpha u^3+J_{\alpha\sigma}u^\sigma)=\tfrac12[g_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda-J_{\alpha\sigma}u^3+\Psi_{\sigma\alpha\mu}u^\mu]\tfrac{\partial u^\sigma}{\partial\xi}\\&\quad-\tfrac12 J_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda u^\sigma+\tfrac12 c_{\alpha\sigma}u^3u^\sigma+\tfrac12\xi b_{\mu\gamma}\overset{*}{\nabla}_\alpha b^{\gamma}_{\nu}u^\nu u^\mu\\&\quad+\tfrac12\tfrac{\partial u^3}{\partial\xi}(\overset{*}{\nabla}_\alpha u^3+J_{\alpha\sigma}u^\sigma)=\varphi^0_{3\beta}(\mathbf{u})+\varphi^1_{3\alpha}(\mathbf{u})\xi+\varphi^2_{3\alpha}(\mathbf{u})\xi^2,\end{aligned}$$

where

$$\begin{aligned}
&\varphi^0_{3\alpha}(u)=\frac{1}{2}[a_{\lambda\sigma}\overset{0}{\nabla}_\alpha u^\lambda \frac{\partial u^\sigma}{\partial\xi}+\overset{0}{\nabla}_\alpha u^3 u^\sigma)\frac{\partial u^3}{\partial\xi}-b_{\lambda\sigma}\overset{0}{\nabla}_\alpha u^\lambda u^\sigma],\\
&\varphi^1_{3\alpha}(u)=-\frac{1}{2}[b_{\lambda\sigma}\overset{0}{\nabla}_\alpha u^\lambda+\overset{*}{\nabla}_\alpha b_{\sigma\lambda}u^\lambda]\frac{\partial u^\sigma}{\partial\xi}+\frac{1}{2}c_{\lambda\sigma}u^\sigma(\overset{*}{\nabla}_\alpha u^\lambda-\delta^\lambda_\alpha\frac{\partial u^3}{\partial\xi}),\\
&\varphi^2_{3\alpha}(u)=\frac{1}{2}(c_{\lambda\sigma}\overset{*}{\nabla}_\alpha u^\lambda+b_{\gamma\lambda}\overset{*}{\nabla}_\alpha b^\gamma_\sigma u^\lambda)\frac{\partial u^\sigma}{\partial\xi},\\
&D_{33}(u)=\frac{1}{2}g_{km}\nabla_3 u^k\nabla_3 u^m=\frac{1}{2}[g_{\alpha\beta}(\frac{\partial u^\alpha}{\partial\xi}+\theta^{-1}I^\alpha_\lambda u^\lambda)(\frac{\partial u^\beta}{\partial\xi}+\theta^{-1}I^\beta_\sigma u^\sigma)+\frac{\partial u^3}{\partial\xi}\frac{\partial u^3}{\partial\xi}].
\end{aligned}$$

Furthermore, from (2.4) and (2.34) we have

$$\begin{aligned}
D_{33}(u)&=\frac{1}{2}[g_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi}-2J_{\alpha\beta}u^\alpha\frac{\partial u^\beta}{\partial\xi}+c_{\alpha\beta}u^\alpha u^\beta+\frac{\partial u^3}{\partial\xi}\frac{\partial u^3}{\partial\xi}]\\
&=\varphi_{33}(u)+\xi\varphi^1_{33}(u)+\xi^2\varphi^2_{33}(u),
\end{aligned}$$

where

$$\begin{aligned}
&\varphi_{33}(u)=\frac{1}{2}[a_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi}-2b_{\alpha\beta}u^\alpha\frac{\partial u^\beta}{\partial\xi}+c_{\alpha\beta}u^\alpha u^\beta+\frac{\partial u^3}{\partial\xi}\frac{\partial u^3}{\partial\xi}],\\
&\varphi^1_{33}(u)=-[b_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi}+c_{\alpha\beta}u^\alpha\frac{\partial u^\beta}{\partial\xi}],\quad \varphi^2_{33}(u)=c_{\alpha\beta}\frac{\partial u^\alpha}{\partial\xi}\frac{\partial u^\beta}{\partial\xi}.
\end{aligned}$$

Our proof is completed. □

We assume that the elastic material constituting the shell are isotropic and homogenous. The contravariant components of elasticity tensor are given by

$$A^{ijkl}=\lambda g^{ij}g^{kl}+\mu(g^{ik}g^{jl}+g^{il}g^{jk}), \tag{2.55}$$

where $(\lambda\geq 0,\mu>0)$ are the elastic coefficient constants. Let

$$\begin{aligned}
&a^{\alpha\beta\sigma\tau}=\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}),\\
&b^{\alpha\beta\sigma\tau}=\lambda b^{\alpha\beta}b^{\sigma\tau}+\mu(b^{\alpha\sigma}b^{\beta\tau}+b^{\alpha\tau}b^{\beta\sigma}),\\
&c^{\alpha\beta\sigma\tau}=\lambda(a^{\alpha\beta}b^{\sigma\tau}+a^{\sigma\tau}b^{\alpha\beta})+\mu(a^{\alpha\sigma}b^{\beta\tau}+a^{\beta\tau}b^{\alpha\sigma}+a^{\alpha\tau}b^{\beta\sigma}+a^{\beta\sigma}b^{\alpha\tau}).
\end{aligned}\tag{2.56}$$

Lemma 2.7. *The elasticity tensor are a rational polynomials with respect to ξ in the S-coordinate system:*

$$\begin{cases}
A^{\alpha\beta\sigma\tau}=\theta^{-4}[a^{\alpha\beta\sigma\tau}+\sum_{k=1}^4\widetilde{A}^{\alpha\beta\sigma\tau}_k\xi^k]=\sum\limits_{k=0}^{\infty}A^{\alpha\beta\sigma\tau}_k\xi^k,\\
A^{\alpha\beta33}=A^{33\alpha\beta}=\lambda g^{\alpha\beta},\quad A^{3333}=\lambda+2\mu,\quad A^{\alpha3\beta3}=A^{3\alpha3\beta}=A^{\alpha33\beta}=A^{3\alpha\beta3}=\mu g^{\alpha\beta},\\
A^{\alpha\beta\sigma3}=A^{\alpha\beta3\sigma}=A^{\alpha3\beta\sigma}=A^{3\alpha\beta\sigma}=0,\quad A^{\alpha333}=A^{3\alpha33}=A^{33\alpha3}=A^{333\alpha}=0,
\end{cases}\tag{2.57}$$

where

$$\begin{cases}
\widetilde{A}^{\alpha\beta\sigma\tau}_1=2c^{\alpha\beta\sigma\tau}-8Ha^{\alpha\beta\sigma\tau},\\
\widetilde{A}^{\alpha\beta\sigma\tau}_2=2(12H^2-K)a^{\alpha\beta\sigma\tau}-10Hc^{\alpha\beta\sigma\tau}+4b^{\alpha\beta\sigma\tau},\\
\widetilde{A}^{\alpha\beta\sigma\tau}_3=8H(K-4H^2)a^{\alpha\beta\sigma\tau}+(8H^2-2K)c^{\alpha\beta\sigma\tau}-8Hb^{\alpha\beta\sigma\tau},\\
\widetilde{A}^{\alpha\beta\sigma\tau}_4=(4H^2-K)^2a^{\alpha\beta\sigma\tau}+2H(K-4H^2)c^{\alpha\beta\sigma\tau}+4H^2b^{\alpha\beta\sigma\tau}.
\end{cases}\tag{2.58}$$

$$A^{\alpha\beta\lambda\sigma}_0=a^{\alpha\beta\lambda\sigma},\quad A^{\alpha\beta\lambda\sigma}_1=2c^{\alpha\beta\sigma\tau},\quad A^{\alpha\beta\lambda\sigma}_2=-6Ka^{\alpha\beta\sigma\tau}+6Hc^{\alpha\beta\sigma\tau}-4b^{\alpha\beta\sigma\tau}. \tag{2.59}$$

Proof. From (2.55) and (2.18), simple calculation show our results. The proof is complete. $\square$

In what follows, we introduce following invariant scale function

$$\begin{cases} \gamma_0(u)=a^{\alpha\beta}\gamma_{\alpha\beta}(u)=\overset{*}{\mathrm{div}}\,u-2Hu^3, \\ \gamma_1(u)=a^{\alpha\beta}\overset{1}{\gamma}_{\alpha\beta}(u)=-b^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta+(4H^2-2K)u^3-2u^\lambda\overset{*}{\nabla}_\lambda H, \\ \gamma_2(u)=a^{\alpha\beta}\overset{2}{\gamma}_{\alpha\beta}(u)=c^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta+u^\lambda\overset{*}{\nabla}_\lambda(2H^2-K), \end{cases} \tag{2.60}$$

$$\begin{cases} \beta_0(u)=b^{\alpha\beta}\gamma_{\alpha\beta}(u)=b^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta-(4H^2-2K)u^3, \\ \beta_1(u)=b^{\alpha\beta}\overset{1}{\gamma}_{\alpha\beta}(u)=-c^\alpha_\beta\overset{*}{\nabla}_\alpha u^\beta+(8H^3-6HK)u^3+\overset{*}{\nabla}_\lambda(4H^2-2K)u^\lambda, \\ \beta_2(u)=b^{\alpha\beta}\overset{2}{\gamma}_{\alpha\beta}(u)=(2Hc^\alpha_\beta-Kb^\alpha_\beta)\overset{*}{\nabla}_\alpha u^\beta+\frac{1}{2}b^{\alpha\beta}\overset{*}{\nabla}_\lambda c_{\alpha\beta}u^\lambda. \end{cases} \tag{2.61}$$

3 The metric tensors after deformation of the surface in $\Re^3$

In this section, we have to study the exchange of geometry of the surface in $\Re^3$ when the surface occurs deformation. We will give the formula for the exchange of metric tensor, curvatures tensor and normal vector to the surface.

Let $\omega\subset\Re^2$ be a compact domain and a immersion $\vec{\Theta}:\omega\to\Re^3$ is smooth enough, the middle surface $\Im$ of shell defined as the image $\vec{\Theta}$. The deformation of a surface means that at each point on the surface bears a small displacement $\vec{\eta}$ and new surface after deformation denote $\Im(\vec{\eta})$ as the image $\vec{\tilde{\Theta}}=\vec{\Theta}+\vec{\eta}$.

For simplicity, later on, we denote the Gateaux derivative of a geometric tensor (for example, metric tensor) with respect to surface $\vec{\Theta}$ along director $\vec{\eta}$

$$\frac{\mathcal{D}a^{\alpha\beta}}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\lim_{t\to0}[a_{\alpha\beta}(\vec{\Theta}+t\vec{\eta})-a_{\alpha\beta}(\vec{\Theta})].$$

Theorem 3.1. *Assume that surface $\Im$ is burned a deformation $\vec{\Theta}\Rightarrow\vec{\tilde{\Theta}}=\vec{\Theta}+\vec{\eta}$. Then following formulae hold*

$$\begin{cases} a_{\alpha\beta}(\eta)-a_{\alpha\beta}=2\overset{0}{E}_{\alpha\beta}(\eta), \quad \frac{\mathcal{D}a_{\alpha\beta}(\eta)}{\mathcal{D}\Theta}\eta=2\gamma_{\alpha\beta}(\eta), \\ a(\eta)=a(1+2\overset{0}{E}(\eta))+2\det(\overset{0}{E}_{\alpha\beta}(\eta)), \quad \frac{\mathcal{D}a(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2a\gamma_0(\eta), \\ \frac{\mathcal{D}a^{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2(\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}-a^{\alpha\beta}a^{\lambda\sigma})\gamma_{\lambda\sigma}(\eta), \quad \sqrt{\frac{a}{a(\eta)}}=1-\gamma_0(\eta)+o(|\eta|^2), \end{cases} \tag{3.1}$$

$$\begin{cases} \frac{\mathcal{D}\boldsymbol{n}(\eta)}{\mathcal{D}\Theta}\eta=-a^{\lambda\sigma}\overset{0}{\nabla}_\lambda\eta^3\boldsymbol{e}_\sigma, \\ \boldsymbol{n}(\eta)=\sqrt{\frac{a}{a(\eta)}}[(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\boldsymbol{n}+a^{\lambda\gamma}(\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3-\overset{0}{\nabla}_\lambda\eta^3)\boldsymbol{e}_\gamma], \end{cases} \tag{3.2}$$

$$\begin{cases} b_{\alpha\beta}(\eta)=\sqrt{\frac{a}{a(\eta)}}[(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_{\lambda}\eta^3)(1+\gamma_0(\eta)+\det\overset{0}{\nabla}_{\alpha}\eta^{\beta}) \\ \qquad +(\rho^{\sigma}_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_{\lambda}\eta^{\sigma}+\overset{*}{\Gamma}{}^{\sigma}_{\alpha\beta})(\varepsilon^{\nu\mu}\varepsilon_{\sigma\gamma}\overset{0}{\nabla}_{\nu}\eta^{\gamma}\overset{0}{\nabla}_{\mu}\eta^3-\overset{0}{\nabla}_{\sigma}\eta^3)], \\ \frac{\mathcal{D}b_{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho_{\alpha\beta}(\eta), \\ \frac{\mathcal{D}b^{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho^{\alpha\beta}(\eta)+2((\varepsilon^{\alpha\nu}b^{\beta}_{\lambda}+\varepsilon^{\beta\nu}b^{\alpha}_{\lambda})\varepsilon^{\lambda\mu}-2b^{\alpha\beta}a^{\nu\mu})\gamma_{\nu\mu}(\eta), \\ \frac{\mathcal{D}b(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\frac{1}{2}b(\rho_{\alpha\beta}(\eta)b^{\alpha\beta}+\rho_{\lambda\sigma}(\eta)b^{\lambda\sigma}=bb^{\alpha\beta}\rho_{\alpha\beta}(\eta)=b\rho_b(\eta), \end{cases} \tag{3.3}$$

$$\begin{cases} \frac{\mathcal{D}H}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho_0(\eta)+4H\gamma_0(\eta)+2Kb^{\alpha\beta}\gamma_{\alpha\beta}(\eta), \\ \frac{\mathcal{D}K}{\mathcal{D}\vec{\Theta}}\vec{\eta}=K\rho_b(\eta)-2K\gamma_0(\eta), \\ \gamma_0(\eta)=a^{\alpha\beta}\gamma_{\alpha\beta}(\eta), \quad \gamma_b(\eta)=b^{\alpha\beta}\gamma_{\alpha\beta}(\eta). \end{cases} \tag{3.4}$$

The Gateaux derivative of Riemann curvature with respect to surface $\Im$ along direction $\vec{\eta}$ is given by

$$\frac{\mathcal{D}\overset{*}{R}_{\alpha\beta\lambda\sigma}(\eta)}{\mathcal{D}\Theta}=b_{\alpha\sigma}\rho_{\beta\lambda}(\eta)+\rho_{\alpha\sigma}(\eta)b_{\beta\lambda}-b_{\alpha\lambda}\rho_{\beta\sigma}(\eta)-\rho_{\alpha\lambda}(\eta)b_{\beta\sigma}, \tag{3.5}$$

where

$$\begin{cases} \gamma_{\alpha\beta}(\eta)=\overset{*}{e}_{\alpha\beta}(\eta)-b_{\alpha\beta}\eta^3, \quad \overset{0}{E}_{\alpha\beta}(\eta)=\gamma_{\alpha\beta}(\eta)+\varphi_{\alpha\beta}(\eta,\eta), \\ \overset{0}{E}_0(\eta)=a^{\alpha\beta}\overset{0}{E}_{\alpha\beta}(\eta), \quad \gamma_0(\eta)=a^{\alpha\beta}\gamma_{\alpha\beta}(\eta), \quad \varphi_0(\eta)=a^{\alpha\beta}\varphi_{\alpha\beta}(\eta), \\ \rho_{\alpha\beta}(\eta)=\frac{1}{2}(\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\eta^3+\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\eta^3)+b_{\alpha\sigma}\overset{*}{\nabla}_{\beta}\eta^{\sigma}+b_{\beta\sigma}\overset{*}{\nabla}_{\alpha}\eta^{\sigma}-c_{\alpha\beta}\eta^3+\overset{*}{\nabla}_{\lambda}b_{\alpha\beta}\eta^{\lambda}, \\ \theta=1-2H\xi+K\xi^2, \quad p(\xi)=1-4H\xi+(4H^2-K)\xi^2, \quad q(\xi)=2\xi-2H\xi^2, \\ \rho_b(\eta)=b^{\alpha\beta}\rho_{\alpha\beta}(\eta), \quad \rho_0(\eta)=a^{\alpha\beta}\rho_{\alpha\beta}(\eta). \end{cases} \tag{3.6}$$

Proof. (i) **Preliminary**

Assume that the displacement vector and base vectors of S-coordinate system at $\Im$

$$\vec{\eta}=\eta^{\lambda}\mathbf{e}_{\lambda}+\eta^3\mathbf{n}, \quad \mathbf{e}_{\alpha}=\partial_{\alpha}\vec{\theta}, \quad \mathbf{n}=\frac{\mathbf{e}_1\times\mathbf{e}_2}{|\mathbf{e}_1\times\mathbf{e}_2|}=\varepsilon^{\alpha\beta}(\mathbf{e}_{\alpha}\times\mathbf{e}_{\beta})$$

are given. Then

Proposition 3.1. The followings on the $\Im$ are valid

$$\begin{cases} \partial_{\alpha}\vec{\eta}=\overset{0}{\nabla}_{\alpha}\eta^{\beta}\mathbf{e}_{\beta}+\overset{0}{\nabla}_{\alpha}\eta^3\mathbf{n}, \\ \gamma_{\alpha\beta}(\eta)=\frac{1}{2}(\partial_{\alpha}\vec{\eta}\mathbf{e}_{\beta}+\partial_{\beta}\vec{\eta}\mathbf{e}_{\alpha})=\frac{1}{2}(a_{\beta\lambda}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}+a_{\alpha\lambda}\overset{0}{\nabla}_{\beta}\eta^{\lambda}), \\ \partial_{\alpha}\vec{\eta}\partial_{\beta}\vec{\eta}=a_{ij}\overset{0}{\nabla}_{\alpha}\eta^i\overset{0}{\nabla}_{\beta}\eta^j=2\varphi_{\alpha\beta}(\eta,\eta), \end{cases} \tag{3.7}$$

$$\partial_{\alpha}\vec{\eta}\times\partial_{\beta}\vec{\eta}=\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_{\alpha}\eta^{\lambda}\overset{0}{\nabla}_{\beta}\eta^{\sigma}\mathbf{n}+\varepsilon^{\lambda\sigma}a_{\sigma\gamma}(\overset{0}{\nabla}_{\alpha}\eta^{\gamma}\overset{0}{\nabla}_{\beta}\eta^3-\overset{0}{\nabla}_{\alpha}\eta^3\overset{0}{\nabla}_{\beta}\eta^{\gamma})\mathbf{e}_{\lambda}. \tag{3.8}$$

Proof. Indeed, using Gaussian and Weingarten's formula, reads

$$
\begin{aligned}
\partial_\alpha \vec{\eta} &= \partial_\alpha(\eta^\lambda \mathbf{e}_\lambda) = \partial_\alpha \eta^\lambda \mathbf{e}_\lambda + \eta^\lambda \mathbf{e}_{\lambda\alpha} + \partial_\alpha \eta^3 \mathbf{n} + \eta^3 \mathbf{n}_\alpha \\
&= \partial_\alpha \eta^\lambda \mathbf{e}_\lambda + \eta^\lambda (\overset{*}{\Gamma}{}^\nu{}_{\lambda\alpha} \mathbf{e}_\nu + b_{\lambda\alpha}\mathbf{n}) + \partial_\alpha \eta^3 \mathbf{n} + \eta^3(-b^\nu_\lambda \mathbf{e}_\nu) \\
&= (\partial_\alpha \eta^\nu + \overset{*}{\Gamma}{}^\nu{}_{\lambda\alpha}\eta^\lambda)\mathbf{e}_\nu + (b_{\alpha\lambda}\eta^\lambda + \partial_\alpha \eta^3)\mathbf{n} - b^\nu_\alpha \eta^3 \mathbf{e}_\nu \\
&= (\overset{*}{\nabla}_\alpha \eta^\nu - b^\nu_\alpha \eta^3)\mathbf{e}_\nu + (\overset{*}{\nabla}_\alpha \eta^3 + b_{\alpha\lambda}\eta^\lambda)\mathbf{n} \\
&= \overset{0}{\nabla}_\alpha \eta^\nu \mathbf{e}_\nu + \overset{0}{\nabla}_\alpha \eta^3 \mathbf{n}, \\
\partial_\alpha \vec{\eta} \partial_\beta \vec{\eta} &= a_{\nu\mu} \overset{0}{\nabla}_\alpha \eta^\nu \overset{0}{\nabla}_\beta \eta^\mu + \overset{0}{\nabla}_\alpha \eta^3 \overset{0}{\nabla}_\beta \eta^3 = a_{ij} \overset{0}{\nabla}_\alpha \eta^i \overset{0}{\nabla}_\beta \eta^j = 2\varphi_{\alpha\beta}(\vec{\eta}).
\end{aligned}
$$

From above formula (3.7) is obtained.

What follows that we prove (3.8), In fact, by virtue of Weingarten's and Gaussian formula

$$
\mathbf{e}_{\alpha\beta} = \overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta} \mathbf{e}_\lambda + b_{\alpha\beta}\mathbf{n}, \quad \mathbf{n}_\alpha = -b^\beta_\alpha \mathbf{e}_\beta = -b_{\alpha\beta}\mathbf{e}^\beta, \tag{3.9}
$$

and (see [2])

$$
\begin{cases}
\mathbf{a}\times\mathbf{b} = \varepsilon_{ijk} a^i b^j \mathbf{e}^k, \\
\mathbf{e}_\alpha \times \mathbf{e}_\beta = \varepsilon_{\alpha\beta}\mathbf{n}, \quad \mathbf{n}\times\mathbf{e}_\alpha = \varepsilon_{\alpha\beta}\mathbf{e}^\beta, \quad \mathbf{e}^\alpha \mathbf{e}_\beta = \delta^\alpha_\beta, \quad \mathbf{n}\times\mathbf{n} = 0, \\
\varepsilon^{\alpha\beta}\varepsilon_{\beta\sigma} = \delta^\alpha_\sigma.
\end{cases} \tag{3.10}
$$

Therefore,

$$
\begin{aligned}
\partial_\alpha \vec{\eta} \times \partial_\beta \vec{\eta} &= \varepsilon^{ijk}(\partial_\alpha \vec{\eta})_i (\partial_\beta \vec{\eta})_j \vec{e}_k = \varepsilon^{ij\lambda}(\partial_\alpha \vec{\eta})_i (\partial_\beta \vec{\eta})_j \vec{e}_k + \varepsilon^{ij3}(\partial_\alpha \vec{\eta})_i (\partial_\beta \vec{\eta})_j \vec{n} \\
&\quad \text{(by properties of permutation tensor)} \\
&= \varepsilon^{3\sigma\lambda}((\partial_\alpha \vec{\eta})_3 (\partial_\beta \vec{\eta})_\sigma - (\partial_\alpha \vec{\eta})_\sigma (\partial_\beta \vec{\eta})_3)\vec{e}_\lambda + \varepsilon^{\lambda\sigma 3}(\partial_\alpha \vec{\eta})_\lambda (\partial_\beta \vec{\eta})_\sigma \vec{n} \\
&= \varepsilon^{3\sigma\lambda} g_{\sigma\gamma}((\partial_\alpha \vec{\eta})^3 (\partial_\beta \vec{\eta})^\gamma - (\partial_\alpha \vec{\eta})^\gamma (\partial_\beta \vec{\eta})^3)\vec{e}_\lambda + \varepsilon^{\lambda\sigma 3} g_{\lambda\nu} g_{\mu\sigma} (\partial_\alpha \vec{\eta})^\nu (\partial_\beta \vec{\eta})^\mu \vec{n}.
\end{aligned}
$$

Observe that

$$
\varepsilon^{3\alpha\beta} = \sqrt{\frac{a}{g}}\varepsilon^{\alpha\beta} = \theta^{-1}\varepsilon^{\alpha\beta}, \quad (\partial_\beta \vec{\eta})^\gamma = \overset{0}{\nabla}_\beta \eta^\gamma.
$$

$$
\partial_\alpha \vec{\eta} \times \partial_\beta \vec{\eta} = \theta^{-1}\varepsilon^{\sigma\lambda} g_{\sigma\gamma}(\overset{0}{\nabla}_\alpha \eta^3 \overset{0}{\nabla}_\beta \eta^\gamma - \overset{0}{\nabla}_\alpha \eta^\gamma \overset{0}{\nabla}_\beta \eta^3)\vec{e}_\lambda + \theta^{-1}\varepsilon^{\lambda\sigma} g_{\lambda\nu} g_{\sigma\mu} \overset{0}{\nabla}_\alpha \eta^\lambda \overset{0}{\nabla}_\beta \eta^\sigma \vec{n}. \tag{3.11}
$$

Taking into account of

$$
g_{\alpha\beta}|_\Im = a_{\alpha\beta}, \quad \theta|_\Im = 1, \quad \varepsilon^{\lambda\sigma} a_{\lambda\nu} a_{\sigma\mu} = \varepsilon_{\nu\mu},
$$

from (3.11) it infers (3.8) immediately. □

(ii) **Metric tensor and its determinant** $a(\eta) = \det(a_{\alpha\beta}(\eta))$.

Proposition 3.2. The Gateaux derivatives for metric tensors and its determinant with respect to $\Im$ along director $\vec{\eta}$ are given by the followings

$$\begin{cases} a_{\alpha\beta}(\eta)-a_{\alpha\beta}=2\overset{0}{E}_{\alpha\beta}(\eta), \quad \frac{\mathcal{D}a_{\alpha\beta}(\eta)}{\mathcal{D}\Theta}\eta=2\gamma_{\alpha\beta}(\eta), \\ a(\eta)=a(1+2\overset{0}{E}(\eta))+2\det(\overset{0}{E}_{\alpha\beta}(\eta)), \quad \frac{\mathcal{D}a(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2a\gamma_0(\eta), \\ \frac{\mathcal{D}a^{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2(\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}-a^{\alpha\beta}a^{\lambda\sigma})\gamma_{\lambda\sigma}(\eta), \quad \sqrt{\frac{a}{a(\eta)}}=1-\gamma_0(\eta)+o(|\eta|^2), \end{cases} \tag{3.12}$$

where $\overset{0}{E}_{\alpha\beta}(\eta)$ are leading term of Green St.Venant strain tensor $E_{\alpha\beta}(\eta)$ and

$$\overset{0}{E}(\eta)=\gamma_0(\eta)+\varphi_0(\eta), \quad \gamma_0(\eta)=a^{\alpha\beta}\gamma_{\alpha\beta}(\eta), \quad \varphi_0(\eta)=a^{\alpha\beta}\varphi_{\alpha\beta}(\eta).$$

Furthermore,

$$a(\eta)>0$$

if vector $\vec{\eta}$ is small enough.

Proof. The deformed surface $\Im(\eta)$ define as the image $\vec{\Theta}(\eta)=\vec{\Theta}+\vec{\eta}$. Assume vectors

$$\mathbf{e}(\eta):=\partial_\alpha\Theta(\vec{\eta})=\partial_\alpha\vec{\Theta}+\partial_\alpha\vec{\eta}=\mathbf{e}_\alpha+\partial_\alpha\vec{\eta}$$

are linearly independent at all points of $\overline{\omega}\subset\Re^2$. It is obvious that if the vector $\vec{\eta}$ is small enough, $\mathbf{e}(\eta)$ can be as base vectors of two dimensional manifold $\Im(\eta)$. So that $a_{\alpha\beta}(\eta)=\mathbf{e}_\alpha(\eta)\mathbf{e}_\beta(\eta)$ are covariant components of metric tensor of $\Im(\eta)$ which is nonsingular matrix. Indeed

$$a_{\alpha\beta}(\eta)=\partial_\alpha(\vec{\Theta}+\vec{\eta})\partial_\beta(\vec{\Theta}+\vec{\eta})=a_{\alpha\beta}+\partial_\alpha\vec{\eta}\mathbf{e}_\beta+\partial_\beta\vec{\eta}\mathbf{e}_\alpha+\partial_\alpha\vec{\eta}\partial_\beta\vec{\eta}.$$

By (3.7),

$$\begin{aligned} a_{\alpha\beta}(\eta)&=a_{\alpha\beta}+2\gamma_{\alpha\beta}(\eta)+a_{ij}\overset{0}{\nabla}_\alpha\eta^i\overset{0}{\nabla}_\beta\eta^j \\ &=a_{\alpha\beta}+2(\gamma_{\alpha\beta}(\eta)+\varphi_{\alpha\beta}(\eta))=a_{\alpha\beta}+2\overset{0}{E}_{\alpha\beta}(\eta). \end{aligned} \tag{3.13}$$

According to the calculation's principle of the determinant for a matrix

$$\det(A_{\alpha\beta})=\frac{1}{2}\widehat{\varepsilon}^{\alpha\beta}\widehat{\varepsilon}^{\lambda\sigma}A_{\alpha\lambda}A_{\beta\zeta}, \qquad \varepsilon^{\alpha\beta}=\frac{\widehat{\varepsilon}^{\alpha\beta}}{\sqrt{a}},$$

and the formula

$$a^{\alpha\beta}=\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}a_{\lambda\sigma}, \quad aa^{\alpha\beta}=\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}a_{\lambda\sigma}, \quad \gamma_0(\eta)=a^{\alpha\beta}\gamma_{\alpha\beta}(\eta), \quad \varphi_0(\eta)=a^{\alpha\beta}\varphi_{\alpha\beta}(\eta),$$

we assert

$$\begin{aligned}a(\eta)&=\det(a_{\alpha\beta}(\eta))=\tfrac{1}{2}\widehat{\varepsilon}^{\alpha\beta}\widehat{\varepsilon}^{\lambda\sigma}a_{\alpha\lambda}(\eta)a_{\beta\sigma}(\eta)=\tfrac{1}{2}a\varepsilon^{\alpha\beta}\varepsilon^{\lambda\sigma}a_{\alpha\lambda}(\eta)a_{\beta\sigma}(\eta)\\&=\tfrac{1}{2}a\varepsilon^{\alpha\beta}\varepsilon^{\lambda\sigma}(a_{\alpha\lambda}+2(\gamma_{\alpha\lambda}(\eta)+\varphi_{\alpha\lambda}(\eta)))(a_{\beta\sigma}+2(\gamma_{\beta\sigma}(\eta)+\varphi_{\beta\sigma}(\eta)))\\&=\tfrac{1}{2}a\{\varepsilon^{\alpha\beta}\varepsilon^{\lambda\sigma}a_{\alpha\lambda}a_{\beta\sigma}+2\varepsilon^{\alpha\beta}\varepsilon^{\lambda\sigma}\{a_{\alpha\lambda}(\gamma_{\beta\sigma}(\eta)+\varphi_{\beta\sigma}(\eta))+a_{\beta\sigma}(\gamma_{\alpha\lambda}(\eta)+\varphi_{\alpha\lambda}(\eta))\}\\&\quad+4a\varepsilon^{\alpha\beta}\varepsilon^{\lambda\sigma}(\gamma_{\alpha\lambda}(\eta)+\varphi_{\alpha\lambda}(\eta))(\gamma_{\beta\sigma}(\eta)+\varphi_{\beta\sigma}(\eta))\}\\&=a+a^{\beta\sigma}(\gamma_{\beta\sigma}(\eta)+\varphi_{\beta\sigma}(\eta))+a^{\alpha\lambda}(\gamma_{\alpha\lambda}(\eta)+\varphi_{\alpha\lambda}(\eta))+2a\det\gamma_{\alpha\beta}(\eta)+\varphi_{\alpha\beta}(\eta)\\&=a+2a\overset{0}{E}(\eta)+2a\det(\overset{0}{E}_{\alpha\beta}(\eta))>0,\quad(\text{if }\vec{\eta}\text{ is small enough}).\end{aligned}$$

Therefor there exist contravariant components of metric tensor

$$a^{\alpha\beta}(\eta)a_{\beta\sigma}(\eta)=\delta^{\alpha}_{\sigma},\quad a^{\alpha\beta}(\eta)=\varepsilon^{\alpha\lambda}(\eta)\varepsilon^{\beta\sigma}(\eta)a_{\lambda\sigma}(\eta),$$

where the permutation tensor is defined by

$$\varepsilon_{\alpha\beta}(\eta)=\begin{cases}\sqrt{a(\eta)},\\-\sqrt{a(\eta)},\\0,\end{cases}\ \varepsilon^{\alpha\beta}(\eta)=\begin{cases}\frac{1}{\sqrt{a(\eta)}},(\alpha,\beta): & \text{even permutation of (1,2)},\\-\frac{1}{\sqrt{a(\eta)}},(\alpha,\beta): & \text{odd permutation of (1,2)},\\0,\text{ otherwise}.\end{cases}$$

Let the normal unite vector be

$$\mathbf{n}(\eta)=\varepsilon^{\alpha\beta}(\eta)\mathbf{e}_{\alpha}(\eta)\times\mathbf{e}_{\beta}(\eta)=\frac{\mathbf{e}_1(\eta)\times\mathbf{e}_2(\eta)}{|\mathbf{e}_1(\eta)\times\mathbf{e}_2(\eta)|},$$

and the contravariant base vector

$$\mathbf{e}^{\alpha}(\eta)=a^{\alpha\beta}\mathbf{e}_{\beta}(\eta),\quad\mathbf{e}^{\alpha}(\eta)\mathbf{e}_{\beta}(\eta)=\delta^{\alpha}_{\beta},\quad a^{\alpha\beta}(\eta)=\mathbf{e}^{\alpha}(\eta)\mathbf{e}^{\beta}(\eta).$$

From this, of course, it infers

$$a(\eta)-a=2a\gamma_0(\eta)+2a\varphi_0(\eta)+2a\det(\overset{0}{E}_{\alpha\beta}(\eta)).$$

The second and third terms of the above equality are two degree of η, so that with (3.13)

$$\tfrac{\mathcal{D}a}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2a\gamma_0(\eta),\quad\tfrac{\mathcal{D}a_{\alpha\beta}}{\mathcal{D}\vec{\Theta}}\vec{\eta}=2\gamma_{\alpha\beta}(\eta).$$

On the other hand

$$a^{\alpha\beta}(\eta)=\varepsilon^{\alpha\lambda}(\eta)\varepsilon^{\beta\sigma}(\eta)a_{\lambda\sigma}=\frac{1}{a(\eta)}\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}a_{\lambda\sigma}(\eta),$$

$$\begin{aligned}\tfrac{\mathcal{D}a^{\alpha\beta}}{\mathcal{D}\vec{\Theta}}\vec{\eta}&=\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}\big(\tfrac{1}{a}\tfrac{\mathcal{D}a_{\lambda\sigma}}{\mathcal{D}\vec{\Theta}}\vec{\eta}-\tfrac{a_{\lambda\sigma}}{a^2}\tfrac{\mathcal{D}a}{\mathcal{D}\vec{\Theta}}\vec{\eta}\big)\\&=\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}\tfrac{2}{a}(\gamma_{\lambda\sigma}(\eta)-a_{\lambda\sigma}r_0(\eta))=2(\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}\gamma_{\lambda\sigma}(\eta)-a^{\alpha\beta}\gamma_0(\eta)).\end{aligned}$$

The proof of Proposition 3.2 is complete. □

(iii) **Second fundamental form and unit normal vector** $\mathbf{n}(\eta)$ to $\Im(\eta)$

Proposition 3.3. The Gateaux derivatives for unit normal vector with respect to $\Im$ along director $\vec{\eta}$ are given by the followings

$$\begin{cases} \frac{\mathcal{D}\mathbf{n}(\eta)}{\mathcal{D}\Theta}\eta = -a^{\lambda\sigma}\overset{0}{\nabla}_\lambda \eta^3\mathbf{e}_\sigma, \\ \mathbf{n}(\eta) = \sqrt{\frac{a}{a(\eta)}}[(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\mathbf{n}+a^{\lambda\gamma}(\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3-\overset{0}{\nabla}_\lambda\eta^3)\mathbf{e}_\gamma]. \end{cases} \tag{3.14}$$

Proof. Firstly, assume that $a(\eta)\neq 0$ by (3.10),

$$\begin{aligned} \mathbf{n}(\eta) &= \tfrac{1}{2}\varepsilon^{\alpha\beta}(\eta)\mathbf{e}_\alpha(\eta)\times\mathbf{e}_\beta(\eta) = \tfrac{1}{2}\sqrt{\frac{a}{a(\eta)}}\varepsilon^{\alpha\beta}(\mathbf{e}_\alpha+\partial_\alpha\vec{\eta})\times(\mathbf{e}_\beta+\partial_\beta\vec{\eta}) \\ &= \tfrac{1}{2}q(\varepsilon^{\alpha\beta}\mathbf{e}_\alpha\times\mathbf{e}_\beta+\varepsilon^{\alpha\beta}(\mathbf{e}_\alpha\times\partial_\beta\vec{\eta}+\partial_\alpha\vec{\eta}\times\mathbf{e}_\beta)+\varepsilon^{\alpha\beta}\partial_\alpha\vec{\eta}\times\partial_\beta\vec{\eta}) \\ &= q(\mathbf{n}+\varepsilon^{\alpha\beta}\mathbf{e}_\alpha\times\partial_\beta\vec{\eta}+\tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\vec{\eta}\times\partial_\beta\vec{\eta}). \end{aligned} \tag{3.15}$$

From (3.7), (3.10) and the formula

$$\det(\overset{0}{\nabla}_\alpha\eta^\beta) = \frac{1}{2}\Big(\varepsilon^{\nu\mu}\varepsilon_{\tau\lambda}\overset{0}{\nabla}_\nu\eta^\tau\overset{0}{\nabla}_\mu\eta^\lambda\Big), \quad \gamma_0(\eta) = \overset{0}{\nabla}_\beta\eta^\beta = a^{\alpha\beta}\gamma_{\alpha\beta}(\eta),$$

it infers

$$\begin{aligned} \varepsilon^{\alpha\beta}\mathbf{e}_\alpha\times\partial_\beta\vec{\eta} &= \varepsilon^{\alpha\beta}\mathbf{e}_\alpha\times(\overset{0}{\nabla}_\beta\eta^\lambda+\overset{0}{\nabla}_\beta\eta^3\mathbf{n}) = \varepsilon^{\alpha\beta}(\overset{0}{\nabla}_\beta\eta^\lambda\mathbf{e}_\alpha\times\mathbf{e}_\lambda+\overset{0}{\nabla}_\beta\eta^3\mathbf{e}_\alpha\times\mathbf{n}) \\ &= \varepsilon^{\alpha\beta}\varepsilon_{\alpha\lambda}(\overset{0}{\nabla}_\beta\eta^\lambda\mathbf{n}-\overset{0}{\nabla}_\beta\eta^3\mathbf{e}^\lambda). \end{aligned}$$

Here we used formula

$$\mathbf{e}_\alpha\times\mathbf{e}_\lambda = \varepsilon_{\alpha\lambda}\mathbf{n}, \quad \mathbf{e}_\alpha\times\mathbf{n} = -\varepsilon_{\alpha\lambda}\mathbf{e}^\lambda = -a^{\lambda\sigma}\varepsilon_{\alpha\lambda}\mathbf{e}_\sigma.$$

Owing to $\varepsilon^{\alpha\beta}\varepsilon_{\alpha\lambda} = \delta^\beta_\lambda$, we have

$$\varepsilon^{\alpha\beta}\mathbf{e}_\alpha\times\partial_\beta\vec{\eta} = \overset{0}{\nabla}_\beta\eta^\beta\mathbf{n}-\overset{0}{\nabla}_\beta\eta^3\mathbf{e}^\beta = \gamma_0(\eta)\mathbf{n}-\overset{0}{\nabla}_\beta\eta^3\mathbf{e}^\beta.$$

In a similar manner, using (3.8) gives

$$\begin{aligned} \varepsilon^{\alpha\beta}\partial_\alpha\vec{\eta}\times\partial_\beta\vec{\eta} &= \varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^\sigma\mathbf{n}+\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}(\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3-\overset{0}{\nabla}_\alpha\eta^3\overset{0}{\nabla}_\beta\eta^\sigma)\mathbf{e}^\lambda \\ &= \varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\lambda\overset{0}{\nabla}_\beta\eta^\sigma\mathbf{n}+2\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3\mathbf{e}^\lambda \\ &= 2\det(\overset{0}{\nabla}_\alpha\eta^\beta)\mathbf{n}+2\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3\mathbf{e}^\lambda. \end{aligned}$$

Substituting above equalities into (3.15) leads to sixth formula of (3.14).

Using sixth formula of (3.14),

$$\begin{aligned} &\sqrt{\frac{a}{a(\eta)}}[(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\mathbf{n}+a^{\lambda\gamma}(\varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\alpha\eta^\sigma\overset{0}{\nabla}_\beta\eta^3-\overset{0}{\nabla}_\lambda\eta^3)\mathbf{e}_\gamma] \\ &= (1-\gamma_0(\eta)+o(|\eta|^2))[(1+\gamma_0(\eta)+o(|\eta|^2))\mathbf{n}-a^{\lambda\gamma}\overset{0}{\nabla}_\lambda\eta^3\mathbf{e}_\gamma] \\ &= \mathbf{n}-a^{\lambda\gamma}\overset{0}{\nabla}_\lambda\eta^3\mathbf{e}_\gamma+o(|\eta|^2). \end{aligned}$$

Hence we assert

$$\frac{\mathcal{D}\mathbf{n}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=-a^{\lambda\gamma}\overset{0}{\nabla}_\lambda\eta^3\mathbf{e}_\gamma. \qquad \square$$

♠ **Curvature Tensor**

Proposition 3.4. The Gateaux derivatives for curvature tensors and its determinant with respect to $\Im$ along director $\vec{\eta}$ are given by the followings

$$\begin{cases} b_{\alpha\beta}(\eta)=\sqrt{\frac{a}{a(\eta)}}[(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^\lambda_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)(1+\gamma_0(\eta)+\det\overset{0}{\nabla}_\alpha\eta^\beta) \\ \qquad +(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^\lambda_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma+\overset{*}{\Gamma}{}^\sigma_{\alpha\beta})(\varepsilon^{\nu\mu}\varepsilon_{\sigma\gamma}\overset{0}{\nabla}_\nu\eta^\gamma\overset{0}{\nabla}_\mu\eta^3-\overset{0}{\nabla}_\sigma\eta^3)], \\ \frac{\mathcal{D}b_{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho_{\alpha\beta}(\eta), \\ \frac{\mathcal{D}b^{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho^{\alpha\beta}+2((\varepsilon^{\alpha\nu}b^\beta_\lambda+\varepsilon^{\beta\nu}b^\alpha_\lambda)\varepsilon^{\lambda\mu}-2b^{\alpha\beta}a^{\nu\mu})\gamma_{\nu\mu}(\eta), \\ \frac{\mathcal{D}b(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\frac{1}{2}b(\rho_{\alpha\beta}(\eta)b^{\alpha\beta}+\rho_{\lambda\sigma}(\eta)b^{\lambda\sigma}=bb^{\alpha\beta}\rho_{\alpha\beta}(\eta)=b\rho_b(\eta), \end{cases} \tag{3.16}$$

where the tensors of order two are defined by

$$\begin{cases} \rho_{\alpha\beta}(\eta):=\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^3+b_{\alpha\sigma}\overset{0}{\nabla}_\beta\eta^\sigma, \quad \rho^\sigma_{\alpha\beta}(\eta):=\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^\sigma-b^\sigma_\alpha\overset{0}{\nabla}_\beta\eta^3, \\ \rho_0(\eta)=a^{\alpha\beta}\rho_{\alpha\beta}(\eta), \quad \rho_b(\eta)=b^{\alpha\beta}\rho_{\alpha\beta}(\eta). \end{cases} \tag{3.17}$$

Proof. According to the Gaussian formula

$$\partial_\alpha\partial_\beta\theta=\overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta}\partial_\lambda\theta+b_{\alpha\beta}\mathbf{n}, \quad \partial_\alpha\partial_\beta\theta(\eta)=\overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta}(\eta)\partial_\lambda\theta(\eta)+b_{\alpha\beta}(\eta)\mathbf{n}(\eta),$$

we have

$$b_{\alpha\beta}(\eta)=\partial_\alpha\partial_\beta\theta(\eta)\mathbf{n}(\eta)=\mathbf{e}_{\alpha\beta}(\eta)\mathbf{n}(\eta)=(\mathbf{e}_{\alpha\beta}+\partial_\alpha\partial_\beta\vec{\eta})\mathbf{n}(\eta). \tag{3.18}$$

On the other hand, the following formula is held

$$\partial_\alpha\partial_\beta\vec{\eta}=\left(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^\lambda_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma\right)\mathbf{e}_\sigma+\left(\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^\lambda_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3\right)\mathbf{n}. \tag{3.19}$$

Indeed, by the Weingarten's and Gaussian formula (3.9)

$$\begin{aligned} \partial_\alpha\partial_\beta\vec{\eta}&=\partial_\alpha(\overset{0}{\nabla}_\beta\eta^\sigma\mathbf{e}_\sigma+\overset{0}{\nabla}_\beta\eta^3\mathbf{n})=\partial_\alpha\overset{0}{\nabla}_\beta\eta^\sigma\mathbf{e}_\sigma+\overset{0}{\nabla}_\beta\eta^\sigma(\overset{*}{\Gamma}{}^\lambda_{\alpha\sigma}\mathbf{e}_\lambda+b_{\alpha\sigma}\mathbf{n}) \\ &\quad+\partial_\alpha\overset{0}{\nabla}_\beta\eta^3\mathbf{n}+\overset{0}{\nabla}_\beta\eta^3(-b^\lambda_\alpha\mathbf{e}_\lambda) \\ &=(\partial_\alpha\overset{0}{\nabla}_\beta\eta^\sigma+\overset{*}{\Gamma}{}^\sigma_{\alpha\lambda}\overset{0}{\nabla}_\beta\eta^\lambda-b^\sigma_\alpha\overset{0}{\nabla}_\beta\eta^3)\mathbf{e}_\sigma+(\partial_\alpha\overset{0}{\nabla}_\beta\eta^3+b_{\alpha\sigma}\overset{0}{\nabla}_\beta\eta^\sigma)\mathbf{n} \\ &=(\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^\sigma+\overset{*}{\Gamma}{}^\lambda_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma-b^\sigma_\alpha\overset{0}{\nabla}_\beta\eta^3)\mathbf{e}_\sigma+(\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^3+\overset{*}{\Gamma}{}^\sigma_{\alpha\beta}\overset{0}{\nabla}_\sigma\eta^3+b_{\alpha\sigma}\overset{0}{\nabla}_\beta\eta^\sigma)\mathbf{n}. \end{aligned}$$

Hence, we conclude (3.19) by virtue of (3.17). Using the Weingarten's formula (3.9) with (3.19), we assert

$$\begin{aligned}\mathbf{e}_{\alpha\beta}+\partial_\alpha\partial_\beta\eta=&\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}\mathbf{e}_\lambda+b_{\alpha\beta}\mathbf{n}+(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma)\mathbf{e}_\sigma+(\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)\mathbf{n}\\=&(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}(\delta^\sigma_\lambda+\overset{0}{\nabla}_\lambda\eta^\sigma))\mathbf{e}_\sigma+(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)\mathbf{n}.\end{aligned}$$

Coming back (3.18) with (3.7) and (3.9) shows

$$\begin{aligned}b_{\alpha\beta}(\eta)=&\sqrt{\tfrac{a}{a(\eta)}}[(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}(\delta^\sigma_\lambda+\overset{0}{\nabla}_\lambda\eta^\sigma))\mathbf{e}_\sigma+(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)\mathbf{n}]\\&[(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta)\mathbf{n}+a^{\lambda\gamma}(\varepsilon^{\nu\mu}\varepsilon_{\lambda\sigma}\overset{0}{\nabla}_\nu\eta^\sigma\overset{0}{\nabla}_\mu\eta^3-\overset{0}{\nabla}_\lambda\eta^3)\mathbf{e}_\gamma].\end{aligned}$$

Since $a^{\lambda\gamma}\mathbf{e}_\sigma\mathbf{e}_\gamma=\delta^\lambda_\sigma$, we have

$$\begin{aligned}b_{\alpha\beta}(\eta)=&q(\eta)[(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}(\delta^\sigma_\lambda+\overset{0}{\nabla}_\lambda\eta^\sigma))(\varepsilon^{\nu\mu}\varepsilon_{\sigma\gamma}\overset{0}{\nabla}_\nu\eta^\gamma\overset{0}{\nabla}_\mu\eta^3-\overset{0}{\nabla}_\sigma\eta^3)\\&+(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta))]\\=&q(\eta)[(b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^3)(1+\gamma_0(\eta)+\det(\overset{0}{\nabla}_\alpha\eta^\beta))\\&+(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma+\overset{*}{\Gamma}{}^{\sigma}_{\alpha\beta})(\varepsilon^{\nu\mu}\varepsilon_{\sigma\gamma}\overset{0}{\nabla}_\nu\eta^\gamma\overset{0}{\nabla}_\mu\eta^3-\overset{0}{\nabla}_\sigma\eta^3)]\\=&q(\eta)[\phi_{\alpha\beta}(\eta)d_0(\eta)+\phi^\sigma_{\alpha\beta}(\eta)d_\sigma(\eta)].\end{aligned}$$

It can be rewritten in

$$\begin{cases}b_{\alpha\beta}(\eta)=q(\eta)[b_{\alpha\beta}+\rho_\alpha(\eta)+\phi_{\alpha\beta}(\eta)d(\eta)+\phi^\sigma_{\alpha\beta}(\eta)m_\sigma(\eta)-(\rho^\sigma_{\alpha\beta}(\eta)+\overset{*}{\Gamma}{}^{\lambda}_{\alpha\beta}\overset{0}{\nabla}_\lambda\eta^\sigma)\overset{0}{\nabla}_\sigma\eta^3],\\b_{\alpha\beta}(\eta)=b_{\alpha\beta}+\rho_{\alpha\beta}(\eta)+o(|\eta|^2),\\\frac{\mathcal{D}b_{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho_{\alpha\beta}(\eta),\\q(\eta)=\sqrt{\frac{a}{a(\eta)}}=1+-\gamma_0(\eta)-\varphi_0(\eta)+2a^{-1}\det(\overset{0}{E}_{\alpha\beta}(\eta))+\cdots.\end{cases}$$

These are the first second of (3.16). Next we consider the contravariant component

$$b^{\alpha\beta}(\eta)=a^{\alpha\lambda}(\eta)a^{\beta\sigma}(\eta)b_{\lambda\sigma}(\eta).$$

Using above, (3.12) and (3.16), and link chain of derivative

$$\begin{aligned}\frac{\mathcal{D}b^{\alpha\beta}(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=&a^{\alpha\lambda}a^{\beta\sigma}\rho_{\lambda\sigma}(\eta)+2((\varepsilon^{\alpha\nu}\varepsilon^{\lambda\mu}-a^{\alpha\lambda}a^{\nu\mu})b^\beta_\lambda+(\varepsilon^{\beta\nu}\varepsilon^{\sigma\mu}-a^{\beta\sigma}a^{\nu\mu})b^\alpha_\sigma)\gamma_{\nu\mu}(\eta)\\=&\rho^{\alpha\beta}+2((\varepsilon^{\alpha\nu}b^\beta_\lambda+\varepsilon^{\beta\nu}b^\alpha_\lambda)\varepsilon^{\lambda\mu}-2b^{\alpha\beta}a^{\nu\mu})\gamma_{\nu\mu}.\end{aligned}$$

Note that

$$b(\eta)=\frac{1}{2}\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}b_{\alpha\beta}(\eta)b_{\lambda\sigma}(\eta).$$

If $b=\det(b_{\alpha\beta})\neq 0$ then
$$b^{\alpha\beta}(\eta)=b^{-1}(\eta)\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}b_{\lambda\sigma}(\eta).$$
Hence
$$\begin{aligned}\frac{\mathcal{D}b(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}&=\tfrac{1}{2}\widehat{\varepsilon}^{\alpha\lambda}\widehat{\varepsilon}^{\beta\sigma}(\rho_{\alpha\beta}b_{\lambda\sigma}+b_{\alpha\beta}\rho_{\lambda\sigma})\\&=\tfrac{1}{2}b(\rho_{\alpha\beta}(\eta)b^{\alpha\beta}+\rho_{\lambda\sigma}(\eta)b^{\lambda\sigma}=bb^{\alpha\beta}\rho_{\alpha\beta}(\eta)=b\rho_b(\eta).\end{aligned}$$
To sum up, it completes our proof. □

Proposition 3.5. The Gateaux derivatives for $(H,K,c_{\alpha\beta})$ with respect to $\Im$ along director $\vec{\eta}$ are given by the followings
$$\begin{cases}\frac{\mathcal{D}H(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=\rho_0(\eta)+4H\gamma_0(\eta)+2Kb^{\alpha\beta}\gamma_{\alpha\beta}(\eta),\\ \frac{\mathcal{D}K(\eta)}{\mathcal{D}\vec{\Theta}}\vec{\eta}=K\rho_b(\eta)-2K\gamma_0(\eta),\\ \gamma_0(\eta)=a^{\alpha\beta}\gamma_{\alpha\beta}(\eta),\quad \gamma_b(\eta)=b^{\alpha\beta}\gamma_{\alpha\beta}(\eta).\end{cases}\tag{3.20}$$

Proof. Since $H(\eta)=a^{\alpha\beta}(\eta)b_{\alpha\beta}$, $K=\frac{b}{a}$, $c_{\alpha\beta}(\eta)=-Ka_{\alpha\beta}(\eta)+2Hb_{\alpha\beta}(\eta)$, applying Propositions 3.3-3.5, we can derive (3.20) directly. □

Proposition 3.6. The Gateaux derivative of Riemannian curvature tensor with respect to surface $\Im(\eta)$ is give by
$$\frac{\mathcal{D}\overset{*}{R}_{\alpha\beta\lambda\sigma}(\eta)}{\mathcal{D}\Theta}=b_{\alpha\sigma}\rho_{\beta\lambda}(\eta)+\rho_{\alpha\sigma}(\eta)b_{\beta\lambda}-b_{\alpha\lambda}\rho_{\beta\sigma}(\eta)-\rho_{\alpha\lambda}(\eta)b_{\beta\sigma}.\tag{3.21}$$
Indeed, the Riemannian curvature tensor of surface $\Im(\eta)$ is given by
$$\overset{*}{R}_{\alpha\beta\lambda\sigma}(\eta)=b_{\alpha\sigma}(\eta)b_{\beta\lambda}(\eta)-b_{\alpha\lambda}(\eta)b_{\beta\sigma}(\eta).$$
Applying Proposition 3.4 immediately yields to (3.21).

(iv) **Symmetry of indices for $\rho_{\alpha\beta}(\eta)$ and $\rho^{\lambda}_{\alpha\beta}(\eta)$**

Let us define the tensor $\rho_{\alpha\beta}(\eta)$ of order two and $\rho^{\sigma}_{\alpha\beta}(\eta)$ of order three generated by the displacement vector $\vec{\eta}$.

Proposition 3.7. The tensors $\rho_{\alpha\beta}(\eta)$ and $\rho^{\sigma}_{\alpha\beta}(\eta)$ are symmetric tensors with respect to index (α,β):
$$\rho_{\alpha\beta}(\eta)=\rho_{\beta\alpha}(\eta),\quad \rho^{\sigma}_{\alpha\beta}(\eta)=\rho^{\sigma}_{\beta\alpha}(\eta)$$
and have equivalent form
$$\begin{aligned}\rho_{\alpha\beta}(\eta):&=\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^3)+b_{\alpha\sigma}\overset{*}{\nabla}_\beta\eta^\sigma+b_{\beta\sigma}\overset{*}{\nabla}_\alpha\eta^\sigma-c_{\alpha\beta}\eta^3+\overset{*}{\nabla}_\sigma b_{\alpha\beta}\eta^\sigma\\&=\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^3)-\overset{1}{\gamma}_{\alpha\beta}(\eta),\end{aligned}\tag{3.22}$$
$$\begin{aligned}\rho^{\sigma}_{\alpha\beta}(\eta):&=\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma)-\frac{1}{2}(b_{\lambda\beta}b^{\sigma}_{\alpha}+b_{\lambda\alpha}b^{\sigma}_{\beta})\eta^\lambda\\&\quad-(b^{\sigma}_{\beta}\overset{*}{\nabla}_\alpha\eta^3+b^{\sigma}_{\alpha}\overset{*}{\nabla}_\beta\eta^3+a^{\sigma\lambda}\overset{*}{\nabla}_\lambda b_{\alpha\beta}\eta^3).\end{aligned}\tag{3.23}$$

Proof. First, we prove (3.22) and (3.23). Indeed, by virtue of (2.36) and (3.17),

$$\begin{aligned}\rho_{\alpha\beta}(\eta)=&\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^3+b_{\alpha\sigma}\overset{0}{\nabla}_\beta\eta^\sigma=\overset{*}{\nabla}_\alpha(\overset{*}{\nabla}_\beta\eta^3+b_{\beta\sigma}\eta^\sigma)+b_{\alpha\sigma}(\overset{*}{\nabla}_\beta\eta^\sigma-b^\sigma_\beta u^3)\\ =&\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3+\overset{*}{\nabla}_\alpha(b_{\beta\sigma}\eta^\sigma)+b_{\alpha\sigma}\overset{*}{\nabla}_\beta\eta^\sigma-b_{\alpha\sigma}b^\sigma_\beta\eta^3\\ =&\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3+b_{\alpha\sigma}\overset{*}{\nabla}_\beta\eta^\sigma+b_{\beta\sigma}\overset{*}{\nabla}_\alpha\eta^\sigma+\overset{*}{\nabla}_\alpha b_{\beta\sigma}\eta^\sigma-c_{\alpha\beta}\eta^3,\end{aligned}$$

since η^3 is looked as a scale function define on $\Im$ and Goddazi formula we claim

$$\begin{aligned}&\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3=\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^3,\quad \overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3=\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^3+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^3),\\ &\overset{*}{\nabla}_\alpha b_{\beta\sigma}=\overset{*}{\nabla}_\sigma b_{\alpha\beta},\quad b_{\beta\sigma}b^\sigma_\alpha=c_{\alpha\beta}.\end{aligned}$$

From this and (2.47), it implies (3.22). Next we prove (3.23). In fact, by (2.29), we claim

$$\begin{aligned}&\overset{*}{\nabla}_\alpha\overset{0}{\nabla}_\beta\eta^\sigma-b^\sigma_\alpha\overset{0}{\nabla}_\beta\eta^3\\ =&\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma-b^\sigma_\alpha b_{\beta\lambda}\eta^\lambda-(b^\sigma_\beta\overset{*}{\nabla}_\alpha\eta^3+b^\sigma_\alpha\overset{*}{\nabla}_\beta\eta^3+\overset{*}{\nabla}_\alpha b^\sigma_\beta\eta^3).\end{aligned}$$

Since the Godazzi formula and the covariant derivative of metric tensor being vanishing,

$$\overset{*}{\nabla}_\alpha b_{\beta\lambda}=\overset{*}{\nabla}_\beta b_{\alpha\lambda},\quad \overset{*}{\nabla}_\alpha a_{\lambda\sigma}=0\Rightarrow\overset{*}{\nabla}_\alpha b^\sigma_\beta\eta^3=a^{\lambda\sigma}\overset{*}{\nabla}_\lambda b_{\alpha\beta}\eta^3.$$

In addition, by virtue of the Ricci formula

$$\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma-\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma=R^\sigma_{\lambda\alpha\beta}\eta^\lambda, \tag{3.24}$$

where $R^\sigma_{\lambda\alpha\beta}$ is Riemann tensor of the 2D manifold $\Im$, and it can be expressed in terms of curvature tensor $b_{\alpha\beta}$ (see in [2]):

$$R^\sigma_{\lambda\alpha\beta}=b_{\lambda\beta}b^\sigma_\alpha-b_{\lambda\alpha}b^\sigma_\beta. \tag{3.25}$$

Hence

$$\begin{aligned}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma-b^\sigma_\alpha b_{\beta\lambda}\eta^\lambda=&\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma)+\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma-\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma)-b^\sigma_\alpha b_{\beta\lambda}\eta^\lambda\\ =&\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma)+\frac{1}{2}(b_{\lambda\beta}b^\sigma_\alpha-b_{\lambda\alpha}b^\sigma_\beta)\eta^\lambda-b^\sigma_\alpha b_{\beta\lambda}\eta^\lambda\\ =&\frac{1}{2}(\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\eta^\sigma+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\alpha\eta^\sigma)-\frac{1}{2}(b_{\lambda\beta}b^\sigma_\alpha+b_{\lambda\alpha}b^\sigma_\beta)\eta^\lambda.\end{aligned}$$

□

Finally, we end our proof for Theorem 3.1. □

4 Differential operator in the S-coordinate system

Hoge-Laplave operator under S- coordinate system

It is well known that for the Navier-Stokes equations in fluid mechanics or the Lamee-Navier equations in elastic mechanics, their principle part contain the divergence of the strain tensor for velocity vector or displacement vector. In the Riemannian space, they do not have interchangeability with Leray projector on the divergence free subspace Kerdiv, but it is possible to make interchange with the Hoge-Laplave operator. In order to make mix with either self, denote Δ_H by

$$\Delta_H = (d\delta + \delta d), \tag{4.1}$$

where d and δ are the exterior differential operator and the supper differential operator, respectively. According to the Weitzenbock formula, it is equal to Bochner-Laplace (traci-Laplace) plus Ricci operator when it acts on vector field, i.e

$$\Delta_H u^i = (d\delta + \delta d)u^i = \Delta_B u^i + (Ric\cdot u)^i, \tag{4.2}$$

where Bochner-Laplace operator is defined by

$$\Delta_B = \nabla^* \nabla = -g^{ij}\nabla_i \nabla_j = -\Delta. \tag{4.3}$$

It is well known that the conservation of the energy-momentum in physics concern divergence of enrgy-momentum tensor, the constitutive equation in continuum also contain the relationship between strain tensor and stress tensor. It is natural that the divergence of strain tensor play important role. The relationship of strain tensor, Bochner-Lpalace operator and Ricci operator in Riemannian space are given by

$$\nabla_i e^{ij}(u) = -\frac{1}{2}\Delta_B u^j + \frac{1}{2} g^{jk}\nabla_k \mathrm{div} u + \frac{1}{2} g^{jk} Ric_{mk} u^m. \tag{4.4}$$

Combining above notations, the divergence of the strain tensor in higher space and two dimensional surface are given by

$$\begin{cases} \mathrm{div} e(\mathbf{u}) = \frac{1}{2}(-\Delta_B \mathbf{u} + \nabla \mathrm{div}(\mathbf{u}) + Ric\cdot\mathbf{u}) = \frac{1}{2}(-\Delta_H \mathbf{u} + 2Ric\cdot\mathbf{u} + \nabla \mathrm{div}\mathbf{u}), \\ \overset{*}{\mathrm{div}}\, e(\mathbf{u}) = \frac{1}{2}(\overset{*}{\Delta}\mathbf{u} + \overset{*}{\nabla}\overset{*}{\mathrm{div}}\,\mathbf{u} + K\mathbf{u}), \end{cases} \tag{4.5}$$

respectively. It is obvious that it is enough to compute the Bochner-Laplace operator when we have to compute the divergence of strain tensor.

By the way in 3D-Euclidean space E^3, following is given in terms of the operator rotrot to compute divergence of strain tensor

$$\Delta \mathbf{u} = \mathrm{graddiv}\mathbf{u} - \mathrm{rotrot}\mathbf{u}.$$

The following theorem gives the expansion with respect to the transverse variable ξ for the Riemmannian curvature and the Ricci curvature tensors.

Theorem 4.1. *Under S-coordinate in the 3D-Riemmannian space, the Riemannian curvature tensor is a polynomial of degree two with respect to the transverse variable* ξ

$$\left\{\begin{array}{l} R_{\alpha\beta\lambda\sigma}=R_{\alpha\beta\lambda\sigma}(1)\xi+R_{\alpha\beta\lambda\sigma}(2)\xi^2, \quad R_{3\alpha\beta\lambda}=R_{3\alpha\beta\lambda}(1)\xi+R_{3\alpha\beta\lambda}(2)\xi^2,\\ R_{\beta\lambda 3\alpha}=R_{3\alpha\beta\lambda}, \quad R_{\alpha 3\beta\lambda}=R_{\beta\lambda\alpha 3}=-R_{3\alpha\beta\lambda},\\ R_{33\alpha\beta}=R_{\alpha\beta 33}=R_{3\alpha 3\beta}=R_{3\alpha\beta 3}=R_{\alpha 33\beta}=0,\\ R_{333\alpha}=R_{33\alpha 3}=R_{3\alpha 33}=R_{\alpha 333}=0, \quad R_{3333}=0, \end{array}\right. \tag{4.6}$$

where

$$\left\{\begin{array}{l} R_{\alpha\beta\lambda\sigma}(1)=-(\frac{\partial^2 b_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 b_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 b_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 b_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda})-(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}\\ \qquad -c_{\beta\lambda}b_{\alpha\sigma})-[\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}(\overset{*}{\nabla}_\alpha b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}b_{\mu\nu})+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}(\overset{*}{\nabla}_\beta b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\sigma}b_{\mu\nu})\\ \qquad -\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}(\overset{*}{\nabla}_\alpha b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}b_{\mu\nu})-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}(\overset{*}{\nabla}_\beta b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda}b_{\mu\nu})],\\ R_{\alpha\beta\lambda\sigma}(2)=\frac{1}{2}(\frac{\partial^2 c_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 c_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 c_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 c_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda})+(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})\\ \qquad +(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\lambda\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}-(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\sigma\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\\ \qquad +\overset{*}{\nabla}_\beta b_{\sigma\mu}(\overset{*}{\nabla}_\alpha b^\mu_\lambda+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}b^\mu_\nu)-\overset{*}{\nabla}_\beta b_{\lambda\mu}(\overset{*}{\nabla}_\alpha b^\mu_\sigma+\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}), \end{array}\right. \tag{4.7}$$

$$\left\{\begin{array}{l} R_{3\alpha\beta\lambda}(1)=\overset{*}{\nabla}_\beta c_{\alpha\lambda}-\overset{*}{\nabla}_\lambda c_{\alpha\beta},\\ R_{3\alpha\beta\lambda}(2)=b^\mu_\lambda\overset{*}{\nabla}_\alpha b_{\beta\mu}-b_{\beta\mu}\overset{*}{\nabla}_\alpha b^\mu_\lambda. \end{array}\right. \tag{4.8}$$

Proof. At the first, we give the expression for the Riemann curvature tensor of four order covariant components under S-coordinate. To do that, according to

$$\begin{aligned} R_{klij}&=\partial_i\Gamma_{lj,k}-\partial_j\Gamma_{li,k}-\Gamma^p_{li}\Gamma_{kj,p}+\Gamma^p_{lj}\Gamma_{ik,p}\\ &=\frac{1}{2}\left(\frac{\partial^2 g_{ik}}{\partial x^j\partial x^l}+\frac{\partial^2 g_{jl}}{\partial x^i\partial x^k}-\frac{\partial^2 g_{il}}{\partial x^j\partial x^k}-\frac{\partial^2 g_{jk}}{\partial x^i\partial x^l}\right)+\Gamma^p_{jl}\Gamma_{ik,p}-\Gamma^p_{il}\Gamma_{jk,p}, \end{aligned} \tag{4.9}$$

and (2.16)–(2.17), we have

$$\left\{\begin{array}{l} R_{\alpha\beta\lambda\sigma}=\frac{1}{2}(\frac{\partial^2 g_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 g_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 g_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 g_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda})+g_{pq}(\Gamma^q_{\beta\sigma}\Gamma^p_{\alpha\lambda}-\Gamma^q_{\beta\lambda}\Gamma^p_{\alpha\sigma})=I_{\alpha\beta\lambda\sigma}+II_{\alpha\beta\lambda\sigma},\\ I_{\alpha\beta\lambda\sigma}=\frac{1}{2}(\frac{\partial^2 g_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 g_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 g_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 g_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}),\\ II_{\alpha\beta\lambda\sigma}=g_{pq}(\Gamma^q_{\beta\sigma}\Gamma^p_{\alpha\lambda}-\Gamma^q_{\beta\lambda}\Gamma^p_{\alpha\sigma})=g_{\nu\mu}(\Gamma^\nu_{\beta\sigma}\Gamma^\mu_{\alpha\lambda}-\Gamma^\nu_{\beta\lambda}\Gamma^\mu_{\alpha\sigma})\\ \qquad +(\Gamma^3_{\beta\sigma}\Gamma^3_{\alpha\lambda}-\Gamma^3_{\beta\lambda}\Gamma^3_{\alpha\sigma})=II^{(1)}_{\alpha\beta\lambda\sigma}+II^{(2)}_{\alpha\beta\lambda\sigma},\\ II^{(1)}_{\alpha\beta\lambda\sigma}=g_{\nu\mu}(\Gamma^\nu_{\beta\sigma}\Gamma^\mu_{\alpha\lambda}-\Gamma^\nu_{\beta\lambda}\Gamma^\mu_{\alpha\sigma}),\\ II^{(2)}=\Gamma^3_{\beta\sigma}\Gamma^3_{\alpha\lambda}-\Gamma^3_{\beta\lambda}\Gamma^3_{\alpha\sigma}. \end{array}\right. \tag{4.10}$$

By applying

$$
\begin{aligned}
&g_{\alpha\lambda}=a_{\alpha\lambda}-2\xi b_{\alpha\lambda}+\xi^2 c_{\alpha\lambda},\\
&\Gamma^\nu_{\alpha\lambda}=\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}+\Phi^\nu_{\alpha\lambda}=\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}-\xi\overset{*}{\nabla}_\alpha b^\nu_\lambda+\xi^2(2H\delta^\nu_\eta-b^\nu_\eta)\overset{*}{\nabla}_\alpha b^\eta_\lambda,\\
&\Gamma^\nu_{3\alpha}=\theta^{-1}I^\nu_\alpha,\quad \Gamma^3_{\alpha\beta}=J_{\alpha\beta},\quad \Gamma^3_{3\lambda}=\Gamma^\alpha_{33}=0,\\
&\begin{cases}
I_{\alpha\beta\lambda\sigma}=I^a_{\alpha\beta\lambda\sigma}-2\xi I^b_{\alpha\beta\lambda\sigma}+\xi^2 I^c_{\alpha\beta\lambda\sigma},\\
I^a_{\alpha\beta\lambda\sigma}=\frac{1}{2}\big(\frac{\partial^2 a_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 a_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 a_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 a_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\big),\\
I^b_{\alpha\beta\lambda\sigma}=\frac{1}{2}\big(\frac{\partial^2 b_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 b_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 b_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 b_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\big),\\
I^c_{\alpha\beta\lambda\sigma}=\frac{1}{2}\big(\frac{\partial^2 c_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2 c_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2 c_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2 c_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\big),
\end{cases}
\end{aligned}
\tag{4.11}
$$

we have

$$
\begin{aligned}
II^{(2)}_{\alpha\beta\lambda\sigma}&=\Gamma^3_{\beta\sigma}\Gamma^3_{\alpha\lambda}-\Gamma^3_{\beta\lambda}\Gamma^3_{\alpha\sigma}=J_{\beta\sigma}J_{\alpha\lambda}-J_{\beta\lambda}J_{\alpha\sigma}\\
&=b_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}b_{\alpha\sigma}+\xi^2(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})-\xi\{(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma})\}.
\end{aligned}
$$

According to the formula for the Riemann curvature in 2D Surface ([1])

$$
\overset{*}{R}_{\alpha\beta\sigma\lambda}=b_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}b_{\alpha\sigma}. \tag{4.12}
$$

Therefore

$$
II^{(2)}_{\alpha\beta\lambda\sigma}=\overset{*}{R}_{\alpha\beta\sigma\lambda}+\xi^2(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})-\xi(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma}). \tag{4.13}
$$

Next we compute $II^{(1)}_{\alpha\beta\lambda\sigma}$. By (2.27)

$$
\begin{aligned}
II^{(1)}_{\alpha\beta\lambda\sigma}&=g_{\nu\mu}(\Gamma^\nu_{\beta\sigma}\Gamma^\mu_{\alpha\lambda}-\Gamma^\nu_{\beta\lambda}\Gamma^\mu_{\alpha\sigma})\\
&=g_{\nu\mu}\{(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}+\Phi^\nu_{\beta\sigma})(\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}+\Phi^\mu_{\alpha\lambda})-(\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}+\Phi^\nu_{\beta\lambda})(\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}+\Phi^\mu_{\alpha\sigma})\}\\
&=g_{\nu\mu}(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma})+g_{\nu\mu}(\Phi^\nu_{\beta\sigma}\Phi^\mu_{\alpha\lambda}-\Phi^\nu_{\beta\lambda}\Phi^\mu_{\alpha\sigma})\\
&\quad+g_{\nu\mu}(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\Phi^\mu_{\alpha\lambda}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}\Phi^\nu_{\beta\sigma}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\Phi^\mu_{\alpha\sigma}-\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}\Phi^\nu_{\beta\lambda}).
\end{aligned}
$$

By appling (2.27) and (2.28)

$$
\begin{cases}
g_{\nu\mu}\Phi^\mu_{\alpha\lambda}=(-\xi\delta^\eta_\nu+\xi^2 b^\eta_\nu)\overset{*}{\nabla}_\alpha b_{\lambda\eta},\quad g_{\nu\mu}\Phi^\nu_{\beta\sigma}=(-\xi\delta^\eta_\mu+\xi^2 b^\eta_\mu)\overset{*}{\nabla}_\beta b_{\sigma\eta},\\
g_{\nu\mu}\Phi^\mu_{\alpha\sigma}=(-\xi\delta^\eta_\nu+\xi^2 b^\eta_\nu)\overset{*}{\nabla}_\alpha b_{\sigma\eta},\quad g_{\nu\mu}\Phi^\nu_{\beta\lambda}=(-\xi\delta^\eta_\mu+\xi^2 b^\eta_\mu)\overset{*}{\nabla}_\beta b_{\lambda\eta},
\end{cases}
\tag{4.14}
$$

it yields

$$
\begin{aligned}
&g_{\nu\mu}\{\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\Phi^\mu_{\alpha\lambda}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}\Phi^\nu_{\beta\sigma}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\Phi^\mu_{\alpha\sigma}-\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}\Phi^\nu_{\beta\lambda}\}\\
=&(-\xi\delta^\eta_\nu+\xi^2 b^\eta_\nu)\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\eta}+(-\xi\delta^\eta_\mu+\xi^2 b^\eta_\mu)\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}\overset{*}{\nabla}_\beta b_{\sigma\eta}\\
&-(-\xi\delta^\eta_\nu+\xi^2 b^\eta_\nu)\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\nabla}_\alpha b_{\sigma\eta}-(-\xi\delta^\eta_\mu+\xi^2 b^\eta_\mu)\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}\overset{*}{\nabla}_\beta b_{\lambda\eta}\\
=&(-\xi\delta^\eta_\nu+\xi^2 b^\eta_\nu)(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\nabla}_\eta b_{\alpha\lambda}+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}\overset{*}{\nabla}_\eta b_{\beta\sigma}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\nabla}_\eta b_{\alpha\sigma}-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}\overset{*}{\nabla}_\eta b_{\beta\lambda}).
\end{aligned}
$$

Using

$$\Phi^{\mu}_{\alpha\lambda}=\theta^{-1}(-\xi\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}+\xi^2(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})\overset{*}{\nabla}_{\alpha}b^{\eta}_{\lambda})$$

and (4.14), we have

$$\begin{aligned}
g_{\nu\mu}\Phi^{\nu}_{\beta\sigma}\Phi^{\mu}_{\alpha\lambda}=&(-\xi\overset{*}{\nabla}_{\beta}b_{\sigma\mu}+\xi^2b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\sigma\eta})\cdot\theta^{-1}(-\xi\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}+\xi^2(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})\overset{*}{\nabla}_{\alpha}b^{\eta}_{\lambda})\\
=&\theta^{-1}\{\xi^2\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}\overset{*}{\nabla}_{\beta}b_{\sigma\mu}-\xi^3[(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})\overset{*}{\nabla}_{\beta}b_{\sigma\mu}\overset{*}{\nabla}_{\alpha}b^{\eta}_{\lambda}+b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\sigma\eta}\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}]\\
&+\xi^4((2H\delta^{\mu}_{\tau}-b^{\mu}_{\tau})b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\sigma\eta}\overset{*}{\nabla}_{\alpha}b^{\tau}_{\lambda})\}=\theta^{-1}\xi^2(1-2H\xi+K\xi^2)\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}\overset{*}{\nabla}_{\beta}b_{\sigma\mu}\\
=&\xi^2\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}\overset{*}{\nabla}_{\beta}b_{\sigma\mu},\\
g_{\nu\mu}\Phi^{\nu}_{\beta\lambda}\Phi^{\mu}_{\alpha\sigma}=&(-\xi\overset{*}{\nabla}_{\beta}b_{\lambda\mu}+\xi^2b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\lambda\eta})\theta^{-1}(-\xi\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}+\xi^2(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})\overset{*}{\nabla}_{\alpha}b^{\eta}_{\sigma})\\
=&\theta^{-1}\{\xi^2\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}\overset{*}{\nabla}_{\beta}b_{\lambda\mu}-\xi^3[(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})\overset{*}{\nabla}_{\beta}b_{\lambda\mu}\overset{*}{\nabla}_{\alpha}b^{\eta}_{\sigma}+b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\lambda\eta}\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}]\\
&+\xi^4((2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})b^{\tau}_{\mu}\overset{*}{\nabla}_{\beta}b_{\lambda\tau}\overset{*}{\nabla}_{\alpha}b^{\eta}_{\sigma})\}=\xi^2\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}\overset{*}{\nabla}_{\beta}b_{\lambda\mu},
\end{aligned}$$

where we have used the following two equalities

$$\left\{\begin{array}{l}
-b^{\mu}_{\eta}\overset{*}{\nabla}_{\beta}b_{\lambda\mu}\overset{*}{\nabla}_{\alpha}b^{\eta}_{\sigma}+b^{\eta}_{\mu}\overset{*}{\nabla}_{\beta}b_{\lambda\eta}\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}=0,\\
(2H\delta^{\mu}_{\eta}-b^{\mu}_{\eta})b^{\tau}_{\mu}=2Hb^{\tau}_{\eta}-c^{\tau}_{\eta}=K\delta^{\tau}_{\eta}.
\end{array}\right.$$

Finally,

$$g_{\nu\mu}(\Phi^{\nu}_{\beta\sigma}\Phi^{\mu}_{\alpha\lambda}-\Phi^{\nu}_{\beta\lambda}\Phi^{\mu}_{\alpha\sigma})=\xi^2a^{\nu\mu}(\overset{*}{\nabla}_{\nu}b_{\alpha\lambda}\overset{*}{\nabla}_{\mu}b_{\beta\sigma}-\overset{*}{\nabla}_{\nu}b_{\alpha\sigma}\overset{*}{\nabla}_{\mu}b_{\beta\lambda}),\tag{4.15}$$

it still possess anti-symmetric of indices. Combing (4.14) and (4.15) with (4.12) yields

$$\begin{aligned}
II^{(1)}_{\alpha\beta\lambda\sigma}=&(a_{\nu\mu}-2\xi b_{\nu\mu}+\xi^2c_{\nu\mu})(\overset{*}{\Gamma}{}^{\nu}{}_{\beta\sigma}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^{\nu}{}_{\beta\lambda}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\sigma})+\xi^2a^{\nu\mu}(\overset{*}{\nabla}_{\nu}b_{\alpha\lambda}\overset{*}{\nabla}_{\mu}b_{\beta\sigma}-\overset{*}{\nabla}_{\nu}b_{\alpha\sigma}\overset{*}{\nabla}_{\mu}b_{\beta\lambda})\\
&+(-\xi\delta^{\eta}_{\nu}+\xi^2b^{\eta}_{\nu})(\overset{*}{\Gamma}{}^{\nu}{}_{\beta\sigma}\overset{*}{\nabla}_{\eta}b_{\alpha\lambda}+\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\lambda}\overset{*}{\nabla}_{\eta}b_{\beta\sigma}-\overset{*}{\Gamma}{}^{\nu}{}_{\beta\lambda}\overset{*}{\nabla}_{\eta}b_{\alpha\sigma}-\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\sigma}\overset{*}{\nabla}_{\eta}b_{\beta\lambda}).
\end{aligned}\tag{4.16}$$

Substituting (4.11), (4.12) and (4.16) into (4.9) leads to

$$\begin{aligned}
R_{\alpha\beta\lambda\sigma}=&\overset{*}{R}_{\alpha\beta\sigma\lambda}+\frac{1}{2}\Big(\frac{\partial^2a_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2a_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2a_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2a_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\Big)+a_{\nu\mu}(\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\lambda}\overset{*}{\Gamma}{}^{\mu}{}_{\beta\sigma}-\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\sigma}\overset{*}{\Gamma}{}^{\mu}{}_{\beta\lambda})\\
&+\xi\Big\{-\Big(\frac{\partial^2b_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2b_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2b_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2b_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\Big)-2b_{\nu\mu}(\overset{*}{\Gamma}{}^{\nu}{}_{\beta\sigma}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^{\nu}{}_{\beta\lambda}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\sigma})\\
&-(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma})\Big\}+\xi^2\Big\{\frac{1}{2}\Big(\frac{\partial^2c_{\alpha\lambda}}{\partial x^\beta\partial x^\sigma}+\frac{\partial^2c_{\beta\sigma}}{\partial x^\alpha\partial x^\lambda}-\frac{\partial^2c_{\beta\lambda}}{\partial x^\alpha\partial x^\sigma}-\frac{\partial^2c_{\alpha\sigma}}{\partial x^\beta\partial x^\lambda}\Big)\\
&+c_{\nu\mu}(\overset{*}{\Gamma}{}^{\nu}{}_{\beta\sigma}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^{\nu}{}_{\beta\lambda}\overset{*}{\Gamma}{}^{\mu}{}_{\alpha\sigma})+(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})\Big\}\\
&+(-\xi\delta^{\eta}_{\nu}+\xi^2b^{\eta}_{\nu})(\overset{*}{\Gamma}{}^{\nu}{}_{\beta\sigma}\overset{*}{\nabla}_{\alpha}b_{\lambda\eta}+\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\lambda}\overset{*}{\nabla}_{\beta}b_{\sigma\eta}-\overset{*}{\Gamma}{}^{\nu}{}_{\beta\lambda}\overset{*}{\nabla}_{\alpha}b_{\sigma\eta}-\overset{*}{\Gamma}{}^{\nu}{}_{\alpha\sigma}\overset{*}{\nabla}_{\beta}b_{\lambda\eta})\\
&+\xi^2(\overset{*}{\nabla}_{\alpha}b^{\mu}_{\lambda}\overset{*}{\nabla}_{\beta}b_{\sigma\mu}-\overset{*}{\nabla}_{\alpha}b^{\mu}_{\sigma}\overset{*}{\nabla}_{\beta}b_{\lambda\mu}).
\end{aligned}\tag{4.17}$$

Owing to the anti-symmetric of index of Riemann curvature tensor

$$\overset{*}{R}_{\alpha\beta\lambda\sigma}=-\overset{*}{R}_{\alpha\beta\sigma\lambda},$$

the sum of first three terms in (4.17) equal to zero

$$\frac{1}{2}\Big(\frac{\partial^2 a_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 a_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 a_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 a_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big) \\ +a_{\nu\mu}(\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}\overset{*}{\Gamma}{}^\mu{}_{\beta\sigma}-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda})+\overset{*}{R}_{\alpha\beta\sigma\lambda}=\overset{*}{R}_{\alpha\beta\lambda\sigma}+\overset{*}{R}_{\alpha\beta\sigma\lambda}=0.$$

Then (4.17) becomes

$$\begin{aligned}R_{\alpha\beta\lambda\sigma}=\xi\{&-\Big(\frac{\partial^2 b_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 b_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 b_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 b_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)-2b_{\nu\mu}(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma})\\ &-(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma})-(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\nu}+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}\overset{*}{\nabla}_\beta b_{\sigma\nu}\\ &-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\nabla}_\alpha b_{\sigma\nu}-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}\overset{*}{\nabla}_\beta b_{\lambda\nu})\}+\xi^2\Big\{\frac{1}{2}\Big(\frac{\partial^2 c_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 c_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 c_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 c_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)\\ &+c_{\nu\mu}(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma})+(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})+b^\eta_\nu(\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}\overset{*}{\nabla}_\alpha b_{\lambda\eta}+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}\overset{*}{\nabla}_\beta b_{\sigma\eta}\\ &-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\overset{*}{\nabla}_\alpha b_{\sigma\eta}-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}\overset{*}{\nabla}_\beta b_{\lambda\eta})+(\overset{*}{\nabla}_\alpha b^\mu_\lambda\overset{*}{\nabla}_\beta b_{\sigma\mu}-\overset{*}{\nabla}_\alpha b^\mu_\sigma\overset{*}{\nabla}_\beta b_{\lambda\mu})\Big\}.\end{aligned}$$

It can be also expressed by

$$\begin{aligned}R_{\alpha\beta\lambda\sigma}=\xi\{&-\Big(\frac{\partial^2 b_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 b_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 b_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 b_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)-(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma})\\ &-[\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}(\overset{*}{\nabla}_\alpha b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}b_{\mu\nu})+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}(\overset{*}{\nabla}_\beta b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\sigma}b_{\mu\nu})-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}(\overset{*}{\nabla}_\alpha b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}b_{\mu\nu})\\ &-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}(\overset{*}{\nabla}_\beta b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda}b_{\mu\nu})]\}+\xi^2\Big\{\frac{1}{2}\Big(\frac{\partial^2 c_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 c_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 c_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 c_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)\\ &+(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})+(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\lambda\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}-(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\sigma\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\\ &+\overset{*}{\nabla}_\beta b_{\sigma\mu}(\overset{*}{\nabla}_\alpha b^\mu_\lambda+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}b^\mu_\nu)-\overset{*}{\nabla}_\beta b_{\lambda\mu}(\overset{*}{\nabla}_\alpha b^\mu_\sigma+\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma})\Big\}.\end{aligned} \tag{4.18}$$

Let us define a tensor of four order covariant components

$$\begin{aligned}R_{\alpha\beta\lambda\sigma}(1)=&-\Big(\frac{\partial^2 b_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 b_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 b_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 b_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)-(c_{\alpha\lambda}b_{\beta\sigma}+c_{\beta\sigma}b_{\alpha\lambda}-b_{\beta\lambda}c_{\alpha\sigma}-c_{\beta\lambda}b_{\alpha\sigma})\\ &-[\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}(\overset{*}{\nabla}_\alpha b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}b_{\mu\nu})+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}(\overset{*}{\nabla}_\beta b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\sigma}b_{\mu\nu})\\ &-\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}(\overset{*}{\nabla}_\alpha b_{\sigma\nu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}b_{\mu\nu})-\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}(\overset{*}{\nabla}_\beta b_{\lambda\nu}+\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda}b_{\mu\nu})],\\ R_{\alpha\beta\lambda\sigma}(2)=&\frac{1}{2}\Big(\frac{\partial^2 c_{\alpha\lambda}}{\partial x^\beta \partial x^\sigma}+\frac{\partial^2 c_{\beta\sigma}}{\partial x^\alpha \partial x^\lambda}-\frac{\partial^2 c_{\beta\lambda}}{\partial x^\alpha \partial x^\sigma}-\frac{\partial^2 c_{\alpha\sigma}}{\partial x^\beta \partial x^\lambda}\Big)+(c_{\beta\sigma}c_{\alpha\lambda}-c_{\beta\lambda}c_{\alpha\sigma})\\ &+(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\lambda\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\sigma}-(b^\mu_\nu\overset{*}{\nabla}_\alpha b_{\sigma\mu}+\overset{*}{\Gamma}{}^\mu{}_{\alpha\sigma}c_{\nu\mu})\overset{*}{\Gamma}{}^\nu{}_{\beta\lambda}\\ &+\overset{*}{\nabla}_\beta b_{\sigma\mu}(\overset{*}{\nabla}_\alpha b^\mu_\lambda+\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda}b^\mu_\nu)-\overset{*}{\nabla}_\beta b_{\lambda\mu}(\overset{*}{\nabla}_\alpha b^\mu_\sigma+\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma}).\end{aligned} \tag{4.19}$$

Then (4.17) becomes

$$R_{\alpha\beta\lambda\sigma}=R_{\alpha\beta\lambda\sigma}(1)\xi+R_{\alpha\beta\lambda\sigma}(2)\xi^2. \tag{4.20}$$

The remaining is to prove formula (4.8)

$$\begin{aligned}R_{3\alpha\beta\lambda}=&\frac{1}{2}\Big(\frac{\partial^2 g_{3\beta}}{\partial x^\beta \partial x^\lambda}+\frac{\partial^2 g_{\alpha\lambda}}{\partial \xi \partial x^\beta}-\frac{\partial^2 g_{\alpha\beta}}{\partial \xi \partial x^\lambda}-\frac{\partial^2 g_{3\lambda}}{\partial x^\beta \partial x^\alpha}\Big)+g_{\nu\mu}(\Gamma^\nu_{3\beta}\Gamma^\mu_{\alpha\beta}-\Gamma^\nu_{\alpha\beta}\Gamma^\mu_{3\lambda})+g_{33}(\Gamma^3_{3\beta}\Gamma^3_{\alpha\lambda}-\Gamma^3_{\alpha\beta}\Gamma^3_{3\lambda})\\ =&(\text{used}(2.16)(2.27))\{\partial_\beta(-b_{\alpha\lambda}+\xi c_{\alpha\lambda})-\partial_\lambda(-b_{\alpha\beta}+\xi c_{\alpha\beta})\}\\ &+g_{\nu\mu}(\theta^{-1}I^\nu_\beta(\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}+\Phi^\mu_{\alpha\lambda})-\theta^{-1}I^\mu_\lambda(\overset{*}{\Gamma}{}^\nu{}_{\alpha\beta}+\Phi^\nu_{\alpha\beta}).\end{aligned} \tag{4.21}$$

Using

$$\begin{aligned}
&\overset{*}{\nabla}_\beta b_{\alpha\lambda}=\partial_\beta b_{\alpha\lambda}-\overset{*}{\Gamma}{}^\mu{}_{\beta\alpha}b_{\mu\lambda}-\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda}b_{\alpha\mu},\quad \overset{*}{\nabla}_\lambda b_{\alpha\beta}=\partial_\lambda b_{\alpha\beta}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\alpha}b_{\mu\beta}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\beta}b_{\alpha\mu},\\
&\partial_\beta b_{\alpha\lambda}-\partial_\lambda b_{\alpha\beta}=\overset{*}{\nabla}_\beta b_{\alpha\lambda}-\overset{*}{\nabla}_\lambda b_{\alpha\beta}+\overset{*}{\Gamma}{}^\mu{}_{\beta\alpha}b_{\mu\lambda}+\overset{*}{\Gamma}{}^\mu{}_{\beta\lambda}b_{\alpha\mu}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\alpha}b_{\mu\beta}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\beta}b_{\alpha\mu}\\
&\qquad\qquad\qquad\quad=\overset{*}{\nabla}_\beta b_{\alpha\lambda}-\overset{*}{\nabla}_\lambda b_{\alpha\beta}+\overset{*}{\Gamma}{}^\mu{}_{\beta\alpha}b_{\mu\lambda}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\alpha}b_{\mu\beta},\\
&\partial_\beta c_{\alpha\lambda}-\partial_\lambda c_{\alpha\beta}=\overset{*}{\nabla}_\beta c_{\alpha\lambda}-\overset{*}{\nabla}_\lambda c_{\alpha\beta}+\overset{*}{\Gamma}{}^\mu{}_{\beta\alpha}c_{\mu\lambda}-\overset{*}{\Gamma}{}^\mu{}_{\lambda\alpha}c_{\mu\beta},
\end{aligned}$$

and the Godazzi formula

$$\overset{*}{\nabla}_\lambda b_{\alpha\beta}=\overset{*}{\nabla}_\beta b_{\alpha\lambda},\tag{4.22}$$

we obtain

$$\begin{aligned}
&\partial_\beta(-b_{\alpha\lambda}+\xi c_{\alpha\lambda})-\partial_\lambda(-b_{\alpha\beta}+\xi c_{\alpha\beta})=-\{\partial_\beta b_{\alpha\lambda}-\partial_\lambda b_{\alpha\beta}-\xi(\partial_\beta c_{\alpha\lambda}-\partial_\lambda c_{\alpha\beta})\}\\
&\quad=\xi(\overset{*}{\nabla}_\beta c_{\alpha\lambda}-\overset{*}{\nabla}_\lambda c_{\alpha\beta})+\overset{*}{\Gamma}{}^\mu{}_{\lambda\alpha}J_{\mu\beta}-\overset{*}{\Gamma}{}^\mu{}_{\beta\alpha}J_{\mu\lambda}.
\end{aligned}\tag{4.23}$$

On the other hand, by (2.41)

$$\begin{cases}
g_{\nu\mu}\theta^{-1}I^\nu_\beta=-J_{\beta\mu},\quad g_{\nu\mu}\theta^{-1}I^\mu_\lambda=-J_{\lambda\nu},\\
J_{\beta\mu}\Phi^\mu_{\alpha\lambda}=-\xi\overset{*}{\nabla}_\lambda b_{\alpha\beta}+\xi^2 b^\mu_\beta\overset{*}{\nabla}_\lambda b_{\alpha\mu},\quad J_{\lambda\nu}\Phi^\nu_{\alpha\beta}=-\xi\overset{*}{\nabla}_\beta b_{\alpha\lambda}+\xi^2 b^\mu_\lambda\overset{*}{\nabla}_\beta b_{\alpha\mu}.
\end{cases}\tag{4.24}$$

Substituting into (4.24) leads to

$$\begin{aligned}
&g_{\nu\mu}(\theta^{-1}I^\nu_\beta(\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}+\Phi^\mu_{\alpha\lambda})-\theta^{-1}I^\mu_\lambda(\overset{*}{\Gamma}{}^\nu{}_{\alpha\beta}+\Phi^\nu_{\alpha\beta}))=J_{\lambda\nu}\overset{*}{\Gamma}{}^\nu{}_{\alpha\beta}-J_{\beta\nu}\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}+\xi(\overset{*}{\nabla}_\lambda b_{\alpha\beta}-\overset{*}{\nabla}_\beta b_{\alpha\lambda})\\
&\quad+\xi^2(b^\mu_\lambda\overset{*}{\nabla}_\alpha b_{\beta\mu}-b^\mu_\beta\overset{*}{\nabla}_\lambda b_{\alpha\mu})\ \ (\text{Godazzi formula}(4.24))\\
&\quad=J_{\lambda\nu}\overset{*}{\Gamma}{}^\nu{}_{\alpha\beta}-J_{\beta\nu}\overset{*}{\Gamma}{}^\mu{}_{\alpha\lambda}+\xi^2(b^\mu_\lambda\overset{*}{\nabla}_\alpha b_{\beta\mu}-b^\mu_\beta\overset{*}{\nabla}_\lambda b_{\alpha\mu}).
\end{aligned}\tag{4.25}$$

Combining (4.22), (4.24) and (4.25) yields

$$R_{3\alpha\beta\lambda}=\xi(\overset{*}{\nabla}_\beta c_{\alpha\lambda}-\overset{*}{\nabla}_\lambda c_{\alpha\beta})+\xi^2(b^\mu_\lambda\overset{*}{\nabla}_\alpha b_{\beta\mu}-b_{\beta\mu}\overset{*}{\nabla}_\alpha b^\mu_\lambda).\tag{4.26}$$

By symmetry and anti-symmetry of indices for Rimemann curture tensor, we obtain

$$R_{\beta\lambda3\alpha}=R_{3\alpha\beta\lambda},\quad R_{\alpha3\beta\lambda}=-R_{3\alpha\beta\lambda},\quad R_{\beta\lambda\alpha3}=-R_{3\alpha\beta\lambda}.\tag{4.27}$$

In what follows,

$$\begin{aligned}
&R_{33\alpha\beta}=\tfrac12\{\partial^2(g_{3\alpha})_{3\beta}+\partial^2(g_{3\beta})_{3\alpha}-\partial^2(g_{3\alpha})_{3\beta}-\partial^2(g_{3\beta})_{3\alpha}\}+g_{pq}(\Gamma^p_{3\beta}\Gamma^q_{3\alpha}-\Gamma^p_{3\beta}\Gamma^3_{3\alpha})=0,\\
&R_{\alpha\beta33}=0,\\
&R_{3\alpha3\beta}=\tfrac12\{\partial^2(g_{33})_{\alpha\beta}+\partial^2(g_{\alpha\beta})_{33}-\partial^2(g_{\alpha3})_{3\beta}-\partial^2(g_{3\beta})_{\alpha3}\}+g_{pq}(\Gamma^p_{\alpha\beta}\Gamma^q_{33}-\Gamma^p_{3\beta}\Gamma^q_{3\alpha})\\
&\quad=\tfrac12\frac{\partial^2(a_{\alpha\beta}-2\xi b_{\alpha\beta}+\xi^2c_{\alpha\beta})}{\partial\xi^2}+g_{\nu\mu}(0-\theta^{-1}I^\nu_\beta\theta^{-1}I^\mu_\alpha)+(\Gamma^3_{\alpha\beta}\Gamma^3_{33}-\Gamma^3_{3\beta}\Gamma^3_{3\alpha})\\
&\quad=c_{\alpha\beta}-c_{\alpha\beta}=0,\\
&R_{\alpha33\beta}=0,\quad R_{3\alpha\beta3}=0,\quad R_{333\alpha}=R_{33\alpha3}=R_{3\alpha33}=R_{\alpha333}=0,\quad R_{3333}=0.
\end{aligned}\tag{4.28}$$

This gives $\Gamma^{\nu}_{33}=\Gamma^3_{3\alpha}=0$, $g_{\nu\mu}I^{\nu}_{\beta}I^{\mu}_{\alpha}=\theta^2 c_{\alpha\beta}$. From this we complete proof of Theorem 4.1. □

Theorem 4.2. *Under the S-coordinate in the 3D-Riemmannian space, the Ricci curvature tensor is a rational polynomial of degree two with respect to the transverse variable ξ whose Taylor expansion is given by*

$$\begin{cases} R_{ij}=R_{ij}(0)+R_{ij}(1)\xi+R_{ij}(2)\xi^2+\cdots, \\ R_{\alpha\beta}(0)=0, \\ R_{\alpha\beta}(1)=\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\overset{*}{\nabla}_\alpha b^\lambda_\beta)+2K(b_{\alpha\beta}-Ha_{\alpha\beta})+2H_{\alpha\beta}+\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\sigma}\overset{*}{\nabla}_\lambda b^\sigma_\beta+\overset{*}{\Gamma}{}^{\sigma}{}_{\beta\lambda}\overset{*}{\nabla}_\sigma b^\lambda_\alpha, \\ R_{\alpha\beta}(2)=(4HH_{\alpha\beta}+4H_\alpha H_\beta-K_{\alpha\beta})+K((K-4H^2)a_{\alpha\beta}+4Hb_{\alpha\beta}) \\ \qquad +\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}((2H\delta^\lambda_\nu-b^\lambda_\nu)\sqrt{a}\overset{*}{\nabla}_\alpha b^\nu_\beta) \\ \qquad -[(2H\delta^\lambda_\nu-b^\lambda_\nu)(\overset{*}{\Gamma}{}^{\sigma}{}_{\lambda\beta}\overset{*}{\nabla}_\alpha b^\nu_\sigma+\overset{*}{\Gamma}{}^{\sigma}{}_{\lambda\alpha}\overset{*}{\nabla}_\beta b^\nu_\sigma)+\overset{*}{\nabla}_\lambda b^\sigma_\alpha\overset{*}{\nabla}_\sigma b^\lambda_\beta], \\ R_{3\beta}(0)=4\overset{*}{\nabla}_\beta H-2b^\lambda_\sigma\overset{*}{\Gamma}{}^{\sigma}{}_{\lambda\beta}, \\ R_{3\beta}(1)=\overset{*}{\nabla}_\beta(8H^2-2K)+2K\overset{*}{\nabla}_\beta\ln\sqrt{a}-4Hb^\lambda_\sigma\overset{*}{\Gamma}{}^{\sigma}{}_{\lambda\beta}, \\ R_{3\beta}(2)=8(6H^2-K)\overset{*}{\nabla}_\beta H-(4H\delta^\lambda_\beta+2b^\lambda_\beta)\overset{*}{\nabla}_\beta K+4HK\overset{*}{\nabla}_\beta\ln\sqrt{a}-(8H^2-2K)b^\lambda_\sigma\overset{*}{\Gamma}{}^{\sigma}{}_{\lambda\beta}, \\ R_{33}=0, \end{cases} \tag{4.29}$$

where $H_\alpha=\partial_\alpha H$, $H_{\alpha\beta}=\partial^2_{\alpha\beta}H$.

Proof. Applying Lemmas 2.2, 2.3 and 2.5, and $g=\theta^2 a$, we have

$$\begin{aligned} R_{\alpha\beta}&=-\frac{\partial^2}{\partial x^\alpha\partial x^\beta}\ln\sqrt{g}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}(\sqrt{g}\Gamma^\lambda_{\alpha\beta})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^3}(\sqrt{g}\Gamma^3_{\alpha\beta})-\Gamma^\lambda_{\alpha\sigma}\Gamma^\sigma_{\lambda\beta}-\Gamma^\lambda_{\alpha3}\Gamma^3_{\lambda\beta}-\Gamma^3_{\alpha\sigma}\Gamma^\sigma_{3\beta}-\Gamma^3_{\alpha3}\Gamma^3_{3\beta}, \\ R_{\alpha\beta}&=-\frac{\partial^2\ln\theta}{\partial x^\alpha\partial x^\beta}+\frac{\partial^2\ln\sqrt{a}}{\partial x^\alpha\partial x^\beta}+\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Gamma^\lambda_{\alpha\beta})+\Gamma^\lambda_{\alpha\beta}\frac{\partial\ln\theta}{\partial x^\lambda} \\ &\quad+\frac{1}{\sqrt{a}}\frac{\partial}{\partial\xi}(\sqrt{a}\Gamma^3_{\alpha\beta})+\Gamma^3_{\alpha\beta}\frac{\partial\ln\theta}{\partial\xi}-\Gamma^\lambda_{\alpha\sigma}\Gamma^\sigma_{\lambda\beta}-\theta^{-1}(J_{\alpha\lambda}I^\lambda_\beta+J_{\beta\sigma}I^\sigma_\alpha). \end{aligned}$$

On the other hand,

$$\begin{aligned} &\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Gamma^\lambda_{\alpha\beta})=\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta})+\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Phi^\lambda_{\alpha\beta}), \\ &\Gamma^\lambda_{\alpha\beta}\frac{\partial\ln\theta}{\partial x^\lambda}=\theta^{-1}(\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}+\Phi^\lambda_{\alpha\beta})(-2H_\lambda\xi+K_\lambda\xi^2) \\ &\qquad=\theta^{-1}\{-2\overset{*}{\nabla}_\lambda H\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}\xi+(\overset{*}{\nabla}_\lambda K\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}+2\overset{*}{\nabla}_\lambda H\overset{*}{\nabla}_\alpha b^\lambda_\beta)\xi^2 \\ &\qquad\quad-(\overset{*}{\nabla}_\nu(K+2H^2)+b^\lambda_\nu\overset{*}{\nabla}_\lambda H)\overset{*}{\nabla}_\alpha b^\nu_\beta\xi^3+(2H\delta^\lambda_\nu-b^\lambda_\nu)\overset{*}{\nabla}_\lambda K\overset{*}{\nabla}_\alpha b^\nu_\beta\xi^4\}, \\ &\frac{1}{\sqrt{a}}\frac{\partial}{\partial\xi}(\sqrt{a}\Gamma^3_{\alpha\beta})=\frac{\partial J_{\alpha\beta}}{\partial\xi}=-c_{\alpha\beta}, \\ &\Gamma^3_{\alpha\beta}\frac{\partial\ln\theta}{\partial\xi}=J_{\alpha\beta}\theta^{-1}(-2H+2K\xi)=\theta^{-1}\{-2Hb_{\alpha\beta}+2(Kb_{\alpha\beta}+Hc_{\alpha\beta})\xi-2Kc_{\alpha\beta}\xi^2\}, \\ &-\theta^{-1}[J_{\alpha\lambda}I^\lambda_\beta+J_{\beta\sigma}I^\sigma_\alpha]=2c_{\alpha\beta}, \end{aligned}$$

$$
\begin{aligned}
\Gamma^\lambda_{\alpha\sigma}\Gamma^\sigma_{\lambda\beta} =& \overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta} + \overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\Phi^\sigma_{\lambda\beta} + \Phi^\lambda_{\alpha\sigma}\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta} + \Phi^\lambda_{\alpha\sigma}\Phi^\sigma_{\beta\lambda} \\
=& \overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta} + (\overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\nabla}_\lambda b^\sigma_\beta + \overset{*}{\Gamma}{}^\sigma{}_{\beta\lambda}\overset{*}{\nabla}_\sigma b^\lambda_\alpha)\xi \\
&+ [(2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}\overset{*}{\nabla}_\alpha b^\nu_\sigma + \overset{*}{\Gamma}{}^\sigma{}_{\lambda\alpha}\overset{*}{\nabla}_\beta b^\nu_\sigma) + \overset{*}{\nabla}_\lambda b^\sigma_\alpha \overset{*}{\nabla}_\sigma b^\lambda_\beta]\xi^2 \\
&- (2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\nabla}_\alpha b^\sigma_\lambda \overset{*}{\nabla}_\beta b^\nu_\sigma + \overset{*}{\nabla}_\beta b^\sigma_\lambda \overset{*}{\nabla}_\alpha b^\nu_\sigma)\xi^3 \\
&+ (2H\delta^\lambda_\nu - b^\lambda_\nu)(2H\delta^\sigma_\mu - b^\sigma_\mu)\overset{*}{\nabla}_\alpha b^\nu_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda \xi^4.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
R_{\alpha\beta} =& -\frac{\partial^2 \ln\theta}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 \ln\sqrt{a}}{\partial x^\alpha \partial x^\beta} + \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\,\overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta}) + \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Phi^\lambda_{\alpha\beta}) \\
&- \overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta} + c_{\alpha\beta} + \theta^{-1}(-2Hb_{\alpha\beta} + (2Hc_{\alpha\beta} + 2Kb_{\alpha\beta})\xi - 2Kc_{\alpha\beta}\xi^2) \\
&+ (\overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\nabla}_\lambda b^\sigma_\beta + \overset{*}{\Gamma}{}^\sigma{}_{\beta\lambda}\overset{*}{\nabla}_\sigma b^\lambda_\alpha)\xi + [(2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}\overset{*}{\nabla}_\alpha b^\nu_\sigma + \overset{*}{\Gamma}{}^\sigma{}_{\lambda\alpha}\overset{*}{\nabla}_\beta b^\nu_\sigma) \\
&+ \overset{*}{\nabla}_\lambda b^\sigma_\alpha \overset{*}{\nabla}_\sigma b^\lambda_\beta]\xi^2 + (2H\delta^\lambda_\nu - b^\lambda_\nu)(2H\delta^\sigma_\mu - b^\sigma_\mu)\overset{*}{\nabla}_\alpha b^\nu_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda \xi^4 \\
&- (2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\nabla}_\alpha b^\sigma_\lambda \overset{*}{\nabla}_\beta b^\nu_\sigma + \overset{*}{\nabla}_\beta b^\sigma_\lambda \overset{*}{\nabla}_\alpha b^\nu_\sigma)\xi^3.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&-\frac{\partial^2 \ln\sqrt{a}}{\partial x^\alpha \partial x^\beta} + \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\,\overset{*}{\Gamma}{}^\lambda{}_{\alpha\beta}) - \overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta} = \overset{*}{R}_{\alpha\beta}, \\
&c_{\alpha\beta} = -Ka_{\alpha\beta} + 2Hb_{\alpha\beta},
\end{aligned}
$$

we have

$$
\overset{*}{R}_{\alpha\beta} - Ka_{\alpha\beta} = 0.
$$

Furthermore,

$$
\begin{aligned}
R_{\alpha\beta} =& -\frac{\partial^2 \ln\theta}{\partial x^\alpha \partial x^\beta} + \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Phi^\lambda_{\alpha\beta}) + 2Hb_{\alpha\beta} + \theta^{-1}(-2Hb_{\alpha\beta} + 2(Hc_{\alpha\beta} + Kb_{\alpha\beta})\xi - 2Kc_{\alpha\beta}\xi^2) \\
&+ (\overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\nabla}_\lambda b^\sigma_\beta + \overset{*}{\Gamma}{}^\sigma{}_{\beta\lambda}\overset{*}{\nabla}_\sigma b^\lambda_\alpha)\xi + [(2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}\overset{*}{\nabla}_\alpha b^\nu_\sigma + \overset{*}{\Gamma}{}^\sigma{}_{\lambda\alpha}\overset{*}{\nabla}_\beta b^\nu_\sigma) + \overset{*}{\nabla}_\lambda b^\sigma_\alpha \overset{*}{\nabla}_\sigma b^\lambda_\beta]\xi^2 \\
&+ (2H\delta^\lambda_\nu - b^\lambda_\nu)(2H\delta^\sigma_\mu - b^\sigma_\mu)\overset{*}{\nabla}_\alpha b^\nu_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda \xi^4 - (2H\delta^\lambda_\nu - b^\lambda_\nu)(\overset{*}{\nabla}_\alpha b^\sigma_\lambda \overset{*}{\nabla}_\beta b^\nu_\sigma + \overset{*}{\nabla}_\beta b^\sigma_\lambda \overset{*}{\nabla}_\alpha b^\nu_\sigma)\xi^3.
\end{aligned}
$$

Thanks to

$$
\begin{aligned}
&\frac{\partial^2 \ln\theta}{\partial x^\alpha \partial x^\beta} = \theta^{-2}(-2H_{\alpha\beta}\xi + (K_{\alpha\beta} + 4HH_{\alpha\beta} - 4H_\alpha H_\beta)\xi^2 + (2H_\alpha K_\beta + 2H_\beta K_\alpha - 2HK_{\alpha\beta} \\
&\qquad\qquad -2KH_{\alpha\beta})\xi^3 + (K_\alpha K_\beta + KK_{\alpha\beta})\xi^4) = -2H_{\alpha\beta}\xi + (K_{\alpha\beta} - 4HH_{\alpha\beta} - 4H_\alpha H_\beta)\xi^2 + \cdots, \\
&\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Phi^\lambda_{\alpha\beta}) = \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\,\overset{*}{\nabla}_\alpha b^\lambda_\beta)\xi + \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}(2H\delta^\lambda_\nu - b^\lambda_\nu)\overset{*}{\nabla}_\alpha b^\lambda_\beta)\xi^2, \\
&2Hb_{\alpha\beta} + \theta^{-1}(-2Hb_{\alpha\beta} + 2(Hc_{\alpha\beta} + Kb_{\alpha\beta})\xi - 2Kc_{\alpha\beta}\xi^2) \\
&\quad = 2[Hc_{\alpha\beta} + (K - 2H^2)b_{\alpha\beta}]\xi + K((K - 4H^2)a_{\alpha\beta} + 4Hb_{\alpha\beta})\xi^2 + \cdots \\
&\quad = K(b_{\alpha\beta} - Ha_{\alpha\beta})\xi + K((K - 4H^2)a_{\alpha\beta} + 4Hb_{\alpha\beta})\xi^2 + \cdots.
\end{aligned}
$$

Finally

$$\begin{cases} R_{\alpha\beta}(1)=\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\overset{*}{\nabla}_\alpha b^\lambda_\beta)+2K(b_{\alpha\beta}-Ha_{\alpha\beta})+2H_{\alpha\beta}+\overset{*}{\Gamma}{}^\lambda{}_{\alpha\sigma}\overset{*}{\nabla}_\lambda b^\sigma_\beta+\overset{*}{\Gamma}{}^\sigma{}_{\beta\lambda}\overset{*}{\nabla}_\sigma b^\lambda_\alpha, \\ R_{\alpha\beta}(2)=(4HH_{\alpha\beta}+4H_\alpha H_\beta-K_{\alpha\beta})+K((K-4H^2)a_{\alpha\beta}+4Hb_{\alpha\beta}) \\ \quad +\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}((2H\delta^\lambda_\nu-b^\lambda_\nu)\sqrt{a}\overset{*}{\nabla}_\alpha b^\nu_\beta)-[(2H\delta^\lambda_\nu-b^\lambda_\nu)(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}\overset{*}{\nabla}_\alpha b^\nu_\sigma+\overset{*}{\Gamma}{}^\sigma{}_{\lambda\alpha}\overset{*}{\nabla}_\beta b^\nu_\sigma)+\overset{*}{\nabla}_\lambda b^\sigma_\alpha\overset{*}{\nabla}_\sigma b^\lambda_\beta]. \end{cases}$$

In addition

$$\begin{aligned} R_{3\beta}&=-\frac{\partial^2}{\partial\xi\partial x^\beta}\ln\sqrt{g}+\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\lambda}(\sqrt{g}\Gamma^\lambda_{3\beta})+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi}(\sqrt{g}\Gamma^3_{3\beta}) \\ &\quad -\Gamma^\lambda_{3\sigma}\Gamma^3_{\lambda\beta}-\Gamma^\lambda_{33}\Gamma^3_{\lambda\beta}-\Gamma^3_{3\sigma}\Gamma^\sigma_{3\beta}-\Gamma^3_{33}\Gamma^3_{3\beta}, \\ R_{33}&=-\frac{\partial^2\ln\theta}{\partial\xi^2}-\Gamma^\lambda_{3\sigma}\Gamma^\sigma_{3\lambda}=\frac{\partial^2\ln\theta}{\partial\xi^2}-\theta^{-2}I^\lambda_\sigma I^\sigma_\lambda \\ &=\theta^{-2}\{-(2K-4H^2+4HK\xi-2K^2\xi^2)-(4H^2-2K-4HK\xi+2K^2)\}=0, \end{aligned}$$

where we have used $b^\lambda_\sigma b^\sigma_\lambda=c^\lambda_\lambda=4H^2-2K$,

$$\begin{aligned} &R_{3\beta}=-\frac{\partial^2}{\partial\xi\partial x^\beta}\ln\theta+\frac{\partial^2}{\partial\xi\partial x^\beta}\ln\sqrt{a}+\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}\Gamma^\lambda_{3\beta})+\Gamma^\lambda_{3\beta}\frac{\partial\ln\theta}{\partial x^\lambda}+\Gamma^\lambda_{3\sigma}\Gamma^\sigma_{\lambda\beta}-\Gamma^3_{3\sigma}\Gamma^\sigma_{3\beta}-\Gamma^\lambda_{33}\Gamma^3_{\lambda\beta} \\ &\qquad -\Gamma^3_{33}\Gamma^3_{3\beta}=-\frac{\partial}{\partial x^\beta}\frac{-2H+2K\xi}{\theta}+(\frac{\partial\ln\sqrt{a}}{\partial x^\lambda}+\frac{\partial\ln\theta}{\partial x^\lambda})\Gamma^\lambda_{3\beta}+\frac{\partial\Gamma^\lambda_{3\beta}}{\partial x^\lambda}+\theta^{-1}I^\lambda_\sigma(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}+\Phi^\sigma_{\lambda\beta}), \\ &-\frac{\partial}{\partial x^\beta}\frac{-2H+2K\xi}{\theta}=\theta^{-2}(2\overset{*}{\nabla}_\beta H-2\overset{*}{\nabla}_\beta K\xi+(2H\overset{*}{\nabla}_\beta K-2K\overset{*}{\nabla}_\beta H)\xi^2), \\ &(\frac{\partial\ln\sqrt{a}}{\partial x^\lambda}+\frac{\partial\ln\theta}{\partial x^\lambda})\Gamma^\lambda_{3\beta}=\theta^{-1}(-b^\lambda_\beta\partial_\lambda\ln\sqrt{a}+K\partial_\beta\ln\sqrt{a}\xi) \\ &\qquad +\theta^{-2}(2b^\lambda_\beta\overset{*}{\nabla}_\lambda H\xi+(-b^\lambda_\beta\overset{*}{\nabla}_\lambda K-2K\overset{*}{\nabla}_\beta H)\xi^2+K\overset{*}{\nabla}_\beta K\xi^3), \\ &\frac{\partial\Gamma^\lambda_{3\beta}}{\partial x^\lambda}=\frac{\partial}{\partial x^\lambda}(\theta^{-1}I^\lambda_\beta)=\theta^{-1}(-\partial_\lambda b^\lambda_\beta+\partial_\beta K\xi) \\ &\qquad +\theta^{-2}(-2b^\lambda_\beta\overset{*}{\nabla}_\lambda H\xi+(2K\overset{*}{\nabla}_\beta H-b^\lambda_\beta\overset{*}{\nabla}_\lambda K)\xi^2+K\overset{*}{\nabla}_\beta K\xi^3), \\ &I^\lambda_\sigma\Phi^\sigma_{\lambda\beta}=\theta^{-1}(\overset{*}{\nabla}_\beta(2H^2-K)\xi+K\overset{*}{\nabla}_\beta K\xi^3), \\ &\theta^{-1}I^\lambda_\sigma(\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}+\Phi^\sigma_{\lambda\beta})=\theta^{-1}(-b^\lambda_\sigma\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}+K\frac{\partial\ln\sqrt{a}}{\partial x^\beta}\xi)+\theta^{-2}(\overset{*}{\nabla}_\beta(2H^2-K)\xi+K\overset{*}{\nabla}_\beta K\xi^3), \\ &R_{3\beta}=\theta^{-1}(-b^\lambda_\beta\partial_\lambda\ln\sqrt{a}-\partial_\lambda b^\lambda_\beta-b^\lambda_\sigma\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}+(2K\overset{*}{\nabla}_\beta\ln\sqrt{a}+\partial_\beta K)\xi) \\ &\qquad +\theta^{-2}\{(2\overset{*}{\nabla}_\beta H-2\overset{*}{\nabla}_\beta K\xi+(2H\overset{*}{\nabla}_\beta K-2K\overset{*}{\nabla}_\beta H)\xi^2)+(2b^\lambda_\beta\overset{*}{\nabla}_\lambda H\xi \\ &\qquad +(-b^\lambda_\beta\overset{*}{\nabla}_\lambda K-2K\overset{*}{\nabla}_\beta H)\xi^2+K\overset{*}{\nabla}_\beta K\xi^3)+(-2b^\lambda_\beta\overset{*}{\nabla}_\lambda H\xi+(2K\overset{*}{\nabla}_\beta H \\ &\qquad -b^\lambda_\beta\overset{*}{\nabla}_\lambda K)\xi^2+K\overset{*}{\nabla}_\beta K\xi^3)+(\overset{*}{\nabla}_\beta(2H^2-K)\xi+K\overset{*}{\nabla}_\beta K\xi^3)\}. \end{aligned}$$

It follows from

$$\partial_\lambda b^\lambda_\beta=2\overset{*}{\nabla}_\beta H-b^\lambda_\beta\overset{*}{\nabla}_\lambda\ln\sqrt{a}+b^\lambda_\sigma\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta},$$

that

$$\begin{aligned} R_{3\beta}&=\theta^{-1}(2\overset{*}{\nabla}_\beta H-2b^\lambda_\sigma\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}+(2K\overset{*}{\nabla}_\beta\ln\sqrt{a}+\overset{*}{\nabla}_\beta K)\xi)+\theta^{-2}\Big\{(2\overset{*}{\nabla}_\beta H \\ &\quad +\overset{*}{\nabla}_\beta(2H^2-3K)\xi+(2(H\delta^\lambda_\beta-b^\lambda_\beta)\overset{*}{\nabla}_\lambda K-2K\overset{*}{\nabla}_\beta H)\xi^2+3K\overset{*}{\nabla}_\beta K\xi^3\Big\}. \end{aligned}$$

Since

$$\theta^{-1}=1+2H\xi+(4H^2-K)\xi^2+\cdots,\quad \theta^{-2}=1+4H\xi+(12H^2-2K)\xi^2+\cdots,$$

we obtain

$$\left\{\begin{array}{l} R_{3\beta}(0)=4\overset{*}{\nabla}_\beta H-2b^\lambda_\sigma \overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta},\\ R_{3\beta}(1)=\overset{*}{\nabla}_\beta(8H^2-2K)+2K\overset{*}{\nabla}_\beta \ln\sqrt{a}-4Hb^\lambda_\sigma \overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta},\\ R_{3\beta}(2)=8(6H^2-K)\overset{*}{\nabla}_\beta H-(4H\delta^\lambda_\beta+2b^\lambda_\beta)\overset{*}{\nabla}_\beta K+4HK\overset{*}{\nabla}_\beta\ln\sqrt{a}-(8H^2-2K)b^\lambda_\sigma\overset{*}{\Gamma}{}^\sigma{}_{\lambda\beta}. \end{array}\right.$$

The proof is complete. □

Next we consider the relationships of covariant derivatives of order two in 3D-space and on the two dimensional manifolds which are necessary for studying differential operators on the manifolds.

Lemma 4.1. *There are relationships between the covariant derivatives of two order of the vectors in 3D space and on two dimensional manifold*

$$\left\{\begin{array}{l} \nabla_\alpha\nabla_\beta u^i=\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i-J_{\alpha\beta}\frac{\partial u^i}{\partial\xi}+\Pi^{i\mu}_{\alpha\beta,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_{\alpha\beta,k}(\xi)u^k,\\ \nabla_3\nabla_3 u^i=\frac{\partial^2 u^i}{\partial\xi^2}+l^i_j(\xi)\frac{\partial u^j}{\partial\xi},\\ \nabla_\beta\nabla_3 u^i=\overset{*}{\nabla}_\beta\frac{\partial u^i}{\partial\xi}+\Pi^{i3}_{\beta3,k}(\xi)\frac{\partial u^k}{\partial\xi}+\Pi^{i\mu}_{\beta3,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_{\beta3,k}(\xi)u^k, \end{array}\right. \tag{4.30}$$

where

$$\left\{\begin{array}{l} \Pi^{ij}_{\alpha\beta,k}(\xi)=\Pi^{ij}_{\alpha\beta,k}(0)+\Pi^{ij}_{\alpha\beta,k}(1)\xi+\Pi^{ij}_{\alpha\beta,k}(2)\xi^2+\cdots,\\ \Pi^{\gamma\mu}_{\alpha\beta,\nu}(\xi)=\Phi^\gamma_{\alpha\nu}\delta^\mu_\beta-\Phi^\mu_{\alpha\beta}\delta^\gamma_\nu+\Phi^\gamma_{\beta\nu}\delta^\mu_\alpha,\\ \Pi^{\gamma\mu}_{\alpha\beta,3}(\xi)=\theta^{-1}(I^\gamma_\beta\delta^\mu_\alpha+I^\gamma_\alpha\delta^\mu_\beta),\\ \Pi^{\gamma0}_{\alpha\beta,\nu}(\xi)=\Phi^\gamma_{\alpha\mu}\Phi^\mu_{\beta\nu}-\Phi^\mu_{\alpha\beta}\Phi^\gamma_{\mu\nu}+\theta^{-1}(I^\gamma_\alpha J_{\beta\nu}-I^\gamma_\nu J_{\alpha\beta})+\overset{*}{\nabla}_\alpha\Phi^\gamma_{\beta\nu},\\ \Pi^{\gamma0}_{\alpha\beta,3}(\xi)=R^\gamma_{\alpha\beta}(\xi),\\ \Pi^{3\mu}_{\alpha\beta,\nu}(\xi)=J_{\beta\nu}\delta^\mu_\alpha+J_{\alpha\nu}\delta^\mu_\beta,\quad \Pi^{3\mu}_{\alpha\beta,3}(\xi)=-\Phi^\mu_{\alpha\beta},\\ \Pi^{30}_{\alpha\beta,\mu}(\xi)=\overset{*}{\nabla}_\alpha J_{\beta\mu}+\Phi^\nu_{\alpha\mu}J_{\beta\nu}-\Phi^\nu_{\alpha\beta}J_{\nu\mu},\quad \Pi^{30}_{\alpha\beta,3}=-c_{\alpha\beta}, \end{array}\right. \tag{4.31}$$

$$\left\{\begin{array}{l} \Pi^{\gamma3}_{\beta3,\sigma}(\xi)=\Phi^\gamma_{\beta\sigma},\quad \Pi^{\gamma3}_{\beta3,3}=\theta^{-1}I^\gamma_\beta,\\ \Pi^{\gamma\mu}_{\beta3,\nu}(\xi)=\theta^{-1}(-b^\gamma_\nu\delta^\mu_\beta+b^\mu_\beta\delta^\gamma_\nu),\quad \Pi^{\gamma\mu}_{\beta3,3}=0,\\ \Pi^{\gamma0}_{\beta3,\sigma}(\xi)=R^\gamma_{\beta\sigma}(\xi),\\ \Pi^{\gamma0}_{\beta3,3}(\xi)=-\theta^{-1}(c^\gamma_\beta-2Kb^\gamma_\beta\xi+K^2\delta^\gamma_\beta\xi^2),\\ \Pi^{3\mu}_{\beta3,\lambda}=0,\quad \Pi^{3\mu}_{\beta3,3}(\xi)=-\theta^{-1}I^\sigma_\beta,\quad \Pi^{33}_{\beta3,\sigma}=J_{\beta\sigma},\quad \Pi^{33}_{\beta3,3}=0,\\ \Pi^{33}_{\beta3,k}=\Pi^{30}_{\beta3,k}=0,\\ l^\alpha_\beta(\xi)(\xi)=2\theta^{-1}I^\alpha_\beta=-2b^\alpha_\beta-2c^\alpha_\beta\xi+2(Kb^\alpha_\beta-2Hc^\alpha_\beta)\xi^2+\cdots,\\ l^3_\beta(\xi)=l^\alpha_3=l^3_3=0, \end{array}\right. \tag{4.32}$$

and

$$\left\{\begin{array}{l} R^{\alpha}_{\beta\sigma}(\xi):=\overset{*}{\nabla}_{\beta}(\theta^{-1}I^{\alpha}_{\sigma})+\theta^{-1}(I^{\lambda}_{\sigma}\Phi^{\alpha}_{\beta\lambda}-I^{\lambda}_{\beta}\Phi^{\alpha}_{\lambda\sigma})=\theta^{-2}(-\overset{*}{\nabla}_{\beta}b^{\alpha}_{\sigma}-\overset{*}{\nabla}_{\sigma}c^{\alpha}_{\beta}\xi+r^{\alpha}_{\beta\sigma}\xi^2),\\ r^{\alpha}_{\beta\sigma}=\overset{*}{\nabla}_{\beta}(2H(K\delta^{\alpha}_{\sigma}-c^{\alpha}_{\sigma}))+(2H\delta^{\alpha}_{\mu}-b^{\alpha}_{\mu})\overset{*}{\nabla}_{\sigma}c^{\alpha}_{\beta}-2H\overset{*}{\nabla}_{\beta}c^{\alpha}_{\sigma}+c^{\mu}_{\sigma}\overset{*}{\nabla}_{\beta}b^{\alpha}_{\mu}). \end{array}\right.$$

Proof. In order to prove Lemma 4.1, we repeatedly and alternately to apply Lemmas 2.1-2.7. Indeed, for example, according to the definition of covariant derivative of second order tensor

$$\begin{aligned}\nabla_{\lambda}\nabla_{\sigma}u^{\alpha}&=\tfrac{\partial}{\partial x^{\lambda}}\nabla_{\sigma}u^{\alpha}+\Gamma^{\alpha}_{\lambda k}\nabla_{\sigma}u^{k}-\Gamma^{k}_{\lambda\sigma}\nabla_{k}u^{\alpha}\\ &=\tfrac{\partial}{\partial x^{\lambda}}\nabla_{\sigma}u^{\alpha}+\Gamma^{\alpha}_{\lambda\mu}\nabla_{\sigma}u^{\mu}+\Gamma^{\alpha}_{\lambda 3}\nabla_{\sigma}u^{3}-\Gamma^{\nu}_{\lambda\sigma}\nabla_{\nu}u^{\alpha}-\Gamma^{3}_{\lambda\sigma}\nabla_{3}u^{\alpha}\\ &=(\text{by Lemma 2.3})=\tfrac{\partial}{\partial x^{\lambda}}\nabla_{\sigma}u^{\alpha}+\overset{*}{\Gamma}{}^{\alpha}{}_{\lambda\mu}\nabla_{\sigma}u^{\mu}-\overset{*}{\Gamma}{}^{\mu}{}_{\lambda\sigma}\nabla_{\mu}u^{\alpha}\\ &\quad+\Phi^{\alpha}_{\lambda\mu}\nabla_{\sigma}u^{\mu}+\Gamma^{\alpha}_{\lambda 3}\nabla_{\sigma}u^{3}-\Phi^{\nu}_{\lambda\sigma}\nabla_{\nu}u^{\alpha}-\Gamma^{3}_{\lambda\sigma}\nabla_{3}u^{\alpha}\\ &=\overset{*}{\nabla}_{\lambda}\nabla_{\sigma}u^{\alpha}+(\Phi^{\alpha}_{\lambda\mu}\delta^{\mu}_{\sigma}-\Phi^{\nu}_{\lambda\sigma}\delta^{\alpha}_{\mu})\nabla_{\nu}u^{\mu}+\theta^{-1}I^{\alpha}_{\lambda}\nabla_{\sigma}u^{3}-J_{\lambda\sigma}\nabla_{3}u^{\alpha}\\ &=(\text{by Lemma 2.4})\,\overset{*}{\nabla}_{\lambda}(\overset{*}{\nabla}_{\sigma}u^{\alpha}+\theta^{-1}I^{\alpha}_{\sigma}u^{3}+\Phi^{\alpha}_{\sigma\mu}u^{\mu})\\ &\quad+(\Phi^{\alpha}_{\lambda\mu}\delta^{\nu}_{\sigma}-\Phi^{\nu}_{\lambda\sigma}\delta^{\alpha}_{\mu})(\overset{*}{\nabla}_{\nu}u^{\mu}+\theta^{-1}I^{\mu}_{\nu}u^{3}+\Phi^{\mu}_{\nu\gamma}u^{\gamma})\\ &\quad+\theta^{-1}I^{\alpha}_{\lambda}(\overset{*}{\nabla}_{\sigma}u^{3}+J_{\sigma\gamma}u^{\gamma})-J_{\lambda\sigma}(\tfrac{\partial u^{\alpha}}{\partial\xi}+\theta^{-1}I^{\alpha}_{\gamma}u^{\gamma}).\end{aligned}$$

Making rearrangement to obtain

$$\begin{aligned}\nabla_{\lambda}\nabla_{\sigma}u^{\alpha}=&\overset{*}{\nabla}_{\lambda}\overset{*}{\nabla}_{\sigma}u^{\alpha}+(\Phi^{\alpha}_{\lambda\mu}\delta^{\nu}_{\sigma}-\Phi^{\nu}_{\lambda\sigma}\delta^{\alpha}_{\mu})\overset{*}{\nabla}_{\nu}u^{\mu}+\theta^{-1}I^{\alpha}_{\lambda}\overset{*}{\nabla}_{\sigma}u^{3}-J_{\lambda\sigma}\tfrac{\partial u^{\alpha}}{\partial\xi}+\overset{*}{\nabla}_{\lambda}(\theta^{-1}I^{\alpha}_{\sigma}u^{3}+\Phi^{\alpha}_{\sigma\mu}u^{\mu})\\ &+(\Phi^{\alpha}_{\lambda\mu}\delta^{\nu}_{\sigma}-\Phi^{\nu}_{\lambda\sigma}\delta^{\alpha}_{\mu})(\theta^{-1}I^{\mu}_{\nu}u^{3}+\Phi^{\mu}_{\nu\gamma}u^{\gamma})+\theta^{-1}(I^{\alpha}_{\lambda}J_{\sigma\gamma}-J_{\lambda\sigma}I^{\alpha}_{\gamma})u^{\gamma},\end{aligned}$$

$$\begin{aligned}\nabla_{\lambda}\nabla_{\sigma}u^{\alpha}=&\overset{*}{\nabla}_{\lambda}\overset{*}{\nabla}_{\sigma}u^{\alpha}+(\Phi^{\alpha}_{\lambda\mu}\delta^{\nu}_{\sigma}-\Phi^{\nu}_{\lambda\sigma}\delta^{\alpha}_{\mu}+\Phi^{\alpha}_{\sigma\mu}\delta^{\nu}_{\lambda})\overset{*}{\nabla}_{\nu}u^{\mu}+\theta^{-1}(I^{\alpha}_{\lambda}\delta^{\nu}_{\sigma}+I^{\alpha}_{\sigma}\delta^{\nu}_{\lambda})\overset{*}{\nabla}_{\nu}u^{3}\\ &-J_{\lambda\sigma}\tfrac{\partial u^{\alpha}}{\partial\xi}+\{\Phi^{\alpha}_{\lambda\mu}\Phi^{\mu}_{\sigma\gamma}-\Phi^{\alpha}_{\mu\gamma}\Phi^{\mu}_{\lambda\sigma}+\overset{*}{\nabla}_{\lambda}\Phi^{\alpha}_{\sigma\gamma}+\theta^{-1}(I^{\alpha}_{\lambda}J_{\sigma\gamma}-J_{\lambda\sigma}I^{\alpha}_{\gamma})\}u^{\gamma}\\ &+\{\overset{*}{\nabla}_{\lambda}(\theta^{-1}I^{\alpha}_{\sigma})+\theta^{-1}(I^{\mu}_{\sigma}\Phi^{\alpha}_{\lambda\mu}-I^{\alpha}_{\mu}\Phi^{\mu}_{\lambda\sigma})\}u^{3}.\end{aligned}\tag{4.33}$$

Next we consider

$$\begin{aligned}\nabla_{\alpha}\nabla_{\beta}u^{3}&=\tfrac{\partial}{\partial x^{\alpha}}\nabla_{\beta}u^{3}+\Gamma^{3}_{\alpha k}\nabla_{\beta}u^{k}-\Gamma^{k}_{\alpha\beta}\nabla_{k}u^{3}\\ &=(\text{by (2.27)})\,\tfrac{\partial}{\partial x^{\alpha}}\nabla_{\beta}u^{3}+\Gamma^{3}_{\alpha\mu}\nabla_{\beta}u^{\mu}-\Gamma u_{\alpha\beta}\nabla_{\mu}u^{3}+J_{\alpha\mu}\nabla_{\beta}u^{\mu}-J_{\alpha\beta}\tfrac{\partial u^{3}}{\partial\xi}\\ &=\overset{*}{\nabla}_{\alpha}(\overset{*}{\nabla}_{\beta}u^{3}+J_{\beta\mu}u^{\mu})-\Phi^{\mu}_{\alpha\beta}(\overset{*}{\nabla}_{\mu}u^{3}+J_{\mu\nu}u^{\nu})+J_{\alpha\mu}(\overset{*}{\nabla}_{\beta}u^{\mu}+\theta^{-1}I^{\mu}_{\beta}u^{3}+\Phi^{\mu}_{\beta\nu}u^{\nu})-J_{\alpha\beta}\tfrac{\partial u^{3}}{\partial\xi}\\ &=\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}u^{3}+(J_{\beta\mu}\delta^{\nu}_{\alpha}+J_{\alpha\mu}\delta^{\nu}_{\beta})\overset{*}{\nabla}_{\nu}u^{\mu}-\Phi^{\mu}_{\alpha\beta}\overset{*}{\nabla}_{\mu}u^{3}-J_{\alpha\beta}\tfrac{\partial u^{3}}{\partial\xi}\\ &\quad+(\overset{*}{\nabla}_{\alpha}J_{\beta\mu}+J_{\alpha\nu}\Phi^{\nu}_{\beta\mu}-J_{\nu\mu}\Phi^{\nu}_{\alpha\beta})u^{\mu}+\theta^{-1}J_{\alpha\mu}I^{\mu}_{\beta}u^{3}.\end{aligned}$$

Below we prove

$$\left\{\begin{array}{l}\theta^{-1}J_{\alpha\mu}I^{\mu}_{\beta}=-c_{\alpha\beta},\\ \overset{*}{\nabla}_{\alpha}J_{\beta\mu}+J_{\alpha\nu}\Phi^{\nu}_{\beta\mu}-J_{\nu\mu}\Phi^{\nu}_{\alpha\beta}=\overset{*}{\nabla}_{\mu}J_{\alpha\beta}.\end{array}\right.\tag{4.34}$$

In fact, in view of (2.41)

$$\theta^{-1}J_{\alpha\mu}I^{\mu}_{\beta}=\theta^{-1}(-\theta c_{\alpha\beta})=-c_{\alpha\beta}.$$

In addition, we have

$$\begin{cases} J_{\alpha\nu}\Phi^{\nu}_{\beta\mu}=-\xi b_{\alpha\gamma}\overset{*}{\nabla}_{\beta}b^{\gamma}_{\mu}, \quad J_{\nu\mu}\Phi^{\nu}_{\alpha\beta}=-\xi b_{\mu\gamma}\overset{*}{\nabla}_{\alpha}b^{\gamma}_{\beta}, \\ \overset{*}{\nabla}_{\alpha}J_{\beta\mu}+J_{\alpha\nu}\Phi^{\nu}_{\beta\mu}-J_{\nu\mu}\Phi^{\nu}_{\alpha\beta}=\overset{*}{\nabla}_{\alpha}b_{\beta\mu}-\xi\overset{*}{\nabla}_{\alpha}c_{\beta\mu}+\xi(-b_{\alpha\gamma}\overset{*}{\nabla}_{\beta}b^{\gamma}_{\mu}+b_{\mu\gamma}\overset{*}{\nabla}_{\alpha}b^{\gamma}_{\beta}). \end{cases} \tag{4.35}$$

Using the Godazzi formula

$$\overset{*}{\nabla}_{\beta}b_{\alpha\gamma}=\overset{*}{\nabla}_{\alpha}b_{\beta\gamma},$$

we assert

$$\begin{aligned} &-b_{\alpha\gamma}\overset{*}{\nabla}_{\beta}b^{\gamma}_{\mu}+b_{\mu\gamma}\overset{*}{\nabla}_{\alpha}b^{\gamma}_{\beta}=-b_{\alpha\gamma}\overset{*}{\nabla}_{\beta}b^{\gamma}_{\mu}+\overset{*}{\nabla}_{\alpha}c_{\beta\mu}-b^{\gamma}_{\beta}\overset{*}{\nabla}_{\alpha}b_{\mu\gamma} \\ =&-b_{\alpha\gamma}\overset{*}{\nabla}_{\mu}b^{\gamma}_{\beta}-b^{\gamma}_{\beta}\overset{*}{\nabla}_{\mu}b_{\alpha\gamma}+\overset{*}{\nabla}_{\alpha}c_{\beta\mu}=-\overset{*}{\nabla}_{\mu}c_{\alpha\beta}+\overset{*}{\nabla}_{\alpha}c_{\beta\mu}. \end{aligned}$$

Substituting above formula into (4.35) leads to

$$\begin{aligned} &\overset{*}{\nabla}_{\alpha}J_{\beta\mu}+J_{\alpha\nu}\Phi^{\nu}_{\beta\mu}-J_{\nu\mu}\Phi^{\nu}_{\alpha\beta}=\overset{*}{\nabla}_{\alpha}b_{\beta\mu}-\xi\overset{*}{\nabla}_{\alpha}c_{\beta\mu}+\xi(-b_{\alpha\gamma}\overset{*}{\nabla}_{\beta}b^{\gamma}_{\mu}+b_{\mu\gamma}\overset{*}{\nabla}_{\alpha}b^{\gamma}_{\beta}) \\ =&\overset{*}{\nabla}_{\alpha}b_{\beta\mu}-\xi\overset{*}{\nabla}_{\alpha}c_{\beta\mu}+\xi(-\overset{*}{\nabla}_{\mu}c_{\alpha\beta}+\overset{*}{\nabla}_{\alpha}c_{\beta\mu}\Omega) \\ =&\overset{*}{\nabla}_{\alpha}b_{\beta\mu}-\xi\overset{*}{\nabla}_{\mu}c_{\alpha\beta}=\overset{*}{\nabla}_{\mu}(b_{\alpha\beta}-\xi c_{\alpha\beta})=\overset{*}{\nabla}_{\mu}J_{\alpha\beta}. \end{aligned}$$

From this it yields (4.34). Finally we obtain

$$\begin{aligned} \nabla_{\alpha}\nabla_{\beta}u^3=&\tfrac{\partial}{\partial x^{\alpha}}\nabla_{\beta}u^3+\Gamma^3_{\alpha k}\nabla_{\beta}u^k-\Gamma^k_{\alpha\beta}\nabla_k u^3=\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}u^3 \\ &+(J_{\beta\mu}\delta^{\nu}_{\alpha}+J_{\alpha\mu}\delta^{\nu}_{\beta})\overset{*}{\nabla}_{\nu}u^{\mu}-\Phi^{\mu}_{\alpha\beta}\overset{*}{\nabla}_{\mu}u^3-J_{\alpha\beta}\tfrac{\partial u^3}{\partial\xi}+\overset{*}{\nabla}_{\mu}J_{\alpha\beta}u^{\mu}-c_{\alpha\beta}u^3. \end{aligned} \tag{4.36}$$

Combing (4.33) and (4.36) we claim

$$\nabla_{\alpha}\nabla_{\beta}u^i=\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}u^i-J_{\alpha\beta}\frac{\partial u^i}{\partial\xi}+\Pi^{i\mu}_{\alpha\beta,k}(\xi)\overset{*}{\nabla}_{\mu}u^k+\Pi^{i0}_{\alpha\beta,k}u^k, \tag{4.37}$$

$$\begin{cases} \Pi^{i\mu}_{\alpha\beta,k}(\xi)=\begin{cases} \Phi^{\gamma}_{\alpha\nu}\delta^{\mu}_{\beta}-\Phi^{\mu}_{\alpha\beta}\delta^{\gamma}_{\nu}+\Phi^{\gamma}_{\beta\nu}\delta^{\mu}_{\alpha}, & i=\gamma,k=\nu \\ \theta^{-1}(I^{\gamma}_{\alpha}\delta^{\mu}_{\beta}+I^{\gamma}_{\beta}\delta^{\mu}_{\alpha}), & i=\gamma,k=3, \\ J_{\beta\nu}\delta^{\mu}_{\alpha}+J_{\alpha\nu}\delta^{\mu}_{\beta}, & i=3,k=\nu, \\ -\Phi^{\mu}_{\alpha\beta}, & i=3,k=3, \end{cases} \\ \Pi^{i0}_{\alpha\beta,k}(\xi)=\begin{cases} \Phi^{\gamma}_{\alpha\mu}\Phi^{\mu}_{\beta\nu}-\Phi^{\gamma}_{\mu\nu}\Phi^{\mu}_{\alpha\beta}+\overset{*}{\nabla}_{\alpha}\Phi^{\gamma}_{\beta\nu}+\theta^{-1}(I^{\gamma}_{\alpha}J_{\beta\nu}-J_{\alpha\beta}I^{\gamma}_{\nu}), & i=\gamma,k=\nu, \\ \overset{*}{\nabla}_{\alpha}(\theta^{-1}I^{\gamma}_{\beta})+\theta^{-1}(I^{\mu}_{\beta}\Phi^{\gamma}_{\alpha\mu}-I^{\gamma}_{\mu}\Phi^{\mu}_{\alpha\gamma}), & i=\gamma,k=3, \\ \overset{*}{\nabla}_{\nu}J_{\alpha\beta}, & i=3,k=\nu, \\ -c_{\alpha\beta}, & i=3,k=3. \end{cases} \end{cases} \tag{4.38}$$

Next we consider

$$\nabla_3\nabla_3 u^\alpha=\frac{\partial}{\partial\xi}\nabla_3 u^\alpha+\Gamma^\alpha_{3k}\nabla_3 u^k-\Gamma^k_{33}\nabla_k u^\alpha.$$

Owing to (2.27):

$$\Gamma^k_{33}=\Gamma^3_{3k}=\Gamma^3_{k3}=0,\quad k=1,2,3, \tag{4.39}$$

it infers

$$\begin{aligned}\nabla_3\nabla_3 u^\alpha&=\tfrac{\partial}{\partial\xi}\nabla_3 u^\alpha+\Gamma^\alpha_{3\beta}\nabla_3 u^\beta=(\delta^\alpha_\beta\tfrac{\partial}{\partial\xi}+\theta^{-1}I^\alpha_\beta)\nabla_3 u^\beta=(\delta^\alpha_\beta\tfrac{\partial}{\partial\xi}+\theta^{-1}I^\alpha_\beta)(\tfrac{\partial u^\beta}{\partial\xi}+\theta^{-1}I^\beta_\lambda u^\lambda)\\&=\tfrac{\partial^2 u^\alpha}{\partial\xi^2}+\tfrac{\partial}{\partial\xi}(\theta^{-1}I^\alpha_\lambda u^\lambda)+\theta^{-1}I^\alpha_\beta(\tfrac{\partial u^\beta}{\partial\xi}+\theta^{-1}I^\beta_\lambda u^\lambda)\\&=\tfrac{\partial^2 u^\alpha}{\partial\xi^2}+2\theta^{-1}I^\alpha_\beta\tfrac{\partial u^\beta}{\partial\xi}+(\tfrac{\partial}{\partial\xi}(\theta^{-1}I^\alpha_\lambda)+\theta^{-2}I^\alpha_\beta I^\beta_\lambda)u^\lambda.\end{aligned} \tag{4.40}$$

The following equality is very useful later on

$$\frac{\partial}{\partial\xi}(\theta^{-1}I^\alpha_\lambda)+\theta^{-2}I^\alpha_\beta I^\beta_\lambda=0. \tag{4.41}$$

To obtain that, we first show that

$$\begin{aligned}\tfrac{\partial}{\partial\xi}(\theta^{-1}I^\alpha_\lambda)&=\theta^{-2}(2H-2K\xi)I^\alpha_\lambda+\theta K\delta^\alpha_\lambda)\\&=\theta^{-2}\{(2H-2K\xi)(-b^\alpha_\lambda+K\xi\delta^\alpha_\lambda)+K\delta^\alpha_\lambda(1-2H\xi+K\xi^2)\}\\&=\theta^{-2}(K\delta^\alpha_\lambda-2Hb^\alpha_\lambda+2Kb^\alpha_\lambda\xi-K^2\delta^\alpha_\lambda\xi^2).\end{aligned}$$

On the other hand,

$$\begin{aligned}I^\alpha_\beta I^\beta_\lambda&=(-b^\alpha_\lambda+K\xi\delta^\alpha_\beta)(-b^\beta_\lambda+K\xi\delta^\beta_\lambda)=b^\alpha_\beta b^\beta_\lambda-2Kb^\alpha_\lambda\xi+K^2\delta^\alpha_\lambda\xi^2\\&(\text{by Lemma 2.1 } K\delta^\alpha_\lambda-2Hb^\alpha_\lambda+c^\alpha_\lambda=0)=c^\alpha_\lambda-2Kb^\alpha_\lambda+K^2\delta^\alpha_\lambda\xi^2,\end{aligned}$$

so that

$$\frac{\partial}{\partial\xi}(\theta^{-1}I^\alpha_\lambda)+\theta^{-2}I^\alpha_\beta I^\beta_\lambda=\theta^{-2}(c^\alpha\alpha_\lambda-2Hb^\alpha_\lambda+K\delta^\alpha_\lambda)=0.$$

This infers (4.41). Coming back to (4.40)

$$\nabla_3\nabla_3 u^\alpha=\tfrac{\partial^2 u^\alpha}{\partial\xi^2}+2\theta^{-1}I^\alpha_\beta\tfrac{\partial u^\beta}{\partial\xi}. \tag{4.42}$$

In addition, in view of (4.39) we have

$$\nabla_3\nabla_3 u^3=\tfrac{\partial}{\partial\xi}\nabla_3 u^3+\Gamma^3_{3k}\nabla_3 u^k-\Gamma^k_{33}\nabla_k u^3=\tfrac{\partial^2 u^3}{\partial\xi^2}. \tag{4.43}$$

Combining (4.42) and (4.43) gives

$$\begin{aligned}&\nabla_3\nabla_3 u^i=\tfrac{\partial^2 u^3}{\partial\xi^2}+l^i_j(\xi)\tfrac{\partial u^\beta}{\partial\xi},\\&l^i_j=\begin{cases}2\theta^{-1}I^\alpha_\beta, & i=\alpha,j=\beta,\\0, & \text{otherwise}.\end{cases}\end{aligned} \tag{4.44}$$

Next we compute

$$\begin{aligned}\nabla_\beta\nabla_3 u^\alpha &= \frac{\partial}{\partial x^\beta}\nabla_3 u^\alpha + \Gamma^\alpha_{\beta k}\nabla_3 u^k - \Gamma^k_{\beta 3}\nabla_k u^\beta (\text{by (6.15)})\\ &= \frac{\partial}{\partial x^\beta}\nabla_3 u^\alpha + \Gamma^\alpha_{\beta\lambda}\nabla_3 u^\lambda + \Gamma^\alpha_{\beta 3}\nabla_3 u^3 - \Gamma^\lambda_{\beta 3}\nabla_\lambda u^\alpha\\ &= \frac{\partial}{\partial x^\beta}\nabla_3 u^\alpha + (\overset{*}{\Gamma}{}^\alpha_{\beta\lambda} + \Phi^\alpha_{\beta\lambda})\nabla_3 u^\lambda + \theta^{-1} I^\alpha_\beta \frac{\partial u^3}{\partial \xi} - \theta^{-1} I^\lambda_\beta (\overset{*}{\nabla}_\lambda u^\alpha + \theta^{-1} I^\alpha_\beta u^3 + \Phi^\alpha_{\beta\sigma} u^\sigma)\\ &= \overset{*}{\nabla}_\beta \nabla_3 u^\alpha + \Phi^\alpha_{\beta\lambda}\nabla_3 u^\lambda + \theta^{-1} I^\alpha_\beta \frac{\partial u^3}{\partial\xi} - \theta^{-1} I^\lambda_\beta \overset{*}{\nabla}_\lambda u^\alpha \Phi^\alpha_{\lambda\sigma}.\end{aligned}$$

Finally we find

$$\begin{aligned}\nabla_\beta\nabla_3 u^\alpha =& \overset{*}{\nabla}_\beta \frac{\partial u^\alpha}{\partial\xi} + \Phi^\alpha_{\beta\lambda}\frac{\partial u^\lambda}{\partial \xi} + \theta^{-1} I^\alpha_\beta \frac{\partial u^3}{\partial\xi} + \theta^{-1}(I^\alpha_\lambda \delta^\sigma_\beta - I^\sigma_\beta \delta^\alpha_\lambda)\overset{*}{\nabla}_\sigma u^\lambda\\ &+ (\overset{*}{\nabla}_\beta(\theta^{-1} I^\alpha_\sigma) + \theta^{-1}(I^\lambda_\sigma \Phi^\alpha_{\beta\lambda} - I^\lambda_\beta \Phi^\alpha_{\lambda\sigma}))u^\sigma - \theta^{-1} I^\alpha_\lambda I^\lambda_\beta u^3.\end{aligned} \tag{4.45}$$

Straightforward calculasions show

$$\begin{aligned}\theta^{-1} I^\alpha_\lambda I^\lambda_\beta &= \theta^{-2}(-b^\alpha_\lambda + K\delta^\alpha_\lambda \xi)(-b^\lambda_\beta + K\delta^\lambda_\beta \xi) = \theta^{-1}(b^\alpha_\lambda b^\lambda_\beta + K^2\delta^\alpha_\beta \xi^2 - 2Kb^\alpha_\beta \xi)\\ &= \theta^{-1}(c^\alpha_\beta + K^2 \delta^\alpha_\beta \xi^2 - 2Kb^\alpha_\beta \xi),\end{aligned}$$

$$\begin{aligned}\theta^{-1}(I^\lambda_\sigma \Phi^\alpha_{\beta\lambda} - I^\lambda_\beta \Phi^\alpha_{\lambda\sigma}) &= \theta^{-2}\{(-b^\lambda_\sigma + K\delta^\lambda_\sigma \xi)(-\overset{*}{\nabla}_\beta b^\alpha_\sigma \xi + (2H\delta^\alpha_\mu - b^\alpha_\mu)\overset{*}{\nabla}_\beta b^\mu_\lambda \xi^2)\\ &\quad -(-b^\lambda_\beta + K\delta^\lambda_\beta \xi)(-\overset{*}{\nabla}_\lambda b^\alpha_\sigma \xi + (2H\delta^\alpha_\mu - b^\alpha_\mu)\overset{*}{\nabla}_\lambda b^\mu_\sigma \xi^2)\}\\ &= \theta^{-2}\{(I^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - I^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda)\xi + (2H\delta^\alpha_\mu - b^\alpha_\mu)(I^\lambda_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda - I^\lambda_\beta \overset{*}{\nabla}_\lambda b^\mu_\sigma)\xi^2\}\\ &= \theta^{-2}(I^\lambda_\beta \overset{*}{\nabla}_\lambda b^\mu_\sigma - I^\lambda_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda)\{\delta^\alpha_\mu \xi - (2H\delta^\alpha_\mu - b^\alpha_\mu)\xi^2\}.\end{aligned}$$

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$$\begin{aligned}&(I^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - I^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda) = -(b^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - b^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda) + K(\delta^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - \delta^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda)\xi,\\ &(\delta^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - \delta^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda) = \overset{*}{\nabla}_\beta b^\alpha_\sigma - \overset{*}{\nabla}_\beta b^\alpha_\sigma = 0,\\ &b^\lambda_\beta \overset{*}{\nabla}_\lambda b^\alpha_\sigma - b^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda = (\text{by Godazzi formula}) b^\lambda_\beta \overset{*}{\nabla}_\sigma b^\alpha_\lambda - b^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda\\ &\quad = \overset{*}{\nabla}_\sigma (c^\alpha_\beta) - b^\alpha_\lambda \overset{*}{\nabla}_\sigma b^\lambda_\beta - b^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda (\text{using Godazzi formula again})\\ &\quad = \overset{*}{\nabla}_\sigma (c^\alpha_\beta) - b^\alpha_\lambda \overset{*}{\nabla}_\beta b^\lambda_\sigma - b^\lambda_\sigma \overset{*}{\nabla}_\beta b^\alpha_\lambda = \overset{*}{\nabla}_\sigma (c^\alpha_\beta) - \overset{*}{\nabla}_\beta (c^\alpha_\sigma).\end{aligned}$$

Therefore

$$\begin{cases}\theta^{-1} I^\alpha_\lambda I^\lambda_\beta = \theta^{-1}(-b^\alpha_\lambda + K\delta^\alpha_\lambda \xi)(-b^\lambda_\beta + K\delta^\lambda_\beta \xi) = \theta^{-1}(c^\alpha_\beta + K^2\delta^\alpha_\beta \xi^2 - 2Kb^\alpha_\beta \xi),\\ I^\lambda_\beta \overset{*}{\nabla}_\lambda b^\mu_\sigma - I^\lambda_\sigma \overset{*}{\nabla}_\beta b^\mu_\lambda = -\overset{*}{\nabla}_\sigma (c^\mu_\beta) + \overset{*}{\nabla}_\beta (c^\mu_\sigma),\\ \theta^{-1}(I^\lambda_\sigma \Phi^\alpha_{\beta\lambda} - I^\lambda_\beta \Phi^\alpha_{\lambda\sigma}) = \theta^{-2}(\overset{*}{\nabla}_\beta (c^\mu_\sigma) - \overset{*}{\nabla}_\sigma (c^\mu_\beta))\{\delta^\alpha_\mu \xi - (2H\delta^\alpha_\mu - b^\alpha_\mu)\xi^2\}.\end{cases} \tag{4.46}$$

On the other hand

$$\begin{aligned}
\overset{*}{\nabla}_\beta(\theta^{-1}I^\alpha_\sigma)=&\theta^{-2}\{\theta(-\overset{*}{\nabla}_\beta b^\alpha_\sigma+\overset{*}{\nabla}_\beta K\delta^\alpha_\sigma)+(2\overset{*}{\nabla}_\beta H\xi-\overset{*}{\nabla}_\beta K\xi^2)I^\alpha_\sigma\}\\
=&\theta^{-2}\{-\overset{*}{\nabla}_\beta b^\alpha_\sigma+(\overset{*}{\nabla}_\beta K\delta^\alpha_\sigma-2H\overset{*}{\nabla}_\beta b^\alpha_\sigma-2b^\alpha_\sigma\overset{*}{\nabla}_\beta H)\xi\\
&+(2H\overset{*}{\nabla}_\beta K-K\overset{*}{\nabla}_\beta b^\alpha_\sigma+\overset{*}{\nabla}_\beta HK\delta^\alpha_\sigma+b^\alpha_\sigma\overset{*}{\nabla}_\beta K)\xi^2\}\\
=&\theta^{-2}\{-\overset{*}{\nabla}_\beta b^\alpha_\sigma+\overset{*}{\nabla}_\beta(K\delta^\alpha_\sigma-2Hb^\alpha_\sigma)\xi+(\overset{*}{\nabla}_\beta(2HK)\delta^\alpha_\sigma+b^\alpha_\sigma\overset{*}{\nabla}_\beta K-K\overset{*}{\nabla}_\beta b^\alpha_\sigma)\xi^2\}\\
=&\theta^{-2}\{-\overset{*}{\nabla}_\beta b^\alpha_\sigma-\overset{*}{\nabla}_\beta c^\alpha_\sigma\xi+(\overset{*}{\nabla}_\beta(2HK)\delta^\alpha_\sigma+b^\alpha_\sigma\overset{*}{\nabla}_\beta K-K\overset{*}{\nabla}_\beta b^\alpha_\sigma)\xi^2\}.
\end{aligned}\tag{4.47}$$

Combining (4.46) and (4.47) leads to

$$\begin{aligned}
&\overset{*}{\nabla}_\beta(\theta^{-1}I^\alpha_\sigma)+\theta^{-1}(I^\lambda_\sigma\Phi^\alpha_{\beta\lambda}-I^\lambda_\beta\Phi^\alpha_{\lambda\sigma})=\theta^{-2}\{-\overset{*}{\nabla}_\beta b^\alpha_\sigma-\overset{*}{\nabla}_\sigma c^\alpha_\beta\xi\\
&\quad+(\overset{*}{\nabla}_\beta(2H(K\delta^\alpha_\sigma-c^\alpha_\sigma))+(2H\delta^\alpha_\mu-b^\alpha_\mu)\overset{*}{\nabla}_\sigma c^\alpha_\beta-2H\overset{*}{\nabla}_\beta c^\alpha_\sigma+c^\mu_\sigma\overset{*}{\nabla}_\beta b^\alpha_\mu)\xi^2\}.
\end{aligned}\tag{4.48}$$

Owing to

$$I^\alpha_\lambda\delta^\sigma_\beta-I^\sigma_\beta\delta^\alpha_\lambda=b^\sigma_\beta\delta^\alpha_\lambda-b^\alpha_\lambda\delta^\sigma_\beta$$

and applying (4.46), (4.45) becomes

$$\begin{aligned}
\nabla_\beta\nabla_3u^\alpha=&\overset{*}{\nabla}_\beta\tfrac{\partial u^\alpha}{\partial\xi}+\Phi^\alpha_{\beta\lambda}\tfrac{\partial u^\lambda}{\partial\xi}+\theta^{-1}I^\alpha_\beta\tfrac{\partial u^3}{\partial\xi}+\theta^{-1}(b^\sigma_\beta\delta^\alpha_\lambda-b^\alpha_\lambda\delta^\sigma_\beta)\overset{*}{\nabla}_\sigma u^\lambda\\
&+\theta^{-2}\{-\overset{*}{\nabla}_\beta b^\alpha_\sigma-\overset{*}{\nabla}_\sigma c^\alpha_\beta\xi+(\overset{*}{\nabla}_\beta(2H(K\delta^\alpha_\sigma-c^\alpha_\sigma))+(2H\delta^\alpha_\mu-b^\alpha_\mu)\overset{*}{\nabla}_\sigma c^\alpha_\beta\\
&-2H\overset{*}{\nabla}_\beta c^\alpha_\sigma+c^\mu_\sigma\overset{*}{\nabla}_\beta b^\alpha_\mu)\xi^2\}u^\sigma-\theta^{-1}(c^\alpha_\beta-2Kb^\alpha_\beta\xi+K^2\delta^\alpha_\beta\xi^2)u^3.
\end{aligned}\tag{4.49}$$

By a similar manner, we find

$$\begin{aligned}
\nabla_\beta\nabla_3u^3=&\tfrac{\partial}{\partial x^\beta}\nabla_3u^3+\Gamma^3_{\beta k}\nabla_3u^k-\Gamma^k_{\beta3}\nabla_ku^3=\tfrac{\partial^2u^3}{\partial x^\beta\partial\xi}+\Gamma^3_{\beta\lambda}\nabla_3u^\lambda-\Gamma^\lambda_{\beta3}\nabla_\lambda u^3\\
=&\tfrac{\partial^2u^3}{\partial x^\beta\partial\xi}+J_{\beta\lambda}(\tfrac{\partial u^\lambda}{\partial\xi}+\theta^{-1}I^\lambda_\sigma u^\sigma)-\theta^{-1}I^\lambda_\beta(\overset{*}{\nabla}_\lambda\lambda u^3+J_{\lambda\sigma}u^\sigma)\\
=&\tfrac{\partial}{\partial\xi}\overset{*}{\nabla}_\beta u^3+J_{\beta\lambda}\tfrac{\partial u^\lambda}{\partial\xi}-\theta^{-1}I^\lambda_\beta\overset{*}{\nabla}_\lambda u^3+\theta^{-1}(J_{\beta\lambda}I^\lambda_\sigma-J_{\lambda\sigma}I^\lambda_\beta)u^\sigma.
\end{aligned}$$

Note that

$$J_{\beta\lambda}I^\lambda_\sigma=-\theta c_{\beta\sigma}=-\theta c_{\sigma\beta}=J_{\lambda\sigma}I^\lambda_\beta,$$

we have

$$\nabla_\beta\nabla_3u^3=\overset{*}{\nabla}_\beta\tfrac{\partial u^3}{\partial\xi}+J_{\beta\lambda}\tfrac{\partial u^\lambda}{\partial\xi}-\theta^{-1}I^\lambda_\beta\overset{*}{\nabla}_\lambda u^3.$$

With (4.45) we assert

$$\nabla_\beta\nabla_3u^i=\overset{*}{\nabla}_\beta\tfrac{\partial u^i}{\partial\xi}+\Pi^{i3}_{\beta3,k}(\xi)\tfrac{\partial u^k}{\partial\xi}+\Pi^{i\mu}_{\beta3,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_{\beta3,k}(\xi)u^k,\tag{4.50}$$

where

$$\left\{\begin{array}{l}
\Pi^{i3}_{\beta 3,k}(\xi)=\begin{cases}\Phi^{\alpha}_{\beta\lambda}(\xi), & i=\alpha,\ k=\lambda,\\ \theta^{-1}I^{\alpha}_{\beta}, & i=\alpha,\ k=3,\\ J_{\beta\lambda}, & i=3,\ k=\lambda,\\ 0, & i=3,\ k=3,\end{cases}\\
\Pi^{i\mu}_{\beta 3,k}(\xi)=\begin{cases}\theta^{-1}(I^{\alpha}_{\nu}\delta^{\mu}_{\beta}-I^{\mu}_{\beta}\delta^{\alpha}_{\nu}), & i=\alpha,\ k=\nu,\\ 0, & i=\alpha,\ k=3,\\ 0, & i=3,\ k=\nu,\\ -\theta^{-1}I^{\mu}_{\beta}, & i=3,\ k=3,\end{cases}\\
\Pi^{i0}_{\beta 3,k}(\xi)=\begin{cases}\overset{*}{\nabla}_{\beta}(\theta^{-1}I^{\alpha}_{\nu})+\theta^{-1}(I^{\lambda}_{\nu}\Phi^{\alpha}_{\beta\lambda}-I^{\lambda}_{\beta}\Phi^{\alpha}_{\lambda\nu}), & i=\alpha,\ k=\nu\\ -\theta^{-1}I^{\alpha}_{\lambda}I^{\lambda}_{\beta}, & i=\alpha,\ k=3,\\ 0, & i=3,\ k=\nu,\\ 0, & i=3,\ k=3.\end{cases}
\end{array}\right. \tag{4.51}$$

To sum up we verify (4.30). □

Theorem 4.3. *Under the S-coordinate system in the 3D Riemannian space, the Bochner-Lplace operator acting on a vector field can be expressed in a rational polynomial with respect to the transverse variable (the length of geodesic curve) ξ.*

$$\Delta u^i=g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}u^i+\frac{\partial^2u^i}{\partial\xi^2}+\Pi^{i3}_{j}(\xi)\frac{\partial u^j}{\partial\xi}+\Pi^{i\mu}_{k}(\xi)\overset{*}{\nabla}_{\mu}u^k+\Pi^{i0}_{k}(\xi)u^k, \tag{4.52}$$

where

$$\left\{\begin{array}{l}
\Pi^{i3}_{j}(\xi)=\begin{cases}\theta^{-1}(-2b^{\alpha}_{\beta}+2K\xi\delta^{\alpha}_{\beta}), & i=\alpha,\ j=\beta,\\ \theta^{-1}(-2H+2K\xi), & i=3,\ j=3,\\ 0,\quad \textit{otherwise},\end{cases}\\
\Pi^{i\mu}_{k}(\xi)=\begin{cases}2g^{\alpha\mu}\Phi^{\gamma}_{\alpha\nu}-g^{\alpha\beta}\Phi^{\mu}_{\alpha\beta}\delta^{\gamma}_{\nu}, & i=\gamma,\ k=\nu\\ 2\theta^{-1}g^{\alpha\mu}I^{\gamma}_{\alpha}, & i=\gamma,\ k=3,\\ -2\theta^{-1}I^{\mu}_{\nu}, & i=3,\ k=\nu,\\ -g^{\alpha\beta}\Phi^{\mu}_{\alpha\beta}, & i=3,\ k=3,\end{cases}\\
\Pi^{i0}_{k}(\xi)=\begin{cases}g^{\alpha\beta}(Phi^{\gamma}_{\alpha\mu}\Phi^{\mu}_{\beta\nu}-\Phi^{\gamma}_{\mu\nu}\Phi^{\mu}_{\alpha\beta}+\overset{*}{\nabla}_{\alpha}\Phi^{\gamma}_{\beta\nu})+\theta^{-1}K\delta^{\gamma}_{\nu}, & i=\gamma,\ k=\nu,\\ g^{\alpha\beta}R^{\gamma}_{\alpha\beta}(\xi), & i=\gamma,\ k=3,\\ g^{\alpha\beta}\overset{*}{\nabla}_{\nu}J_{\alpha\beta}, & i=3,\ k=\nu,\\ -g^{\alpha\beta}c_{\alpha\beta}, & i=3,\ k=3.\end{cases}
\end{array}\right. \tag{4.53}$$

Remark 4.1. The Taylor expansions in (4.53) are given by

$$\begin{cases} \Pi_{\beta}^{\alpha 3}(\xi)=-2b_{\beta}^{\alpha}-2c_{\beta}^{\alpha}\xi-(Kb_{\beta}^{\alpha}+4Hc_{\beta}^{\alpha})\xi^2+\cdots, \\ \Pi_3^{33}(\xi)=-2H+(2K-4H^2)\xi+2H(3K-4H^2)\xi^2+\cdots, \\ \Pi_{\nu}^{\alpha\beta}(\xi)=\mu(\overset{*}{\nabla}_{\nu} b^{\alpha\beta}-2\delta_{\nu}^{\alpha}\alpha a^{\beta\lambda}\overset{*}{\nabla}_{\lambda} H)\xi+2\mu(b_{\mu}^{\alpha}\overset{*}{\nabla}_{\alpha} b^{\beta\mu}-\delta_{\nu}^{\alpha}b^{\beta\mu}\overset{*}{\nabla}_{\mu} H \\ \qquad +2b^{\beta\lambda}\overset{*}{\nabla}_{\lambda} b_{\nu}^{\alpha}-b_{\nu}^{\alpha}a^{\beta\mu}\overset{*}{\nabla}_{\mu}(2H^2-K))\xi^2+\cdots, \\ \Pi_3^{\alpha\beta}(\xi)=-2b^{\alpha\beta}-6c^{\alpha\beta}\xi+6(-2Hc^{\alpha\beta}+Kb^{\alpha\beta})\xi^2+\cdots, \\ \Pi_{\nu}^{3\mu}(\xi)=2b_{\nu}^{\mu}+2c_{\nu}^{\mu}\xi+(4Hc_{\nu}^{\mu}-2Kb_{\nu}^{\mu})\xi^2+\cdots, \\ \Pi_3^{3\mu}(\xi)=2a^{\lambda\mu}\overset{*}{\nabla}_{\lambda} H\xi+(2b^{\lambda\mu}\overset{*}{\nabla}_{\lambda} H-2a^{\lambda\mu}\overset{*}{\nabla}_{\lambda}(2H^2-K))\xi^2+\cdots, \\ \Pi_{\nu}^{\gamma 0}(\xi)=K\delta_{\nu}^{\gamma}+(2HK\delta_{\nu}^{\gamma}-\overset{*}{\Delta} b_{\nu}^{\gamma})\xi+(K(4H^2-K)\delta_{\nu}^{\gamma}-2b^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta} b_{\nu}^{\gamma} \\ \qquad -b_{\mu}^{\gamma}\overset{*}{\Delta} b_{\nu}^{\mu}-2a^{\lambda\mu}\overset{*}{\nabla}_{\mu} b_{\nu}^{\gamma}\overset{*}{\nabla}_{\lambda} H)\xi^2+\cdots, \\ \Pi_3^{\gamma 0}(\xi)=-2a^{\gamma\mu}\overset{*}{\nabla}_{\mu} H-(2a^{\gamma\mu}\overset{*}{\nabla}_{\mu}(2H^2-K)+a^{\alpha\beta}\overset{*}{\nabla}_{\alpha} c_{\beta}^{\gamma}+8Ha^{\gamma\mu}\overset{*}{\nabla}_{\mu} H)\xi \\ \qquad +\{2(K-2H^2)a^{\gamma\mu}\overset{*}{\nabla}_{\mu} H-4Ha^{\alpha\beta}\overset{*}{\nabla}_{\alpha} c_{\beta}^{\gamma}+a^{\alpha\beta}r_{\alpha\beta}^{\gamma} \\ \qquad -2b^{\alpha\beta}(\overset{*}{\nabla}_{\alpha} c_{\beta}^{\gamma}+4H\overset{*}{\nabla}_{\beta} b_{\alpha}^{\gamma})-3c^{\alpha\beta}\overset{*}{\nabla}_{\beta} b_{\alpha}^{\gamma}\}\xi^2+\cdots, \\ \Pi_{\nu}^{30}(\xi)=2\overset{*}{\nabla}_{\nu} H+(24H^2\overset{*}{\nabla}_{\nu} H-6\overset{*}{\nabla}_{\nu}(HK)-2b^{\alpha\beta}\overset{*}{\nabla}_{\nu} c_{\alpha\beta})\xi^2+\cdots, \\ \Pi_3^{30}(\xi)=2K-4H^2+(6HK-8H^3)\xi+(16H^2K-16H^4-2K^2)xi^2+\cdots. \end{cases}$$

Proof. Applying Lemma 2.2 and Lemma 4.1

$$\begin{aligned} \Delta u^i &= g^{lm}\nabla_m\nabla_l u^i = g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}u^i+\nabla_3\nabla_3 u^i \\ &= g^{\alpha\beta}(\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta} u^i - J_{\alpha\beta}\tfrac{\partial u^i}{\partial\xi}+\Pi_{\alpha\beta,k}^{i\mu}(\xi)\overset{*}{\nabla}_{\mu} u^k+\Pi_{\alpha\beta,k}^{i0}(\xi)u^k)+\tfrac{\partial^2 u^i}{\partial\xi^2}+l_j^i(\xi)\tfrac{\partial u^j}{\partial\xi} \\ &= g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta} u^i+\tfrac{\partial^2 u^i}{\partial\xi^2}+\Pi_j^{i3}(\xi)\tfrac{\partial u^j}{\partial\xi}+\Pi_k^{i\mu}(\xi)\overset{*}{\nabla}_{\mu} u^k+\Pi_k^{i0}(\xi)u^k. \end{aligned} \tag{4.54}$$

Since $g^{\alpha\beta}J_{\alpha\beta}=-\theta^{-1}I_{\alpha}^{\alpha}$, we obtain

$$\begin{cases} \Pi_j^{i3}(\xi)=l_j^i(\xi)-g^{\alpha\beta}J_{\alpha\beta}\delta_j^i \\ =\begin{cases} \theta^{-1}(-2b_{\beta}^{\alpha}+2K\xi\delta_{\beta}^{\alpha}), & i=\alpha,\ j=\beta, \\ \theta^{-1}(-2H+2K\xi), & i=3,\ j=3, \\ 0, \quad \text{otherwise}, \end{cases} \\ \Pi_k^{i\mu}(\xi)=g^{\alpha\beta}\Pi_{\alpha\beta,k}^{i\mu}(\xi), \quad \Pi_k^{i0}(\xi)=g^{\alpha\beta}\Pi_{\alpha\beta,k}^{i0}(\xi). \end{cases} \tag{4.55}$$

By virtue (4.31) and Lemma 2.5

$$\begin{cases} g^{\alpha\beta}\Pi^{i\mu}_{\alpha\beta,k}(\xi)=\begin{cases} 2g^{\alpha\mu}\Phi^{\gamma}_{\alpha\nu}-g^{\alpha\beta}\Phi^{\mu}_{\alpha\beta}\delta^{\gamma}_{\nu}, & i=\gamma,\ k=\nu\\ 2\theta^{-1}g^{\alpha\mu}I^{\gamma}_{\alpha}, & i=\gamma,\ k=3,\\ -2\theta^{-1}I^{\mu}_{\nu}, & i=3,\ k=\nu,\\ -g^{\alpha\beta}\Phi^{\mu}_{\alpha\beta}, & i=3,\ k=3,\end{cases}\\ g^{\alpha\beta}\Pi^{i0}_{\alpha\beta,k}(\xi)=\begin{cases} g^{\alpha\beta}(\Phi^{\gamma}_{\alpha\mu}\Phi^{\mu}_{\beta\nu}-\Phi^{\gamma}_{\mu\nu}\Phi^{\mu}_{\alpha\beta}+\overset{*}{\nabla}_{\alpha}\Phi^{\gamma}_{\beta\nu})+\theta^{-2}(I^{\gamma}_{\nu}I^{\alpha}_{\alpha}-I^{\gamma}_{\alpha}I^{\alpha}_{\nu}), & i=\gamma,\ k=\nu,\\ g^{\alpha\beta}R^{\gamma}_{\alpha\beta}(\xi), & i=\gamma,\ k=3,\\ g^{\alpha\beta}\overset{*}{\nabla}_{\nu}J_{\alpha\beta}, & i=3,k=\nu,\\ -g^{\alpha\beta}c_{\alpha\beta}, & i=3,k=3.\end{cases}\end{cases} \tag{4.56}$$

Applying (4.46) and Lemma 2.1 $K\delta^{\alpha}_{\beta}-2Hb^{\alpha}_{\beta}+c^{\alpha}_{\beta}=0$, it yield

$$I^{\gamma}_{\nu}I^{\alpha}_{\alpha}-I^{\gamma}_{\alpha}I^{\alpha}_{\nu}=2Hb^{\gamma}_{\nu}-c^{\gamma}_{\nu}-2HK\delta^{\gamma}_{\nu}+K^2\delta^{\gamma}_{\nu}\xi^2=K\delta^{\gamma}_{\nu}-2HK\delta^{\gamma}_{\nu}+K^2\delta^{\gamma}_{\nu}\xi^2=\theta K\delta^{\gamma}_{\nu}.$$

Hence, we obtain the 2nd and 3rd parts of (4.53). □

Theorem 4.4. *Under the S-cooedinate system in the 3D Riemann space, the Betrimi-Laplace operator $\Delta=g^{ij}\nabla_i\nabla_j$ is a polynomial with respect to transfers variable ξ, which can be made Taylor expansion with respect to ξ, i.e. for a two times differential function φ,*

$$\begin{cases} \Delta\varphi=g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi+g^{\alpha\beta}\Phi^{\lambda}_{\alpha\beta}\overset{*}{\nabla}_{\lambda}\varphi-2\theta^{-1}(K\xi-H)\frac{\partial\varphi}{\partial\xi}+\frac{\partial^2\varphi}{\partial\xi^2}\\ \quad=\frac{\partial^2\varphi}{\partial\xi^2}-(2H+(4H^2-2K)\xi+(8H^3-6HK)\xi^2)\frac{\partial\varphi}{\partial\xi}+\overset{0}{\Delta}\varphi+\overset{1}{\Delta}\varphi\xi+\overset{2}{\Delta}\varphi\xi^2+\cdots,\\ \overset{0}{\Delta}\varphi=\overset{*}{\Delta}\varphi:=a^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi,\\ \overset{1}{\Delta}\varphi=2b^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi-2(a^{\lambda\sigma}\overset{*}{\nabla}_{\sigma}H)\overset{*}{\nabla}_{\lambda}\varphi,\\ \overset{2}{\Delta}\varphi=3c^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi-2[(b^{\lambda\sigma}\overset{*}{\nabla}_{\sigma}H+a^{\lambda\sigma}\overset{*}{\nabla}_{\sigma}(2H^2-K))]\overset{*}{\nabla}_{\lambda}\varphi.\end{cases} \tag{4.57}$$

Proof. Indeed, by (2.34),

$$\begin{aligned}\Delta\varphi&=g^{ij}\nabla_i\nabla_j\varphi=g^{ij}(\partial_i\partial_j\varphi-\Gamma^k_{ij}\partial_k\varphi)=g^{ij}(\partial^2_{ij}\varphi-\Gamma^{\lambda}_{ij}\partial_{\lambda}\varphi-\Gamma^3_{ij}\partial_{\xi}\varphi)\\ &=g^{\alpha\beta}(\partial^2_{\alpha\beta}\varphi-\Gamma^{\lambda}_{\alpha\beta}\partial_{\lambda}\varphi-\Gamma^3_{\alpha\beta}\partial\xi\varphi)+g^{33}(\partial^2_{33}\varphi-\Gamma^{\lambda}_{33}\partial_{\lambda}\varphi-\Gamma^3_{33}\partial_{\xi}\varphi)\\ &=g^{\alpha\beta}(\partial^2_{\alpha\beta}\varphi-(\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}+\Phi^{\lambda}_{\alpha\beta})\partial_{\lambda}\varphi-\Gamma^3_{\alpha\beta}\partial_{\xi}\varphi)+\frac{\partial^2\varphi}{\partial\xi^2}\\ &=g^{\alpha\beta}(\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi+\Phi^{\lambda}_{\alpha\beta}\overset{*}{\nabla}_{\lambda}\varphi-J_{\alpha\beta}\frac{\partial\varphi}{\partial\xi})+\frac{\partial^2\varphi}{\partial\xi^2}\\ &=g^{\alpha\beta}\overset{*}{\nabla}_{\alpha}\overset{*}{\nabla}_{\beta}\varphi+g^{\alpha\beta}\Phi^{\lambda}_{\alpha\beta}\overset{*}{\nabla}_{\lambda}\varphi-g^{\alpha\beta}J_{\alpha\beta}\frac{\partial\varphi}{\partial\xi}+\frac{\partial^2\varphi}{\partial\xi^2}.\end{aligned}$$

Since

$$a^{\alpha\beta}b_{\alpha\beta}=2H,\quad a^{\alpha\beta}c_{\alpha\beta}=4H^2-2K,\quad b^{\alpha\beta}b_{\alpha\beta}=4H^2-2K,\quad b^{\alpha\beta}c_{\alpha\beta}=8H^3-6HK,$$

$$\begin{aligned}&a^{\alpha\beta}\overset{*}{\nabla}_{\alpha}b^{\lambda}_{\beta}=a^{\lambda\sigma}\overset{*}{\nabla}_{\sigma}(2H),\quad b^{\alpha\beta}\overset{*}{\nabla}_{\sigma}b_{\alpha\beta}=\tfrac{1}{2}\overset{*}{\nabla}_{\sigma}(b^{\alpha\beta}b_{\alpha\beta})=\overset{*}{\nabla}_{\sigma}(2H^2-K),\\ &b^{\alpha\beta}\overset{*}{\nabla}_{\alpha}b^{\lambda}_{\beta}=a^{\lambda\sigma}b^{\alpha\beta}\overset{*}{\nabla}_{\sigma}b_{\alpha\beta}=a^{\lambda\sigma}\tfrac{1}{2}\overset{*}{\nabla}_{\sigma}(b^{\alpha\beta}b_{\alpha\beta})=a^{\lambda\sigma}\overset{*}{\nabla}_{\sigma}(2H^2-K),\end{aligned}$$

it is not difficult to prove that

$$\begin{aligned}&g^{\alpha\beta}=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^2+\cdots,\\&g^{\alpha\beta}J_{\alpha\beta}=\theta^{-1}(2H-2K\xi)=2H+(4H^2-2K)\xi+(8H^3-6HK)\xi^2+\cdots,\\&g^{\alpha\beta}\Phi^{\lambda}_{\alpha\beta}=-2[a^{\lambda\sigma}\overset{*}{\nabla}_\sigma H\xi+(b^{\lambda\sigma}\overset{*}{\nabla}_\sigma H+a^{\lambda\sigma}\overset{*}{\nabla}_\sigma(2H^2-K))\xi^2]+\cdots.\end{aligned}$$

Therefore

$$\begin{aligned}\Delta\varphi=&\tfrac{\partial^2\varphi}{\partial\xi^2}-(2H+(4H^2-2K)\xi+(8H^3-6HK)\xi^2)\tfrac{\partial\varphi}{\partial\xi}+\overset{*}{\Delta}\varphi+2b^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\varphi\xi\\&+3c^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta\varphi\xi^2-2[a^{\lambda\sigma}\overset{*}{\nabla}_\sigma H\xi+(b^{\lambda\sigma}\overset{*}{\nabla}_\sigma H+a^{\lambda\sigma}\overset{*}{\nabla}_\sigma(2H^2-K))\xi^2]\overset{*}{\nabla}_\lambda\varphi+\cdots.\end{aligned}$$

The proof is completed. □

5 A dimensional splitting form for linearly elastic equations in 3D shell in $\Re^3$

As well know that the initial and boundary value problem of linearly elastic mechanics are give by

$$\mathcal{L}^i(u):=\nabla_j(A^{ijkl}e_{kl}(u))=f.$$

Since A^{ijkl} is defined by (2.55) and $g^{kl}e_{kl}(u)=\text{div}u$, we claim

$$\begin{aligned}A^{ijkl}e_{kl}(u)&=\lambda g^{ij}g^{kl}e_{kl}(u)+\mu\{g^{ik}g^{jl}e_{kl}(u)+g^{il}g^{jk}e_{kl}(u)\}\\&=\lambda g^{ij}\text{div}u+\mu(e^{ij}(u)+e^{ji}(u))=\lambda g^{ij}\text{div}u+2\mu e^{ij}(u).\end{aligned}$$

Therefore

$$\begin{aligned}&\mathcal{L}^i(u):=\nabla_j(A^{ijkl}e_{kl}(u))=\lambda g^{ij}\nabla_j\text{div}u+2\mu\nabla_je^{ij}(u),\\&\nabla_je^{ij}(u)=\tfrac{1}{2}\nabla_j(\nabla^iu^j+\nabla^ju^i)=\tfrac{1}{2}\nabla_j\nabla^iu^j+\tfrac{1}{2}\nabla_j\nabla^ju^i,\\&\nabla_j\nabla^ju^i=g^{jk}\nabla_j\nabla_ku^i=\Delta u^i.\end{aligned}$$

Applying the Ricci formula

$$\nabla_j\nabla^iu^j=g^{ik}\nabla_j\nabla_ku^j=g^{ik}\nabla_k\nabla_ju^j-g^{ik}R^j_{mkj}u^m=g^{ik}\nabla_k\text{div}u-R^j_{mkj}u^m,$$

where R^j_{mkl} are Riemannian curvature tensor of three dimensional Riemannian space E^3, if E^3 is Euclidian space, $R^j_{mkl}=0$. Furthermore

$$R^j_{mkj}=-R^j_{mjk}=-R_{mk},$$

where R_{mk} are Ricci curvature tensor.

In this case, the linear elasticity operator is given by

$$\mathcal{L}^i(u)=-\mu\Delta u^i-(\lambda+\mu)g^{ij}\nabla_j\text{div}u+g^{ik}R_{mk}u^m.$$

Finally elasticity equations in Euclidian space are given by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}-\mu\Delta u-(\lambda+\mu)\nabla\nabla\cdot u=f, & \text{in } \Omega\subset R^3,\\ u=0, & \text{on } \Gamma_{01},\\ \sigma\cdot n|_{\Gamma_{02}}=h, \sigma\cdot n|_{\Gamma_t\cup\Gamma_b}=0, & \\ u_{t=0}=U_0, \quad \frac{\partial u}{\partial t}=U_1, & \text{in } \Omega, \end{cases} \tag{5.1}$$

where the lateral surface $\Gamma_0=\Gamma_{01}\cup\Gamma_{02}$, σ is a stress tensor.

Theorem 5.1. *Under the S-coordinate system in E^3, Eq.* (5.1) *can be expressed as*

$$\begin{cases} \frac{\partial^2 u^i}{\partial t^2}+\mathcal{L}^i(u)=f^i,\\ \mathcal{L}^i(u)=-\mu\Delta u^i-(\lambda+\mu)g^{ij}\partial_j(\overset{*}{div}\boldsymbol{u}+\frac{\partial u^3}{\partial\xi}+d_k u^k), \end{cases}$$

in details,

$$\begin{cases} \mathcal{L}^\alpha(u)=-\mu\frac{\partial^2 u^\alpha}{\partial\xi^2}-m^{\alpha 3}_\beta(\xi)\frac{\partial u^\beta}{\partial\xi}-(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\frac{\partial u^3}{\partial\xi}-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^\alpha\\ \qquad -(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{div}\,u+m^{\alpha\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+m^{\alpha 0}_k(\xi)u^k,\\ \mathcal{L}^3(u)=-(\lambda+2\mu)\frac{\partial^2 u^3}{\partial\xi^2}+m^{33}_k(\xi)\frac{\partial u^k}{\partial\xi}-(\lambda+\mu)\overset{*}{div}\frac{\partial u}{\partial\xi}\\ \qquad -\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^3+m^{3\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+m^{30}_k(\xi)u^k, \end{cases} \tag{5.2}$$

where

$$\begin{cases} m^{\alpha\beta}_k(\xi)=-\mu\Pi^{\alpha\beta}_k(\xi)-(\lambda+\mu)g^{\alpha\beta}d_k(\xi),\\ m^{\alpha 0}_k(\xi)=-\mu\Pi^{\alpha 0}_k(\xi)-(\lambda+\mu)g^{\alpha\beta}d_{\beta k}(\xi),\\ m^{3\beta}_k(\xi)=-\mu\Pi^{3\beta}_k(\xi),\\ m^{30}_k(\xi)=-\mu\Pi^{30}_k(\xi)-(\lambda+\mu)d_{3k}(\xi),\\ m^{33}_k(\xi)=\mu(2H-2K\xi)\delta^3_k-(\lambda+\mu)d_k(\xi),\\ m^{\alpha 3}_\beta(\xi)=\mu\theta^{-1}(-2b^\alpha_\beta+2K\xi\delta^\alpha_\beta), \quad m^{\alpha 3}_3=0. \end{cases} \tag{5.3}$$

Matrices d_{ij} are given by

$$\begin{cases} d_\beta(\xi)=\theta^{-1}(-2\overset{*}{\nabla}_\beta H\xi+\overset{*}{\nabla}_\beta K\xi^2)=-\overset{*}{\nabla}_\beta H\xi+\overset{*}{\nabla}_\beta(K-2H^2)\xi^2+\cdots,\\ d_3(\xi)=\theta^{-1}(-2H+2K\xi)=-2H+(2K-4H^2)\xi+(6HK-8H^3)\xi^2+\cdots,\\ d_{33}(\xi)=\frac{\partial d_3}{\partial\xi}=\theta^{-2}(-4H^2+12HK\xi-6K^2\xi^2)\\ \qquad =-4H^2+4H(3K-4H^2)\xi+2K(4H^2-3K)\xi^2+\cdots,\\ d_{3\beta}(\xi)=\frac{\partial d_\beta}{\partial\xi}=\theta^{-2}(-2\overset{*}{\nabla}_\beta H+2\overset{*}{\nabla}_\beta K\xi+(2K\overset{*}{\nabla}_\beta H-2H\overset{*}{\nabla}_\beta K)\xi^2)\\ \qquad =-2\overset{*}{\nabla}_\beta H+2\overset{*}{\nabla}_\beta(K_2H^2)\xi+(8H\overset{*}{\nabla}_\beta K+\overset{*}{\nabla}_\beta(2HK-8H^3)\xi^2+\cdots,\\ d_{\beta\lambda}(\xi)=\overset{*}{\nabla}_\beta d_\lambda(\xi)=\theta^{-2}\{-\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda H\xi+(2H\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda H+\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda K-4\overset{*}{\nabla}_\beta H\overset{*}{\nabla}_\lambda H)\xi^2\\ \qquad +(-2K\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda H-2H\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda K+2\overset{*}{\nabla}_\beta K\overset{*}{\nabla}_\lambda H+2\overset{*}{\nabla}_\beta H\overset{*}{\nabla}_\mu K)\xi^3\\ \qquad +(K\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda K-\overset{*}{\nabla}_\beta K\overset{*}{\nabla}_\lambda K)\xi^4\}\\ \qquad =-\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda H\xi+(\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda K-2H\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\lambda H-4\overset{*}{\nabla}_\lambda H\overset{*}{\nabla}_\beta H)\xi^2+\cdots. \end{cases} \tag{5.4}$$

Remark 5.1. Taylor expansions of (5.3) are given by

$$
\left\{
\begin{aligned}
& m^{ij}_k(\xi)=m^{ij}_k(0)+m^{ij}_k(1)\xi+m^{ij}_k(2)\xi^2+\cdots,\\
& m^{\alpha\beta}_\nu(0)=0,\quad m^{\alpha\beta}_\nu(1)=\mu\overset{*}{\nabla}_\nu b^{\alpha\beta}+((\lambda+\mu)a^{\alpha\beta}\delta^\mu_\nu-2\mu a^{\beta\mu}\delta^\alpha_\nu)\overset{*}{\nabla}_\mu H,\\
& m^{\alpha\beta}_\nu(2)=(\lambda+\mu)(2b^{\alpha\beta}\overset{*}{\nabla}_\nu H+a^{\alpha\beta}\overset{*}{\nabla}_\nu(2H^2-K))\\
&\qquad\qquad +\mu\{2\overset{*}{\nabla}_\nu c^{\alpha\beta}+2b^{\beta\lambda}\overset{*}{\nabla}_\nu b^\alpha_\lambda-2H\overset{*}{\nabla}_\nu b^{\alpha\beta}-\delta^\alpha_\nu a^{\beta\lambda}\overset{*}{\nabla}_\lambda(11H^2-4K)\},\\
& m^{\alpha\beta}_3(0)=2\mu b^{\alpha\beta}+2(\lambda+\mu)Ha^{\alpha\beta},\quad m^{\alpha\beta}_3(1)=2(\lambda+4\mu)c^{\alpha\beta},\\
& m^{\alpha\beta}_3(2)=(14\mu+6\lambda)Hc^{\alpha\beta}-(16\mu+4\lambda)Kb^{\alpha\beta},\\
& m^{\alpha0}_\nu(0)=\mu K\delta^\alpha_\nu,\quad m^{\alpha0}_\nu(1)=\mu(\overset{*}{\Delta}b^\alpha_\nu-2HK\delta^\alpha_\nu+(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu H,\\
& m^{\alpha0}_\nu(2)=\mu\{K(K-4H^2)\delta^\alpha_\nu+2b^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma b^\alpha_\nu+b^\alpha_\lambda\overset{*}{\Delta}b^\lambda_\nu\}=2a^{\lambda\sigma}\overset{*}{\nabla}_\sigma b^\alpha_\nu\overset{*}{\nabla}_\lambda H\\
&\qquad\qquad +(\lambda+\mu)\{2Ha^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu H-a^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu K+4a^{\alpha\beta}\overset{*}{\nabla}_\beta H\overset{*}{\nabla}_\nu K\},\\
& m^{\alpha0}_3(0)=(\lambda+3\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta H,\\
& m^{\alpha0}_3(1)=\mu(2a^{\alpha\mu}\overset{*}{\nabla}_\mu(2H^2-K)+a^{\lambda\beta}\overset{*}{\nabla}_\lambda c^\alpha_\beta+8Ha^{\alpha\mu}\overset{*}{\nabla}_\mu H)\\
&\qquad\qquad -(\lambda+\mu)\{4(b^{\alpha\beta}-2Ha^{\alpha\beta})\overset{*}{\nabla}_\beta H+2a^{\alpha\beta}\overset{*}{\nabla}_\beta K\},
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
& m^{\alpha0}_3(2)=\mu\{2(2H^2-K)a^{\gamma\mu}\overset{*}{\nabla}_\mu H+4Ha^{\alpha\beta}\overset{*}{\nabla}_\alpha c^\gamma_\beta-a^{\alpha\beta}r^\gamma_{\alpha\beta}+2b^{\alpha\beta}(\overset{*}{\nabla}_\alpha c^\gamma_\beta+4H\overset{*}{\nabla}_\beta b^\gamma_\alpha)\\
&\qquad\qquad +3c^{\alpha\beta}\overset{*}{\nabla}_\beta b^\gamma_\alpha\}+(\lambda+\mu)\{6c^{\alpha\beta}\overset{*}{\nabla}_\beta H+2b^{\alpha\beta}\overset{*}{\nabla}_\gamma\beta(8H^2-2K)\\
&\qquad\qquad +a^{\alpha\beta}(\overset{*}{\nabla}_\beta(\tfrac{64}{3}H^3-10HK)-2H\overset{*}{\nabla}_\beta K)\},
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
& m^{3\beta}_\nu(\xi)=\mu\{-2b^\beta_\nu-2c^\beta_\nu\xi+2(kb^\beta_\nu-2Hc^\beta_n u)\xi^2\}+\cdots,\\
& m^{3\beta}_3(\xi)=\mu\{-2a^{\beta\lambda}\overset{*}{\nabla}_\lambda H\xi+(2a^{\beta\lambda}\overset{*}{\nabla}_\lambda(2H^2-K)-2b^{\beta\lambda}\overset{*}{\nabla}_\lambda H)\xi^2\}+\cdots,\\
& m^{30}_\nu(\xi)=2\lambda\overset{*}{\nabla}_\nu H+2(\lambda+\mu)\overset{*}{\nabla}_\nu(K-2H^2)\xi\\
&\qquad\qquad +\{\lambda(\overset{*}{\nabla}_\nu(8H^3-2HK)-4H\overset{*}{\nabla}_\nu K)+\mu(2b^{\alpha\beta}\overset{*}{\nabla}_\nu c_{\alpha\beta}+4K\overset{*}{\nabla}_\nu H)\}\xi^2+\cdots,\\
& m^{30}_3(\xi)=4\lambda H^2+2\mu(4H^2-K)+(4H\lambda+6\mu H)(4H^2-3K)\xi\\
&\qquad\qquad +\{2\lambda(3K^2-28H^2K+24H^4)+8\mu(K^2-9H^2K+8H^4)\}\xi^2+\cdots,\\
& m^{33}_\nu(\xi)=(\lambda+\mu)(2\overset{*}{\nabla}_\nu H\xi+\overset{*}{\nabla}_\nu(2H^2-K)\xi^2),\\
& m^{33}_3(\xi)=(\lambda+2\mu)2H+\{(\lambda+\mu)(4H^2-2K)-2\mu K\}\xi+(\lambda+\mu)2H(4H^2-3K)\xi^2+\cdots,\\
& m^{\alpha3}_\nu(\xi)=-2\mu b^\alpha_\nu+2\mu(K\delta^\alpha_\nu-2Hb^\alpha_\nu)\xi+2\mu(2HK\delta^\alpha_\nu+(K-4H^2)b^\alpha_\nu)\xi+\cdots,\\
& m^{\alpha3}_3(\xi)=0.
\end{aligned}
\right.
$$

Proof. At first we prove (5.2). To do that, since (4.30) and (2.35), rewritten (5.1) into components form

$$
\begin{aligned}
&\frac{\partial^2u^i}{\partial t^2}-\mu\Big\{\frac{\partial^2u^i}{\partial\xi^2}+\Pi^{i3}_j\frac{\partial u^j}{\partial\xi}+g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^i+\Pi^{i\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+\Pi^{i0}_k(\xi)u^k\Big\}\\
&\qquad -(\lambda+\mu)g^{ij}\partial_j\mathrm{div}u=f^i.
\end{aligned}
\tag{5.5}
$$

In addition,

$$
\begin{aligned}
&\mathrm{div} u = \frac{\partial u^3}{\partial \xi} + \overset{*}{\mathrm{div}}\, u + d_k(\xi) u^k,\\
&d_\lambda(\xi) = \theta^{-1}(-2\overset{*}{\nabla}_\lambda H\xi + \overset{*}{\nabla}_\lambda K\xi^2),\;\; d_3(\xi) = \theta^{-1}(-2H + 2K\xi),
\end{aligned} \tag{5.6}
$$

$$
\begin{cases}
g^{ij}\partial_j \mathrm{div}(u) = \{g^{\alpha\beta}\overset{*}{\nabla}_\beta \mathrm{div} u, \frac{\partial}{\partial\xi}\mathrm{div} u\},\\
\overset{*}{\nabla}_\beta \mathrm{div} u = \overset{*}{\nabla}_\beta \frac{\partial u^3}{\partial \xi} + \overset{*}{\nabla}_\beta \overset{*}{\mathrm{div}}\, u + d_{\beta k}(\xi) u^k + d_k(\xi)\overset{*}{\nabla}_\beta u^k,\\
\frac{\partial}{\partial \xi}\mathrm{div} u = \frac{\partial^2 u^3}{\partial \xi^2} + d_k(\xi)\frac{\partial u^k}{\partial \xi} + \overset{*}{\mathrm{div}}\frac{\partial u}{\partial \xi} + d_{3k}(\xi) u^k,\\
d_{mk} = \{\overset{*}{\nabla}_\beta d_k(\xi), \frac{\partial}{\partial \xi} d_k\} \quad (\text{see}(5.4)).
\end{cases} \tag{5.7}
$$

Therefore

$$
g^{ij}\partial_j \mathrm{div}(u) = \begin{cases}
g^{\alpha\beta}[\overset{*}{\nabla}_\beta \frac{\partial u^3}{\partial \xi} + \overset{*}{\nabla}_\beta \overset{*}{\mathrm{div}}\, u + d_{\beta k}(\xi) u^k + d_k(\xi)\overset{*}{\nabla}_\beta u^k], & i = \alpha\\
\frac{\partial^2 u^3}{\partial \xi^2} + d_k(\xi)\frac{\partial u^k}{\partial \xi} + \overset{*}{\mathrm{div}}\frac{\partial u}{\partial \xi} + d_{3k}(\xi) u^k, & i = 3.
\end{cases} \tag{5.8}
$$

Substituting (5.6) leads to (5.2), we end our proof. □

Theorem 5.2. *Under the S-coordinate system in E^3, if the solution of* (5.1) *in neighborhood of surface $\Im$ and right hand f can be made Taylor expansions with respect to transverse variable ξ*

$$
\begin{cases}
u(x,\xi) = u_0(x) + u_1(x)\xi + u_2(x)\xi^2 + \cdots,\\
f(x,\xi) = f_0(x) + f_1(x)\xi + f_2(x)\xi^2 + \cdots,
\end{cases} \tag{5.9}
$$

then the linear elasticity operators cam be made Taylor expansions as

$$
\begin{cases}
\mathcal{L}^i(u) = \mathcal{L}^i_0(u_0,u_1,u_2) + \mathcal{L}^i_1(u_0,u_1.u_2)\xi + \mathcal{L}^i_2(u_0,u_1,u_2)\xi^2 + \cdots,\\
\mathcal{L}^i_0(u_0,u_1,u_2) = \mathcal{K}^i_0(u_0) + L^i_0(u_1,u_2),\\
\mathcal{L}^i_1(u_0,u_1.u_2) = \mathcal{K}^i_0(u_1) + \mathcal{K}^i_1(u_0) + L^i_1(u_1,u_2),\\
\mathcal{L}^i_2(u_0,u_1.u_2) = \mathcal{K}^i_0(u_2) + \mathcal{K}^i_1(u_1) + \mathcal{K}^i_2(u_0) + L^i_2(u_1,u_2),
\end{cases} \tag{5.10}
$$

where $u_0(x), u_1(x), u_2(x)$ satisfy following boundary value problems

$$
\begin{cases}
\frac{\partial^2 u^i_0}{\partial t^2} + \mathcal{K}^i_0(u_0) + L^i_0(u_1,u_2) = f^i_0,\\
\frac{\partial^2 u^i_1}{\partial t^2} + \mathcal{K}^i_0(u_1) + \mathcal{K}^i_1(u_0) + L^i_1(u_1,u_2) = f^i_1,\\
\frac{\partial^2 u^i_2}{\partial t^2} + \mathcal{K}^i_0(u_2) + \mathcal{K}^i_1(u_1) + \mathcal{K}^i_2(u_0) + L^i_2(u_1,u_2) = f^i_2,
\end{cases} \tag{5.11}
$$

with boundary conditions in (5.1) *on the boundary $\gamma_1 = \Gamma_{02} \cap \{\xi = 0\}$ of middle surface*

$$
u_k|_{\gamma_0} = 0, k = 1,2,3; \quad \sigma^{ij} n_j|_{\gamma_1} = \sigma^{ij}_0(u_0) n_j = h^i(\xi = 0), \tag{5.12}
$$

where

$$\left\{\begin{array}{l} \mathcal{K}_0^\alpha(u_0)=-\mu\overset{*}{\Delta}u_0^\alpha-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_0)-m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha 0}(0)u_0^k,\\ \mathcal{K}_1^\alpha(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha-2(\lambda+\mu)b^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{div}\,u_0-m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha 0}(1)u_0^k,\\ \mathcal{K}_2^\alpha(u_0)=-3\mu c^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha-3(\lambda+\mu)b^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{div}\,u_0-m_k^{\alpha\beta}(2)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha 0}(2)u_0^k, \end{array}\right. \tag{5.13}$$

$$\left\{\begin{array}{l} \mathcal{K}_0^3(u_0)=-\mu\overset{*}{\Delta}u_0^3+\mu m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_0^k,\\ \mathcal{K}_1^3(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha+m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{30}(1)u_0^k,\\ \mathcal{K}_2^3(u_0)=3\mu c^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^3+m_k^{3\beta}(2)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(2)u_0^k, \end{array}\right. \tag{5.14}$$

$$\left\{\begin{array}{l} L_0^\alpha(u_1,u_2)=-2\mu u_2^\alpha-m_\beta^{\alpha 3}(0)u_1^\beta-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3,\\ L_1^\alpha(u_1,u_2)=-m_\beta^{\alpha 3}(1)u_1^\beta-m_\beta^{\alpha 3}(0)u_2^3-2(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_2^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3,\\ L_2^\alpha(u_1,u_2)=-m_\beta^{\alpha 3}(1)u_2^\beta-m_\beta^{\alpha 3}(2)u_1^\beta, \end{array}\right. \tag{5.15}$$

$$\left\{\begin{array}{l} L_0^3(u_1,u_2)=-2(\lambda+2\mu)u_2^3+m_k^{33}(0)u_1^k-(\lambda+\mu)\overset{*}{div}\,u_1\\ L_1^3(u_1,u_2,u_3)=m_k^{33}(0)u_2^3+m_k^{33}(1)u_1^3-2(\lambda+\mu)\overset{*}{div}\,u_2^3,\\ L_2^3(u_1,u_2,u_3,u_4)=2m_k^{33}(1)u_2^k+m_k^{33}(2)u_1^k. \end{array}\right. \tag{5.16}$$

Here m_k^{ij} are defined by (5.3).

Proof. Consider (5.2)

$$\begin{aligned} \mathcal{L}^\alpha(u)=&-\mu\frac{\partial^2 u^\alpha}{\partial\xi^2}-m_\beta^{\alpha 3}(\xi)\frac{\partial u^\beta}{\partial\xi}-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^\alpha\\ &-(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u+\frac{\partial u^3}{\partial\xi})+m_k^{\alpha\beta}(\xi)\overset{*}{\nabla}_\beta u^k+m_k^{\alpha 0}(\xi)u^k. \end{aligned} \tag{5.17}$$

Using

$$g^{\alpha\beta}=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^2+\cdots,$$

gives

$$\begin{aligned} &-\mu\frac{\partial^2 u^\alpha}{\partial\xi^2}=-2\mu u_2^\alpha+\cdots,\\ &-m_\beta^{\alpha 3}(\xi)\frac{\partial u^\beta}{\partial\xi}=-\{m_\beta^{\alpha 3}(0)u_1^\beta+(m_\beta^{\alpha 3}(1)u_1^\beta+2m_\beta^{\alpha 3}(o)u_2^\beta)\xi\\ &\qquad\qquad+(2m_\beta^{\alpha 3}(1)u_2^\beta+m_\beta^{\alpha 3}(2)u_1^\beta)\xi^2+\cdots\},\\ &-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^\alpha=-\mu\{\overset{*}{\Delta}u_0^\alpha+(\overset{*}{\Delta}u_1^\alpha+2b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha)\xi\\ &\qquad+(\overset{*}{\Delta}u_2^\alpha+2b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_1^\alpha+3c^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha)\xi^2\}+\cdots, \end{aligned}$$

$$
\begin{aligned}
&(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u+\tfrac{\partial u^3}{\partial\xi})=(\lambda+\mu)\{a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0+u_1^3)\\
&\qquad+(a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_1+2u_2^3)+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0+u_1^3))\xi+(a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_2)\\
&\qquad+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_1+2u_2^3)+3c^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0+u_1^3))\xi^2\}+\cdots,\\
&m_k^{\alpha\beta}(\xi)\overset{*}{\nabla}_\beta u^k=m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k+(m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_1^k+m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k)\xi\\
&\qquad+(m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_2^k+m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_1^k+m_k^{\alpha\beta}(2)\overset{*}{\nabla}_\beta u_0^k)+\cdots,\\
&m_k^{\alpha0}(\xi)u^k=m_k^{\alpha0}(0)u_0^k+(m_k^{\alpha0}(0)u_1^k+m_k^{\alpha0}(1)u_0^k)\xi+(m_k^{\alpha0}(0)u_2^k\\
&\qquad+m_k^{\alpha0}(1)u_1^k+m_k^{\alpha0}(2)u_0^k)+\cdots.
\end{aligned}
\tag{5.18}
$$

Denote

$$
\left\{\begin{aligned}
&\mathcal{K}_0^\alpha(u_0)=-\mu\overset{*}{\Delta}u_0^\alpha-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0)-m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha0}(0)u_0^k,\\
&L_0^\alpha(u_1,u_2)=-2\mu u_2^\alpha-m_\beta^{\alpha3}(0)u_1^\beta-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3,
\end{aligned}\right.
\tag{5.19}
$$

$$
\left\{\begin{aligned}
&\mathcal{K}_1^\alpha(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha-2(\lambda+\mu)b^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\mathrm{div}}\,u_0-m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha0}(1)u_0^k,\\
&L_1^\alpha(u_1,u_2)=-m_\beta^{\alpha3}(1)u_1^\beta-m_\beta^{\alpha3}(0)u_2^3-2(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_2^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3,
\end{aligned}\right.
\tag{5.20}
$$

$$
\left\{\begin{aligned}
&\mathcal{K}_2^\alpha(u_0)=-3\mu c^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha-3(\lambda+\mu)b^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\mathrm{div}}\,u_0-m_k^{\alpha\beta}(2)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha0}(2)u_0^k,\\
&L_2^\alpha(u_1,u_2)=-m_\beta^{\alpha3}(1)u_2^\beta-m_\beta^{\alpha3}(2)u_1^\beta.
\end{aligned}\right.
\tag{5.21}
$$

Taking (5.18)-(5.21) into account, (5.17) can be made expansion

$$
\begin{aligned}
\mathcal{L}^\alpha(u)=&\mathcal{K}_0^\alpha(u_0)+L_0^\alpha(u_1,u_2)+\{\mathcal{K}_0^\alpha(u_1)+\mathcal{K}_1(u_0)+L_1^\alpha(u_1,u_2,u_3)\}\xi\\
&+\{\mathcal{K}_0^\alpha(u_2)+\mathcal{K}_1(u_1)+\mathcal{K}_2^\alpha(u_0)+L_2^\alpha(u_1,u_2,u_3,u_4)\}\xi^2+\cdots.
\end{aligned}
$$

So that we obtain following equations

$$
\left\{\begin{aligned}
&\frac{\partial^2u_0^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_0)+L_0^\alpha(u_1,u_2)=f_0^\alpha,\\
&\frac{\partial^2u_1^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_1)+\mathcal{K}_1^\alpha(u_0)+L_1^\alpha(u_1,u_2,u_3)=f_1^\alpha,\\
&\frac{\partial^2u_2^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_2)+\mathcal{K}_1^\alpha(u_1)+\mathcal{K}_2^\alpha(u_0)+L_2^\alpha(u_1,u_2,u_3,u_4)=f_2^\alpha,\\
&\cdots.
\end{aligned}\right.
\tag{5.22}
$$

By similar manner, we assert

$$
\begin{aligned}
\mathcal{L}^3(u)=&-(\lambda+2\mu)\frac{\partial^2u^3}{\partial\xi^2}+m_k^{33}(\xi)\frac{\partial u^k}{\partial\xi}-(\lambda+\mu)\overset{*}{\mathrm{div}}\frac{\partial u}{\partial\xi}\\
&-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^3+m_k^{3\beta}(\xi)\overset{*}{\nabla}_\beta u^k+m_k^{30}(\xi)u^k.
\end{aligned}
\tag{5.23}
$$

Substituting (5.10) into (5.23) leads to

$$\begin{aligned}\mathcal{L}^3(u)=&-2(\lambda+2\mu)u_2^3+m_k^{33}(0)u_1^k-(\lambda+\mu)\overset{*}{\operatorname{div}}u_1-\mu\overset{*}{\Delta}u_0^3+m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k\\&+m_k^{30}(0)u_0^k+\{m_k^{33}(0)2u_2^k+m_k^{33}(1)u_1^k-2(\lambda+\mu)\overset{*}{\operatorname{div}}u_2-\mu(\overset{*}{\Delta}u_1^3\\&+2b^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^3)+m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_1^k+m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_1^k+m_k^{30}(1)u_0^k\}\xi\\&+\{m_k^{33}(1)2u_2^k+m_k^{33}(2)u_1^k-\mu(\overset{*}{\Delta}u_2^3+2b^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_1^3+3c^{\alpha\beta})\overset{*}{\nabla}_\beta\alpha\overset{*}{\nabla}_\beta u_0^3\\&+m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_2^k+m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_1^k+m_k^{3\beta}(2)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_2^k\\&+m_k^{30}(1)u_1^k+m_k^{33}(2)u_0^k\}\xi^2+\cdots.\end{aligned}\tag{5.24}$$

Hence it can be expressed as

$$\begin{aligned}\mathcal{L}^3(u)=&\mathcal{K}_0^\alpha(u_0)+L_0^3(u_1,u_2)+\{\mathcal{K}_0^3(u_1)+\mathcal{K}_1^3(u_0)+L_1^3(u_1,u_2,u_3)\}\xi\\&+\{\mathcal{K}_0^3(u_2)+\mathcal{K}_1^3(u_1)+\mathcal{K}_2^3(u_0)+L_2^3(u_1,u_2,u_3,u_4)\}\xi^2+\cdots.\end{aligned}\tag{5.25}$$

Consequently, we obtain following equations

$$\begin{cases}\frac{\partial^2u_0^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_0)+L_0^\alpha(u_1,u_2)=f_0^\alpha,\\\frac{\partial^2u_1^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_1)+\mathcal{K}_1^\alpha(u_0)+L_1^\alpha(u_1,u_2,u_3)=f_1^\alpha,\\\frac{\partial^2u_2^\alpha}{\partial t^2}+\mathcal{K}_0^\alpha(u_2)+\mathcal{K}_1^\alpha(u_1)+\mathcal{K}_2^\alpha(u_0)+L_2^\alpha(u_1,u_2,u_3,u_4)=f_2^\alpha,\\\cdots,\end{cases}\tag{5.25}$$

where

$$\begin{cases}\mathcal{K}_0^3(u_0)=-\mu\overset{*}{\Delta}u_0^3+m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_0^k,\\\mathcal{K}_1^3(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha+m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{30}(1)u_0^k,\\\mathcal{K}_2^3(u_0)=3\mu c^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^3+m_k^{3\beta}(2)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(2)u_0^k,\\L_0^3(u_1,u_2)=-2(\lambda+2\mu)u_2^3+m_k^{33}(0)u_1^k-(\lambda+\mu)\overset{*}{\operatorname{div}}u_1,\\L_1^3(u_1,u_2,u_3)=m_k^{33}(0)u_2^3+m_k^{33}(1)u_1^3-2(\lambda+\mu)\overset{*}{\operatorname{div}}u_2^3,\\L_2^3(u_1,u_2,u_3,u_4)=2m_k^{33}(1)u_2^k+m_k^{33}(2)u_1^k.\end{cases}\tag{5.26}$$

We then complete our proof. □

Theorem 5.3. *Under S-coordinate system in* E^3, *if* (5.5) *is satisfied then linearly stress tensor* $\sigma^{ij}(u)=A^{ijkl}e_{kl}(u)$ *can be made Taylor expansion with respect to transverse variable* ξ

$$\sigma^{ij}(u)=\sigma_0^{ij}(u_0)+\sigma_1^{ij}(u_0,u_1)\xi+\sigma_2^{ij}(u_0,u_1,u_2)\xi^2+\cdots,\tag{5.27}$$

where

$$\begin{cases}\sigma_0^{\alpha\beta}(u_0)=A_0^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)+\lambda a^{\alpha\beta}u_1^3,\\\sigma_1^{\alpha\beta}(u_0,u_1)=A_0^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_1)+\overset{1}{\gamma}_{\lambda\sigma}(u_0))+A_1^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)+2\lambda a^{\alpha\beta}u_2^3+2\lambda b^{\alpha\beta}u_1^3\\\sigma_2^{\alpha\beta}(u_0,u_1,u_2)=A_0^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_2)+\overset{1}{\gamma}_{\lambda\sigma}(u_1)+\overset{2}{\gamma}_{\lambda\sigma}(u_0))+A_1^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_1)+\overset{1}{\gamma}_{\lambda\sigma}(u_0))\\\qquad+A_2^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)+4\lambda b^{\alpha\beta}u_2^3+3\lambda c^{\alpha\beta}u_1^3\end{cases}\tag{5.28}$$

$$\begin{cases} \sigma_0^{33}(u_0)=\lambda(\overset{*}{div}\,u_0+c_k(0)u_0^k)+(\lambda+2\mu)u_1^3, \\ \sigma_1^{33}(u_0,u_1)=\lambda(\overset{*}{div}\,u_1+c_k(0)u_1^k+c_k(1)u_0^k)+2(\lambda+2\mu)u_2^3, \\ \sigma_2^{33}(u_0,u_1,u_2)=\lambda(\overset{*}{div}\,u_2+c_k(0)u_2^k+c_k(1)u_1^k+c_k(2)u_0^k), \\ \sigma_0^{\alpha3}(u_0)=\mu(u_1^\alpha+a^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3), \\ \sigma_1^{\alpha3}(u_1)=\mu(2u_2^\alpha+a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3), \\ \sigma_2^{\alpha3}(u_0,u_1,u_2)=\mu(a^{\alpha\beta}\overset{*}{\nabla}_\beta u_2^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3+3c^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3), \end{cases} \tag{5.29}$$

where

$$\begin{cases} c_\alpha(0)=0, \quad c_\alpha(1)=-2\overset{*}{\nabla}_\alpha H, \quad c_\alpha(2)=\overset{*}{\nabla}_\alpha(K-2H^2), \\ c_3(0)=-2H, \quad c_3(1)=(2K-4H^2), \quad c_3(2)=(6HK-8H^3). \end{cases} \tag{5.30}$$

The boundary conditions on top and bottom surface of shell are given by

$$\begin{cases} \sigma(u)\cdot n(\varepsilon n)=\sigma^{i3}(u)=\sigma_0^{i3}(u_0)+\sigma_1^{i3}(u_0,u_1)\varepsilon+\sigma_2^{i3}(u_0,u_1,u_2)\varepsilon^2=h^i, \quad on \;\; \Gamma_t, \\ \sigma(u)\cdot n(-\varepsilon n)=-\sigma^{i3}(u)=-\{\sigma_0^{i3}(u_0)-\sigma_1^{\alpha3}(u_0,u_1)\varepsilon+\sigma_2^{i3}(u_0,u_1,u_2)\varepsilon^2\}=h^i, \quad on \;\; \Gamma_b. \end{cases} \tag{5.31}$$

Proof. As well known that linearly stress tensor $\sigma^{ij}(u)$ of isotropic linearly elastic materials corresponding to displacement vector u is given by

$$\sigma^{ij}(u)=A^{ijkm}e_{km}(u).$$

Elastic coefficient tensor of isotropic linearly elastic materials is given by (2.55) and can be made Taylor expansion with respect to transverse variable ξ by (2.57). In addition, owing to Lemma 2.6 and $\overset{k}{\gamma}_{ij}(u)$ are linear form for u, therefore

$$\begin{aligned} e_{\lambda\sigma}(u)&=\gamma_{\lambda\sigma}(u)+\overset{1}{\gamma}_{\lambda\sigma}(u)\xi+\overset{2}{\gamma}_{\lambda\sigma}(u)\xi^2 \\ &=\gamma_{\lambda\sigma}(u_0)+(\gamma_{\lambda\sigma}(u_1)+\overset{1}{\gamma}_{\lambda\sigma}(u_0))\xi+(\gamma_{\lambda\sigma}(u_2)+\overset{1}{\gamma}_{\lambda\sigma}(u_1)+\overset{2}{\gamma}_{\lambda\sigma}(u_0))\xi^2+\cdots, \\ e_{3\sigma}(u)&=\tfrac{1}{2}(g_{\lambda\sigma}\tfrac{\partial u^\lambda}{\partial\xi}+\overset{*}{\nabla}_\sigma u^3) \\ e_{33}(u)&=\tfrac{\partial u^3}{\partial\xi}=u_1^3+2u_2^3\xi+\cdots. \end{aligned}$$

Note non vanishing A^{ijkm} by Lemma 2.7

$$\begin{aligned} &\sigma^{\alpha\beta}(u)=A^{\alpha\beta lm}e_{lm}(u)=A^{\alpha\beta\lambda\sigma}e_{\lambda\sigma}(u)+A^{\alpha\beta33}e_{33}(u)=A^{\alpha\beta\lambda\sigma}e_{\lambda\sigma}(u)+\lambda g^{\alpha\beta}e_{33}(u), \\ &\sigma^{3\alpha}(u)=A^{3\alpha lm}e_{lm}(u)=A^{3\alpha3\beta}e_{3\beta}(u)+A^{3\alpha\beta3}E_{\beta3}(u)=2\mu g^{\alpha\beta}e_{3\alpha}(u)=\mu(\tfrac{\partial u^\alpha}{\partial\xi}+g^{\alpha\beta}\overset{*}{\nabla}_\beta u^3), \\ &\sigma^{33}=A^{33lm}e_{lm}(u)=A^{33\lambda\sigma}e_{\lambda\sigma}(u)+A^{3333}e_{33}(u)=\lambda g^{\lambda\sigma}e_{\lambda\sigma}(u)+(\lambda+2\mu)\tfrac{\partial u^3}{\partial\xi} \\ &\quad=\lambda(\overset{*}{\mathrm{div}}\,u+c_k(\xi)u^k)+(\lambda+2\mu)\tfrac{\partial u^3}{\partial\xi}, \end{aligned}$$

where $c_k(\xi)$ defined by (2.34). Thanks to (2.20)

$$g^{\alpha\beta}=a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi+\cdots.$$

We assert that

$$\begin{aligned}
\sigma^{\alpha\beta}(u) &= A^{\alpha\beta\lambda\sigma}e_{\lambda\sigma}(u)+A^{\alpha\gamma 33}e_{33}(u)\\
&=(A_0^{\alpha\beta\lambda\sigma}+A_1^{\alpha\beta\lambda\sigma}\xi+A_2^{\alpha\beta\lambda\sigma}\xi^2+\cdots)(\gamma_{\lambda\sigma}(u_0+u_1\xi+u_2\xi^2+\cdots)+\overset{1}{\gamma}_{\lambda\sigma}(u_0+u_1\xi+u_2\xi^2\\
&\quad +\cdots)\xi+\overset{2}{\gamma}_{\lambda\sigma}(u_0+u_1\xi+u_2\xi^2+\cdots)\xi^2)+\lambda g^{\alpha\beta}\tfrac{\partial}{\partial\xi}(u_0^3+u_1^3\xi+u_2^3\xi^2+\cdots)\\
&=A_0^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)+\{A_0^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_1)+\overset{1}{\gamma}_{\lambda\sigma}(u_0))A_1^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)\}\xi+\{A_0^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_2)\\
&\quad +\overset{1}{\gamma}_{\lambda\sigma}(u_1)+\overset{2}{\gamma}_{\lambda\sigma}(u_0))A_1^{\alpha\beta\lambda\sigma}(\gamma_{\lambda\sigma}(u_1)+\overset{1}{\gamma}_{\lambda\sigma}(u_0))+A_2^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(u_0)\}\xi^2\\
&\quad +\lambda\{a^{\alpha\beta}u_1^3+2(b^{\alpha\beta}u_1^3+a^{\alpha\beta}u_2^3)\xi+(3c^{\alpha\beta}u_1^3+4b^{\alpha\beta}u_2^3)\xi^2\}+\cdots.
\end{aligned}$$

From this it yields (5.29). Moreover

$$\begin{cases}
\sigma^{33}(u)=\lambda\{\overset{*}{\mathrm{div}}\,u+c_k(\xi)u^k\}+(\lambda+2\mu)\frac{\partial u^3}{\partial\xi}\\
\qquad\quad =\lambda\{\overset{*}{\mathrm{div}}\,u_0+c_k(0)u_0^k+(\overset{*}{\mathrm{div}}\,u_1+c_k(1)u_0^k+c_k(0)u_1^k)\xi\\
\qquad\qquad +(\overset{*}{\mathrm{div}}\,u_2+c_k(0)u_2^k+c_k(1)u_1^k+c_k(2)u_0^k)\xi^2\}+(\lambda+2\mu)(u_1^3+2u_2^3\xi)+\cdots,\\
\sigma_0^{33}(u_0)=\lambda(\overset{*}{\mathrm{div}}\,u_0+c_k(0)u_0^k)+(\lambda+2\mu)u_1^3,\\
\sigma_1^{33}(u_0,u_1)=\lambda(\overset{*}{\mathrm{div}}\,u_1+c_k(0)u_1^k+c_k(1)u_0^k)+2(\lambda+2\mu)u_2^3,\\
\sigma_2^{33}(u_0,u_1,u_2)=\lambda(\overset{*}{\mathrm{div}}\,u_2+c_k(0)u_2^k+c_k(1)u_1^k+c_k(2)u_0^k),
\end{cases}$$

$$\begin{cases}
\sigma^{\alpha 3}(u)=\mu(\frac{\partial u^\alpha}{\partial\xi}+g^{\alpha\beta}\overset{*}{\nabla}_\beta u^3)\\
\qquad\quad =\mu\{u_1^\alpha+2u_2^\alpha\xi+(a^{\alpha\beta}+2b^{\alpha\beta}\xi+3c^{\alpha\beta}\xi^2)\overset{*}{\nabla}_\beta(u_0^3+u_1^3\xi+u_2^3\xi^2)\}+\cdots,\\
\sigma_0^{\alpha 3}(u_0)=\mu(u_1^\alpha+a^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3),\\
\sigma_1^{\alpha 3}(u_1)=\mu(2u_2^\alpha+a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3),\\
\sigma_2^{\alpha 3}(u_0,u_1,u_2)=\mu(a^{\alpha\beta}\overset{*}{\nabla}_\beta u_2^3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3+3c^{\alpha\beta}\overset{*}{\nabla}_\beta u_0^3),
\end{cases}$$

where $c_k(\xi)$ are defined by (2.34)

$$\begin{aligned}
c_\alpha(\xi)&=\theta^{-1}(-2\overset{*}{\nabla}_\alpha H\xi+\overset{*}{\nabla}_\alpha K\xi^2)=(1+2H\xi+(4H^2-K)\xi^2+\cdots)(-2\overset{*}{\nabla}_\alpha H\xi+\overset{*}{\nabla}_\alpha K\xi^2)\\
&=-2\overset{*}{\nabla}_\alpha H\xi+\overset{*}{\nabla}_\alpha(K-2H^2)\xi^2+\cdots=c_\alpha(0)+c_\alpha(1)\xi+c_\alpha(2)\xi^2+\cdots\\
c_3(\xi)&=\theta^{-1}(-2H+2K\xi)=-2H+(2K-4H^2)\xi+(6HK-8H^3)\xi^2+\cdots\\
&=c_3(0)+c_3(1)\xi+c_3(2)\xi\alpha^2+\cdots.
\end{aligned}$$

Hence

$$\begin{aligned}
&c_\alpha(0)=0,\quad c_\alpha(1)=-2\overset{*}{\nabla}_\alpha H,\quad c_\alpha(2)=\overset{*}{\nabla}_\alpha(K-2H^2),\\
&c_3(0)=-2H,\quad c_3(1)=(2K-4H^2),\quad c_3(2)=(6HK-8H^3).
\end{aligned}$$

Finally, boundary conditions on the top and bottom of shell are nature boundary conditions

$$g_{jm}\sigma^{ij}n^m|_{\Gamma_t\cup\Gamma_b}=h^i.$$

Note, in Theorem 3.1, displacement $\vec{\eta}=\varepsilon\vec{n}=(0,0,1)$ at top surface of shell, it yields from (3.2)

$$\vec{n}(\varepsilon n)=\vec{n}-a^{\lambda\varrho}(\overset{*}{\nabla}_\lambda \eta^3+b_{\lambda\beta}\eta^\beta)\vec{e}_\sigma=n+0(\varepsilon^2)=\vec{n}+0(\varepsilon^2).$$

Therefore, by Theorem 5.2, we have

$$\begin{aligned} h^\alpha &= g_{jm}\sigma^{\alpha j}n^m(\varepsilon n)=g_{33}\sigma^{\alpha 3}(u)=\sigma^{\alpha 3}(u)\\ &=\sigma_0^{\alpha 3}(u_0)+\sigma_1^{\alpha 3}(u_0,u_1)\varepsilon+\sigma_2^{\alpha 3}(u_0,u_1,u_2)\varepsilon^2, \qquad \text{on } \Gamma_t,\\ h^3&=\sigma^{33}(u)=\sigma_0^{33}(u_0)+\sigma_1^{33}(u_0,u_1)\varepsilon+\sigma_2^{33}(u_0,u_1,u_2)\varepsilon^2, \quad \text{on } \Gamma_t. \end{aligned}$$

Similarly, since $n(-\varepsilon n)=-n$, $h^i=g_{jm}\sigma^{ij}(u)n^m(-\varepsilon n)$ on the bottom surface of shell, so that

$$\begin{aligned} h^\alpha&=-\sigma^{\alpha 3}(u)=-\{\sigma_0^{\alpha 3}(u)-\sigma_1^{\alpha 3}(u_0,u_1)\varepsilon+\sigma_2^{\alpha 3}(u_0,u_1,u_2)\varepsilon^2\}, \quad \text{on } \Gamma_b,\\ h^3&=-\sigma^{33}(u)=-\{\sigma_0^{33}(u_0)-\sigma_1^{33}(u_0,u_1)\varepsilon+\sigma_2^{33}(u_0,u_1,u_2)\varepsilon^2\}, \quad \text{on } \Gamma_b. \end{aligned}$$

This completes our proof. □

6 A dimensional splitting form for nonlinearly elastic equations in 3D shell in $\Re^3$

In this section we study nonlinearly elastic equations for isomeric and isomorphic St.Venant-Kirchhoff materials. Nonlinearly elastic equations for 3D elastic shell are given by: find $u=(u^i):\overline{\Omega}\to\Re^3$ such that :

$$\begin{cases} \mathcal{N}^i(u):=-\nabla_j\widetilde{\Sigma}^{ij}(u)=f^i, & \text{in } \Omega,\\ u=0, & \text{on } \Gamma_0,\\ \widetilde{\Sigma}^{ij}(u)n_j=h^i, & \text{on } \Gamma_1, \end{cases} \tag{6.1}$$

where stress tensor σ^{ij}, second Piola-Kirchhoff stress tensor Σ^{ij} and first Piola-Kirchhoff stress tensor $\widetilde{\Sigma}^{ij}(u)$ are given respectively by:

$$\begin{cases} \sigma^{ij}(u)=A^{ijkl}e_{kl}(u), \quad \Sigma^{ij}(u)=A^{ijkl}E_{kl}(u),\\ \widetilde{\Sigma}^{ij}(u)=\Sigma^{ij}(u)+\Sigma^{kj}(u)\nabla_k u^i=(\delta_k^i+\nabla_k u^i)\Sigma^{kj}(u)=(\delta_k^i+\nabla_k u^i)A^{kjml}E_{ml}(u), \end{cases} \tag{6.2}$$

where $e_{ij}(u)$, $E_{ij}(u)$ are strain tensor (2.42) and Green-St Venant strain tensor (2.45).

In the followings, we have to consider first Piola-Kirchhoff stress tensor $\widetilde{\Sigma}^{ij}(u)$. The covariant derivatives of first Piola-Kirchhoff stress tensor are given by

$$\begin{aligned} \nabla_j\widetilde{\Sigma}^{ij}(u)&=\nabla_j((\delta_k^i+\nabla_k u^i)A^{kjlm}E_{lm}(u))=A^{kjlm}\nabla_j((\delta_k^i+\nabla_k u^i)E_{lm}(u))\\ &=A^{kjlm}\{(\delta_k^i+\nabla_k u^i)\nabla_j E_{lm}(u)+E_{lm}(u)\nabla_j\nabla_k u^i\}, \end{aligned} \tag{6.3}$$

where we consider the materials are isomeric and isomorphic, and covariant derivatives of metric tensor in Euclisean space are vanising (i.e. $\nabla_j A^{kjlm}=0$). As well known that linearly elastic operator

$$\begin{cases} \mathcal{L}^i(u)=-\nabla_j\sigma^{ij}(u)=-\nabla_l(A^{ijkm}e_{Km}(u))=-A^{ijkm}\nabla_j e_{km}(u),\\ E_{ij}(u)=e_{ij}(u)+D_{ij}(u). \end{cases} \tag{6.4}$$

Therefore nonlinearly elastic operator is given by

$$\begin{aligned} &\mathcal{N}^i(u)=-\nabla_j\widetilde{\Sigma}^{ij}(u)\\ =&(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-A^{kjlm}\{(\delta^i_k+\nabla_k u^i)\nabla_j D_{lm}(u)+E_{lm}(u)\nabla_j\nabla_k u^i\}. \end{aligned} \tag{6.5}$$

As well known that isotropic and homogenous elstic coefficient tensor of four order are given by (2.55) and satisfiy (2.57), in particular all components A^{ijlm} are vanissh except

$$\begin{aligned} &A^{\alpha\beta\lambda\sigma}=\lambda g^{\alpha\beta}g^{\lambda\sigma}+\mu(g^{\alpha\lambda}g^{\beta\sigma}+g^{\alpha\sigma}g^{\beta\lambda}),\quad A^{\alpha\beta 33}=A^{33\alpha\beta}=\lambda g^{\alpha\beta},\\ &A^{\alpha 3\beta 3}=A^{3\alpha 3\beta}=A^{\alpha 33\beta}=A^{3\alpha\beta 3}=\mu g^{\alpha\beta},\quad A^{3333}=\lambda+2\mu. \end{aligned}$$

In addition

$$\nabla_j g^{lm}=0,\quad g^{lm}=g^{ml},\quad g^{3\alpha}=g^{\alpha 3}=0, g^{33}=1,\quad \Delta u^i=g^{lm}\nabla_l\nabla_m u^i.$$

So that it assert that

$$\begin{aligned} A^{kjlm}&\{(\delta^i_k+\nabla_k u^i)\nabla_j D_{lm}(u)+E_{lm}(u)\nabla_j\nabla_k u^i\}=A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)\\ &+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}+A^{33\lambda\sigma}\{(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{\lambda\sigma}(u)+E_{\lambda\sigma}(u)\nabla_3\nabla_3 u^i\}\\ &+A^{\alpha\beta 33}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{33}(u)+E_{33}(u)\nabla_\beta\nabla_\alpha u^i\}+A^{3\beta 3\sigma}\{(\delta^i_3+\nabla_3 u^i)\nabla_\beta D_{3\sigma}(u)\\ &+E_{3\sigma}(u)\nabla_3\nabla_\alpha u^i\}+A^{\alpha 3\lambda 3}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_3 D_{\lambda 3}(u)+E_{\lambda 3}(u)\nabla_3\nabla_\alpha u^i\}\\ &+A^{\alpha 33\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_3 D_{3\sigma}(u)+E_{3\sigma}(u)\nabla_3\nabla_\alpha u^i\}+A^{3\beta\lambda 3}\{(\delta^i_3+\nabla_3 u^i)\nabla_\beta D_{\lambda 3}(u)\\ &+E_{\lambda 3}(u)\nabla_\beta\nabla 3u^i\}+A^{3333}\{(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{33}(u)+E_{33}(u)\nabla_3\nabla_3 u^i\}. \end{aligned}$$

It can be reads as

$$\begin{aligned} A^{kjlm}&\{(\delta^i_k+\nabla_k u^i)\nabla_j D_{lm}(u)+E_{lm}(u)\nabla_j\nabla_k u^i\}=A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)\\ &+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}+\lambda g^{\alpha\beta}\{(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{\alpha\beta}(u)+E_{\alpha\beta}(u)\nabla_3\nabla_3 u^i\}\\ &+\lambda g^{\alpha\beta}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{33}(u)+E_{33}(u)\nabla_\beta\nabla_\alpha u^i\}+\mu g^{\alpha\beta}\{(\delta^i_3+\nabla_3 u^i)\nabla_\alpha D_{3\beta}(u)\\ &+E_{3\beta}(u)\nabla_3\nabla_\alpha u^i\}+\mu g^{\alpha\beta}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_3 D_{\beta 3}(u)+E_{\beta 3}(u)\nabla_3\nabla_\alpha u^i\}\\ &+\mu g^{\alpha\beta}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_3 D_{\beta 3}(u)+E_{\beta 3}(u)\nabla_3\nabla_\alpha u^i\}+\mu g^{\alpha\beta}\{(\delta^i_3+\nabla_3 u^i)\nabla_\alpha D_{\beta 3}(u)\\ &+E_{\beta 3}(u)\nabla_\alpha\nabla_3 u^i\}+A^{3333}\{(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{33}(u)+E_{33}(u)\nabla_3\nabla_3 u^i\}. \end{aligned}$$

According Ricci Theorem and Riemannian Curvature tensor are vanish in Euclidian space, therefore

$$\nabla_3\nabla_\alpha u^i=\nabla_\alpha\nabla_3 u^i$$

and symmetry of indices of strain tensor, previous equality becomes

$$\begin{aligned}&A^{kjlm}\{(\delta^i_k+\nabla_k u^i)\nabla_j D_{lm}(u)+E_{lm}(u)\nabla_j\nabla_k u^i\}=A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)\\&\quad+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}+\lambda\{(\delta^i_3+\nabla_3 u^i)\nabla_3(g^{\alpha\beta}D_{\alpha\beta}(u))+(g^{\alpha\beta}E_{\alpha\beta}(u))\nabla_3\nabla_3 u^i\}\\&\quad+\lambda\{(\delta^i_\alpha+\nabla_\alpha u^i)g^{\alpha\beta}\nabla_\beta D_{33}(u)+E_{33}(u)g^{\alpha\beta}\nabla_\beta\nabla_\alpha u^i\}+2\mu(\delta^i_3+\nabla_3 u^i)g^{\alpha\beta}\nabla_\alpha D_{3\beta}(u)\\&\quad+2\mu g^{\alpha\beta}(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_3 D_{\beta 3}(u)+4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i\\&\quad+(\lambda+2\mu)\{(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{33}(u)+E_{33}(u)\nabla_3\nabla_3 u^i\}.\end{aligned}$$

Substitute it into (6.5) leads to

$$\begin{aligned}\mathcal{N}^i(u)=&(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}\\&-\lambda(\delta^i_3+\nabla_3 u^i)\nabla_3(g^{\alpha\beta}D_{\alpha\beta}(u)+D_{33}(u))-(\lambda g^{\alpha\beta}E_{\alpha\beta}(u)+2\mu E_{33}(u))\nabla_3\nabla_3 u^i\\&-\lambda g^{\alpha\beta}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{33}(u)-\lambda E_{33}(u)(g^{\alpha\beta}\nabla_\beta\nabla_\alpha u^i+\nabla_3\nabla_3 u^i)\\&-2\mu(\delta^i_3+\nabla_3 u^i)g^{\alpha\beta}\nabla_\alpha D_{3\beta}(u)-4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i\}\\&-2\mu(\delta^i_\alpha+\nabla_\alpha u^i)g^{\alpha\beta}\nabla_3 D_{\beta 3}(u)-2\mu(\delta^i_3+\nabla_3 u^i)\nabla_3 D_{33}(u).\end{aligned}$$

Since

$$\Delta u^i=g^{ij}\nabla_i\nabla_j u^i=g^{\alpha\beta}\nabla_\beta\nabla_\alpha u^i+\nabla_3\nabla_3 u^i,\quad \mathrm{Div}(u)=g^{ij}D_{ij}(u)=g^{\alpha\beta}D_{\alpha\beta}+D_{33}(u).$$

Hence

$$\begin{aligned}\mathcal{N}^i(u)=&(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}\\&-\lambda(\delta^i_3+\nabla_3 u^i)\nabla_3\mathrm{Div}u-(\lambda g^{\alpha\beta}E_{\alpha\beta}(u)+2\mu E_{33}(u))\nabla_3\nabla_3 u^i\\&-\lambda g^{\alpha\beta}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{33}(u)-\lambda E_{33}(u)\Delta u^i-2\mu(\delta^i_3+\nabla_3 u^i)(g^{\alpha\beta}\nabla_\alpha D_{3\beta}(u)\\&+\nabla_3 D_{33}(u))-4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i\}-2\mu(\delta^i_\alpha+\nabla_\alpha u^i)g^{\alpha\beta}\nabla_3 D_{\beta 3}(u).\end{aligned}\tag{6.6}$$

Reads otherwise

$$\begin{aligned}\mathcal{N}^i(u)=&(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}\\&-\lambda E_{33}(u)\Delta u^i-(\lambda g^{\alpha\beta}E_{\alpha\beta}(u)+2\mu E_{33}(u))\nabla_3\nabla_3 u^i-4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i\\&-\lambda(\delta^i_3+\nabla_3 u^i)\nabla_3\mathrm{Div}u-g^{\alpha\beta}(\delta^i_\alpha+\nabla_\alpha u^i)\{\lambda\nabla_\beta D_{33}(u)+2\mu\nabla_3 D_{\beta 3}(u)\}\\&-2\mu(\delta^i_3+\nabla_3 u^i)(g^{\alpha\beta}\nabla_\alpha D_{3\beta}(u)+\nabla_3 D_{33}(u)).\end{aligned}\tag{6.7}$$

Furthermore, by similar manner, we assert

$$\begin{aligned}&-A^{\alpha\beta\lambda\sigma}\{(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta D_{\lambda\sigma}(u)+E_{\lambda\sigma}(u)\nabla_\beta\nabla_\alpha u^i\}=-(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta(A^{\alpha\beta\lambda\sigma}D_{\lambda\sigma}(u))\\&\quad+A^{\alpha\beta\lambda\sigma}E_{\lambda\sigma}(u)\nabla_\alpha\nabla_\beta u^i=-(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta(A^{\alpha\beta\lambda\sigma}D_{\lambda\sigma}(u))\\&\quad-2\mu E^{\alpha\beta}(u)\nabla_\alpha\nabla_\beta u^i-\lambda g^{\lambda\sigma}E_{\lambda\sigma}(u)(\Delta u^i-\nabla_3\nabla_3 u^i).\end{aligned}$$

Substitutin it into (6.7) leads to

$$\begin{aligned}\mathcal{N}^i(u)=&(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-\lambda g^{mk}E_{mk}(u)\Delta u^i-2\mu E_{33}(u)\nabla_3\nabla_3 u^i\\&-2\mu E^{\alpha\beta}(u)\nabla_\alpha\nabla_\beta u^i-4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i-(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta(A^{\alpha\beta\lambda\sigma}D_{\lambda\sigma}(u))\\&-g^{\alpha\beta}(\delta^i_\alpha+\nabla_\alpha u^i)\{\lambda\nabla_\beta D_{33}(u)+2\mu\nabla_3 D_{\beta 3}(u)\}\\&-(\delta^i_3+\nabla_3 u^i)(2\mu(g^{mk}\nabla_m D_{3k}(u))+\lambda\nabla_3(g^{ml}D_{ml}(u))).\end{aligned}\tag{6.8}$$

Next we need to consider relationship of covariant derivatives of nonlinear stain tensor tensor.

Lemma 6.1. *The covariant derivatives of symmetric tensor of stain tensor D_{ij} in S-coordinate system are given by*

$$\begin{cases}\nabla_\alpha D_{\lambda\sigma}=\overset{*}{\nabla}_\alpha D_{\lambda\sigma}+\Lambda^{\nu 3}_{\alpha\lambda\sigma}D_{\nu 3}+\Lambda^{\nu\mu}_{\alpha\lambda\sigma}D_{\nu\mu},\\ \nabla_3 D_{\lambda\sigma}=\frac{\partial}{\partial\xi}D_{\lambda\sigma}+\Lambda^{\nu\mu}_{3\lambda\sigma}D_{\nu\mu},\\ \nabla_\alpha D_{3\lambda}=\overset{*}{\nabla}_\alpha D_{3\lambda}-\Phi^\beta_{\alpha\lambda}D_{3\beta}-\theta^{-1}I^\beta_\alpha D_{\beta\lambda}+J_{\alpha\lambda}D_{33},\\ \nabla_3 D_{3\sigma}=\frac{\partial}{\partial\xi}D_{3\sigma}-\theta^{-1}I^\lambda_\sigma D_{3\lambda},\\ \nabla_\alpha D_{33}=\overset{*}{\nabla}_\alpha D_{33}-2\theta^{-1}I^\beta_\alpha D_{\beta 3},\quad \nabla_3 D_{33}=\frac{\partial}{\partial\xi}D_{33},\end{cases}\tag{6.9}$$

where

$$\begin{cases}\Lambda^{\mu\nu}_{\alpha\lambda\sigma}(\xi):=-(\Phi^\mu_{\alpha\lambda}\delta^\nu_\sigma+\Phi^\mu_{\alpha\sigma}\delta^\nu_\lambda)=\Lambda^{\mu\nu}_{\alpha\lambda\sigma}(1)\xi+\Lambda^{\mu\nu}_{\alpha\lambda\sigma}(2)\xi^2+\cdots,\\ \Lambda^{3\nu}_{\alpha\lambda\sigma}(\xi)=-(J_{\alpha\lambda}\delta^\nu_\sigma+J_{\alpha\sigma}\delta^\nu_\lambda)=\Lambda^{3\nu}_{\alpha\lambda\sigma}(0)+\Lambda^{3\nu}_{\alpha\lambda\sigma}(1)\xi,\\ \Lambda^{\mu\nu}_{3\lambda\sigma}(\xi)=-\theta^{-1}(I^\mu_\lambda\delta^\nu_\sigma+I^\mu_\sigma\delta^\nu_\lambda)=\Lambda^{\mu\nu}_{3\lambda\sigma}(0)+\Lambda^{\mu\nu}_{3\lambda\sigma}(1)\xi+\Lambda^{\mu\nu}_{3\lambda\sigma}(2)\xi^2+\cdots,\\ \Lambda^{\mu\nu}_{\alpha\lambda\sigma}(1)=\overset{*}{\nabla}_\alpha(b^\mu_\lambda\delta^\nu_\sigma+b^\mu_\sigma\delta^\nu_\lambda),\quad \Lambda^{\mu\nu}_{\alpha\lambda\sigma}(2)=(2H\delta^\mu_\eta-b^\mu_\eta)\overset{*}{\nabla}_\alpha(b^\eta_\lambda\delta^\nu_\sigma+b^\eta_\sigma\delta^\nu_\lambda),\\ \Lambda^{3\nu}_{\alpha\lambda\sigma}(0)=-(b_{\alpha\lambda}\delta^\nu_\sigma+b_{\alpha\sigma}\delta^\nu_\lambda),\quad \Lambda^{3\nu}_{\alpha\lambda\sigma}(1)=c_{\alpha\lambda}\delta^\nu_\sigma+c_{\alpha\sigma}\delta^\nu_\lambda,\\ \Lambda^{\mu\nu}_{3\lambda\sigma}(0)=b^\mu_\lambda\delta^\nu_\sigma+b^\mu_\sigma\delta^\nu_\lambda,\quad \Lambda^{\mu\nu}_{3\lambda\sigma}(1)=c^\mu_\lambda\delta^\nu_\sigma+c^\mu_\sigma\delta^\nu_\lambda,\\ \Lambda^{\mu\nu}_{3\lambda\sigma}(2)=(2Hc^\mu_\lambda-Kb^\mu_\lambda)\delta^\nu_\sigma+(2Hc^\mu_\sigma-Kb^\mu_\sigma)\delta^\nu_\lambda.\end{cases}\tag{6.10}$$

Proof. According to definition of covariant derivative,

$$\begin{aligned}\nabla_\alpha D_{\lambda\sigma}&=\frac{\partial}{\partial x^\alpha}D_{\lambda\sigma}-\Gamma^k_{\alpha\lambda}D_{k\sigma}-\Gamma^k_{\alpha\sigma}D_{\lambda k}\\&=\nabla_\alpha D_{\lambda\sigma}=\frac{\partial}{\partial x^\alpha}D_{\lambda\sigma}-\Gamma^\nu_{\alpha\lambda}D_{\nu\sigma}-\Gamma^\nu_{\alpha\sigma}D_{\lambda\nu}-\Gamma^3_{\alpha\lambda}D_{3\sigma}-\Gamma^3_{\alpha\sigma}D_{\lambda 3}.\end{aligned}$$

By virtue of Lemma 2.3

$$\Gamma^\nu_{\alpha\lambda}=\overset{*}{\Gamma}{}^\nu_{\alpha\lambda}+\Phi^\nu_{\alpha\lambda},\quad \Gamma^3_{\alpha\lambda}=J_{\alpha\lambda},\quad \overset{*}{\nabla}_\alpha D_{\lambda\sigma}=\frac{\partial}{\partial\xi^\alpha}D_{\lambda\sigma}-\overset{*}{\Gamma}{}^\nu_{\alpha\lambda}D_{\nu\sigma}-\overset{*}{\Gamma}{}^\nu_{\alpha\sigma}D_{\lambda\nu}.$$

Therefore, since symmetry with respect to subscript of $D_{ij}=D_{ji}$,

$$\begin{aligned}\nabla_\alpha D_{\lambda\sigma} &= \frac{\partial}{\partial x^\alpha} D_{\lambda\sigma} - (\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda} + \Phi^\nu_{\alpha\lambda}) D_{\nu\sigma} - (\overset{*}{\Gamma}{}^\nu{}_{\alpha\sigma} + \Phi^\nu_{\alpha\sigma}) D_{\lambda\nu} - J_{\alpha\lambda} D_{3\sigma} - J_{\alpha\sigma} D_{3\lambda} \\ &= \overset{*}{\nabla}_\alpha D_{\lambda\sigma} - \Phi^\nu_{\alpha\lambda} D_{\nu\sigma} - \Phi^\nu_{\alpha\sigma} D_{\nu\lambda} - J_{\alpha\lambda} D_{3\sigma} - J_{\alpha\sigma} D_{3\lambda} \\ &= \overset{*}{\nabla}_\alpha D_{\lambda\sigma} + \Lambda^{\nu\mu}_{\alpha\lambda\sigma} D_{\nu\sigma} + \Lambda^{\gamma\mu}_{\alpha\lambda\sigma} D_{\nu\mu}.\end{aligned}$$

In similar manner, by Lemma 2.3

$$\begin{aligned}\nabla_3 D_{\lambda\sigma} &= \frac{\partial}{\partial\xi} D_{\lambda\sigma} - \Gamma^k_{3\lambda} D_{k\sigma} - \Gamma^k_{3\sigma} D_{\lambda k} = \frac{\partial}{\partial\xi} D_{\lambda\sigma} - \Gamma^\nu_{3\lambda} D_{\nu\sigma} - \Gamma^n u_{3\sigma} D_{\lambda\nu} - \Gamma^3_{3\lambda} D_{3\sigma} - \Gamma^3_{3\sigma} D_{\lambda 3} \\ &= \frac{\partial}{\partial\xi} D_{\lambda\sigma} - \theta^{-1}(I^\nu_\lambda D_{\nu\sigma} - I^\nu_\sigma D_{\nu\lambda}) = \frac{\partial}{\partial\xi} D_{\lambda\sigma} + \Lambda^{\nu\mu}_{3\lambda\sigma} D_{\nu\mu},\end{aligned}$$

$$\begin{aligned}\nabla_\alpha D_{3\lambda} &= \partial_\alpha D_{3\lambda} - \Gamma^k_{\alpha 3} D_{k\lambda}(u) - \Gamma^k_{\alpha\lambda} D_{3k}(u) = \partial_\alpha D_{3\lambda} - \Gamma^\nu_{\alpha 3} D_{\nu\lambda}(u) - \Gamma^\nu_{\alpha\lambda} D_{3\nu}(u) - \Gamma^3_{\alpha\lambda} D_{33}(u) \\ &= \partial_\alpha D_{3\lambda} - \theta^{-1} I^\nu_\alpha D_{\nu\lambda}(u) - (\overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda} + \Phi^\nu_{\alpha\lambda}) D_{3\nu}(u) - J_{\alpha\lambda} D_{33}(u) \\ &= \partial_\alpha D_{3\lambda} - \overset{*}{\Gamma}{}^\nu{}_{\alpha\lambda} D_{3\nu}(u) - \theta^{-1} I^\nu_\alpha D_{\nu\lambda}(u) - \Phi^\nu_{\alpha\lambda} D_{3\nu}(u) - J_{\alpha\lambda} D_{33}(u) \\ &= \overset{*}{\nabla}_\alpha D_{3\lambda}(u) - \theta^{-1} I^\nu_\alpha D_{\nu\lambda}(u) - \Phi^\nu_{\alpha\lambda} D_{3\nu}(u) - J_{\alpha\lambda} D_{33}(u).\end{aligned}$$

Other equalities of (6.9) can be obtain in same way. End our proof. □

Lemma 6.2. *Assume that the elastic materials is isomeric and isomorphic St.Venaut-Kirchhoff materials. Then under S-coordinate system nonlinear elastic operator defined by* (6.1) *can be expressed as*

$$\begin{cases}\mathcal{N}^i(u) = (-(\lambda+\mu)\delta^i_3 + \mathcal{K}_0(u)) \frac{\partial^2 u^i}{\partial\xi^2} + \mathcal{K}^i_j(u) \frac{\partial u^j}{\partial\xi} - (\lambda+\mu)\{g^{\alpha\beta} \overset{*}{\nabla}_\beta (\overset{*}{div}\, u + \frac{\partial u^3}{\partial\xi})\delta^i_\alpha \\ \qquad + \overset{*}{div} \frac{\partial u}{\partial\xi} \delta^i_3\} + \mathcal{K}^{\lambda 3}(u) \overset{*}{\nabla}_\lambda \frac{\partial u^i}{\partial\xi} + \mathcal{K}^{\lambda\sigma}(u) \overset{*}{\nabla}_\lambda \overset{*}{\nabla}_\sigma u^i + M^{i\mu}_k(u) \overset{*}{\nabla}_\mu u^k \\ \qquad + M^{i0}_k(u) u^k - (\delta^i_\beta + \nabla_\beta u^i)\mathcal{D}^\beta(u) - (\delta^i_3 + \nabla_3 u^i)\mathcal{D}^3(u), \\ Divu = g^{ij} E_{ij}(u),\end{cases} \tag{6.11}$$

where $E_{ij}(u)$ is Green-St-Vennant strain tensor defined by (2.45) *and*

$$\begin{cases}\mathcal{K}_0(u) = -\{\mu + \lambda Divu + 2\mu E_{33}(u)\}, \\ \mathcal{K}^i_j(u) = -\{m^{i3}_j(\xi) + \lambda Divu \Pi^{i3}_j(\xi) + 2\mu E_{33}(u) l^i_j - 2\mu E^{\lambda\sigma}(u) J_{\lambda\sigma}(\xi)\delta^i_j \\ \qquad + 4\mu g^{\lambda\sigma} E_{3\sigma}(u) \Pi^{i3}_{\lambda 3,j}(\xi)\}, \\ \mathcal{K}^{\lambda\sigma}(u) = -((\mu + \lambda Divu) g^{\lambda\sigma} + 2\mu E^{\lambda\sigma}(u)), \\ \mathcal{K}^{\lambda 3}(u) = -4\mu g^{\lambda\sigma} E_{3\sigma}(u), \\ M^{i\mu}_k(u) = m^{i\mu}_k(\xi) - \lambda Divu \Pi^{i\mu}_k(\xi) - 2\mu E^{\lambda\sigma}(u) \Pi^{i\mu}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u) \Pi^{i\mu}_{\lambda 3,k}(\xi), \\ M^{i0}_k(u) = m^{i0}_k(\xi) - \lambda Divu \Pi^{i0}_k(\xi) - 2\mu E^{\lambda\sigma}(u) \Pi^{i0}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u) \Pi^{i0}_{\lambda 3,k}(\xi),\end{cases} \tag{6.12}$$

$$\begin{cases} \mathcal{D}^\alpha(u):=A^{\alpha\beta\lambda\sigma}(\overset{*}{\nabla}_\beta D_{\lambda\sigma}(u)+\Lambda^{\nu3}_{\beta\lambda\sigma}D_{\nu3}(u)+\Lambda^{\nu\mu}_{\beta\lambda\sigma}D_{\nu\mu}(u)) \\ \qquad +g^{\alpha\beta}(\lambda\overset{*}{\nabla}_\beta D_{33}(u)+2\mu\frac{\partial}{\partial\xi}D_{3\beta}(u)-2(\lambda+\mu)I^\lambda_\beta D_{3\lambda}(u)), \\ \mathcal{D}^3(u):=\{2\mu g^{\alpha\beta}\overset{*}{\nabla}_\alpha D_{3\beta}(u)+\lambda\frac{\partial}{\partial\xi}(g^{\alpha\beta}D_{\alpha\beta}(u))+(\lambda+2\mu)\frac{\partial}{\partial\xi}D_{33}(u) \\ \qquad -2\mu g^{\alpha\beta}(\Phi^\lambda_{\alpha\beta}(\xi)D_{3\lambda}(u)+I^\lambda_\alpha D\alpha_{\lambda\beta}(u))-2\mu\theta^{-1}(2H-2K\xi)D_{33}(u)\}, \end{cases} \tag{6.13}$$

where, $\Pi^{i\mu}_{\alpha\beta,k}$, $\Pi^{i\mu}_k$ *and* m^{ij}_k *are defined in Lemma* 4.1, *Theorem* 4.3 *and* (5.3).

Proof. Taking (6.9) into account, we claim

$$\begin{cases} q^i_2(u):=-(\delta^i_\alpha+\nabla_\alpha u^i)\nabla_\beta(A^{\alpha\beta\lambda\sigma}D_{\lambda\sigma}(u))-g^{\alpha\beta}(\delta^i_\alpha+\nabla_\alpha u^i)\{\lambda\nabla_\beta D_{33}(u) \\ \qquad +2\mu\nabla_3 D_{\beta3}(u)\}-(\delta^i_3+\nabla_3 u^i)(2\mu(g^{mk}\nabla_m D_{3k}(u))+\lambda\nabla_3(g^{ml}D_{ml}(u))) \\ \quad =-(\delta^i_\alpha+\nabla_\alpha u^i)\mathcal{D}^\alpha(u)-(\delta^i_3+\nabla_3 u^i)\mathcal{D}^3(u), \\ \mathcal{D}^\alpha(u):=A^{\alpha\beta\lambda\sigma}(\overset{*}{\nabla}_\beta D_{\lambda\sigma}(u)+\Lambda^{\nu3}_{\beta\lambda\sigma}D_{\nu3}(u)+\Lambda^{\nu\mu}_{\beta\lambda\sigma}D_{\nu\mu}(u)) \\ \qquad +g^{\alpha\beta}(\lambda\overset{*}{\nabla}_\beta D_{33}(u)+2\mu\frac{\partial}{\partial\xi}D_{3\beta}(u)-2(\lambda+\mu)I^\lambda_\beta D_{3\lambda}(u)), \\ \mathcal{D}^3(u):=2\mu g^{\alpha\beta}\overset{*}{\nabla}_\alpha D_{3\beta}(u)+\lambda\frac{\partial}{\partial\xi}(g^{\alpha\beta}D\alpha_{\alpha\beta}(u))+(\lambda+2\mu)\frac{\partial}{\partial\xi}D_{33}(u) \\ \qquad -2\mu g^{\alpha\beta}(\Phi^\lambda_{\alpha\beta}(\xi)D_{3\lambda}(u)+I^\lambda_\alpha D\alpha_{\lambda\beta}(u))-2\mu\theta^{-1}(2H-2K\xi)D_{33}(u). \end{cases} \tag{6.14}$$

Similarly, applying Lemma 4.1 and Theorem 4.3 and Theorem 5.1, we assert

$$\begin{cases} q^i_1(u):=(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-\lambda g^{mk}E_{mk}(u)\Delta u^i-2\mu E_{33}(u)\nabla_3\nabla_3 u^i \\ \qquad -2\mu E^{\alpha\beta}(u)\nabla_\alpha\nabla_\beta u^i-4\mu g^{\alpha\beta}E_{3\beta}(u)\nabla_3\nabla_\alpha u^i \\ \quad =(\delta^i_k+\nabla_k u^i)\mathcal{L}^k(u)-\lambda\mathrm{Div}u\{g^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i+\frac{\partial^2}{\partial\xi^2}u^i+\Pi^{i3}_j(\xi)\frac{\partial}{\partial\xi}u^j \\ \qquad +\Pi^{i\mu}_k(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_k(\xi)u^k\}-2\mu E_{33}(u)(\frac{\partial^2}{\partial\xi^2}u^i+l^i_j(\xi)\frac{\partial}{\partial\xi}u^j) \\ \qquad -2\mu E^{\alpha\beta}(u)\{\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i-J_{\alpha\beta}\frac{\partial}{\partial\xi}u^i+\Pi^{i\mu}_{\alpha\beta,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_{\alpha\beta,k}u^k\} \\ \qquad -4\mu g^{\alpha\beta}E_{3\beta}(u)\{\overset{*}{\nabla}_\alpha\frac{\partial}{\partial\xi}u^i+\Pi^{i3}_{\alpha3,j}\frac{\partial}{\partial\xi}u^j+\Pi^{i\mu}_{\alpha3,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{i0}_{\alpha3,k}(\xi)k^k\}, \end{cases} \tag{6.15}$$

where

$$\mathrm{Div}u=g^{ml}E_{ml}(u).$$

Owing to (5.2)

$$\begin{cases} \mathcal{L}^\alpha(u)=-\mu\frac{\partial^2 u^\alpha}{\partial\xi^2}-m^{\alpha3}_\beta(\xi)\frac{\partial u^\beta}{\partial\xi}-(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\frac{\partial u^3}{\partial\xi}-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^\alpha \\ \qquad -(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\mathrm{div}}\,u+m^{\alpha\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+m^{\alpha0}_k(\xi)u^k, \\ \mathcal{L}^3(u)=-(\lambda+2\mu)\frac{\partial^2 u^3}{\partial\xi^2}+m^{33}_k(\xi)\frac{\partial u^k}{\partial\xi}-(\lambda+\mu)\overset{*}{\mathrm{div}}\,\frac{\partial u}{\partial\xi} \\ \qquad -\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^3+m^{3\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+m^{30}_k(\xi)u^k, \end{cases}$$

with (6.8), (6.14) and (6.15), we obtain

$$\begin{aligned}\mathcal{N}^\alpha(u)=&-\mu\tfrac{\partial^2u^\alpha}{\partial\xi^2}-m^{\alpha3}_\beta(\xi)\tfrac{\partial u^\beta}{\partial\xi}-(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\tfrac{\partial u^3}{\partial\xi}-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^\alpha-(\lambda+\mu)g^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\mathrm{div}}\,u\\&+m^{\alpha\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k+m^{\alpha0}_k(\xi)u^k-\lambda\mathrm{Div}u\{g^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^\alpha+\tfrac{\partial^2}{\partial\xi^2}u^\alpha+\Pi^{\alpha3}_j(\xi)\tfrac{\partial}{\partial\xi}u^j\\&+\Pi^{\alpha\mu}_k(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{\alpha0}_k(\xi)u^k\}-2\mu E_{33}(u)(\tfrac{\partial^2}{\partial\xi^2}u^\alpha+l^\alpha_j(\xi)\tfrac{\partial}{\partial\xi}u^j)-2\mu E^{\lambda\sigma}(u)\{\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^\alpha\\&-J_{\lambda\sigma}\tfrac{\partial}{\partial\xi}u^\alpha+\Pi^{\alpha\mu}_{\lambda\sigma,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{\alpha0}_{\lambda\sigma,k}u^k\}-4\mu g^{\lambda\sigma}E_{3\sigma}(u)\{\overset{*}{\nabla}_\lambda\tfrac{\partial}{\partial\xi}u^\alpha+\Pi^{\alpha3}_{\lambda3,j}\tfrac{\partial}{\partial\xi}u^j\\&+\Pi^{\alpha\mu}_{\lambda3,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{\alpha0}_{\lambda3,k}(\xi)u^k\}-(\delta^\alpha_\beta+\nabla_\beta u^\alpha)\mathcal{D}^\beta(u)-(\delta^\alpha_3+\nabla_3u^\alpha)\mathcal{D}^3(u),\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\mathcal{N}^\alpha(u)=&\mathcal{K}_0(u)\tfrac{\partial^2u^\alpha}{\partial\xi^2}+\mathcal{K}^\alpha_j(u)\tfrac{\partial u^j}{\partial\xi}-(\lambda+\mu)g^{\alpha\gamma}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u+\tfrac{\partial u^3}{\partial\xi})\\&+\mathcal{K}^{\lambda3}(u)\overset{*}{\nabla}_\lambda\tfrac{\partial u^\alpha}{\partial\xi}+\mathcal{K}^{\lambda\sigma}(u)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^\alpha+M^{\alpha\mu}_k(u)\overset{*}{\nabla}_\mu u^k+M^{\alpha0}_k(u)u^k\\&-(\delta^\alpha_\beta+\nabla_\beta u^\alpha)\mathcal{D}^\beta(u)-(\delta^\alpha_3+\nabla_3u^\alpha)\mathcal{D}^3(u),\end{aligned}$$

where

$$\begin{cases}\mathcal{K}_0(u)=-\{\mu+\lambda\mathrm{Div}gu+2\mu E_{33}(u)\},\\ \mathcal{K}^\alpha_j(u)=-\{m^{\alpha3}_j(\xi)+\lambda\mathrm{Div}u\Pi^{\alpha3}_j(\xi)+2\mu E_{33}(u)l^\alpha_j-2\mu E^{\lambda\sigma}(u)J_{\lambda\sigma}(\xi)\delta^\alpha_j\\ \qquad\qquad+4\mu g^{\lambda\sigma}E_{3\sigma}(u)\Pi^{\alpha3}_{\lambda3,j}(\xi)\},\\ \mathcal{K}^{\lambda\sigma}(u)=-(\mu g^{\lambda\sigma}+\lambda\mathrm{Div}ug^{\lambda\sigma}+2\mu E^{\lambda\sigma}(u)),\\ M^{\alpha\mu}_k(u)=m^{\alpha\mu}_k(\xi)-\lambda\mathrm{Div}u\Pi^{\alpha\mu}_k(\xi)-2\mu E^{\lambda\sigma}(u)\Pi^{\alpha\mu}_{\lambda\sigma,k}(\xi)-4\mu g^{\lambda\sigma}E_{3\sigma}(u)\Pi^{\alpha\mu}_{\lambda3,k}(\xi),\\ M^{\alpha0}_k(u)=m^{\alpha0}_k(\xi)-\lambda\mathrm{Div}u\Pi^{\alpha0}_k(\xi)-2\mu E^{\lambda\sigma}(u)\Pi^{\alpha0}_{\lambda\sigma,k}(\xi)-4\mu g^{\lambda\sigma}E_{3\sigma}(u)\Pi^{\alpha0}_{\lambda3,k}(\xi).\end{cases}$$

Third component is given by

$$\begin{aligned}\mathcal{N}^3(u)=&-(\lambda+2\mu)\tfrac{\partial^2u^3}{\partial\xi^2}+m^{33}_k(\xi)\tfrac{\partial u^k}{\partial\xi}-(\lambda+\mu)\overset{*}{\mathrm{div}}\tfrac{\partial u}{\partial\xi}-\mu g^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u^3+m^{3\beta}_k(\xi)\overset{*}{\nabla}_\beta u^k\\&+m^{30}_k(\xi)u^k-\lambda\mathrm{Div}u\{g^{\alpha\beta}\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^3+\tfrac{\partial^2}{\partial\xi^2}u^3+\Pi^{33}_j(\xi)\tfrac{\partial}{\partial\xi}u^j+\Pi^{3\mu}_k(\xi)\overset{*}{\nabla}_\mu u^k\\&+\Pi^{30}_k(\xi)u^k\}-2\mu E_{33}(u)(\tfrac{\partial^2}{\partial\xi^2}u^3+l^3_j(\xi)\tfrac{\partial}{\partial\xi}u^j)-2\mu E^{\alpha\beta}(u)\{\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^3-J_{\alpha\beta}\tfrac{\partial}{\partial\xi}u^3\\&+\Pi^{3\mu}_{\alpha\beta,k}(\xi)\overset{*}{\nabla}_\mu u^k+\Pi^{30}_{\alpha\beta,k}u^k\}-4\mu g^{\alpha\beta}E_{3\beta}(u)\{\overset{*}{\nabla}_\alpha\tfrac{\partial}{\partial\xi}u^3+\Pi^{33}_{\alpha3,j}\tfrac{\partial}{\partial\xi}u^j+\Pi^{3\mu}_{\alpha3,k}(\xi)\overset{*}{\nabla}_\mu u^k\\&+\Pi^{30}_{\alpha3,k}(\xi)k^k\}-(\delta^3_\beta+\nabla_\beta u^3)\mathcal{D}^\beta(u)-(\delta^3_3+\nabla_3u^3)\mathcal{D}^3(u),\end{aligned}$$

which also can be expressed as

$$\begin{aligned}\mathcal{N}^3(u)=&-(\lambda+\mu-\mathcal{K}(u))\tfrac{\partial^2u^3}{\partial\xi^2}+\mathcal{K}^3_j(u)\tfrac{\partial u^j}{\partial\xi}-(\lambda+\mu)\overset{*}{\mathrm{div}}\tfrac{\partial u}{\partial\xi}\\&+\mathcal{K}^{\lambda3}(u)\overset{*}{\nabla}_\lambda\tfrac{\partial u^3}{\partial\xi}+\mathcal{K}^{\lambda\sigma}(u)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^3\\&+M^{3\mu}_k(u)\overset{*}{\nabla}_\mu u^k+M^{30}_k(u)u^k-(\delta^3_\beta+\nabla_\beta u^3)\mathcal{D}^\beta(u)-(\delta^3_3+\nabla_3u^3)\mathcal{D}^3(u),\end{aligned}$$

where

$$\begin{cases} \mathcal{K}^3_j(u) = -\{m^{33}_j(\xi) + \lambda \mathrm{Div} u \Pi^{33}_j(\xi) + 2\mu E_{33}(u) l^3_j - 2\mu E^{\lambda\sigma}(u) J_{\lambda\sigma}(\xi)\delta^3_j \\ \qquad\qquad + 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{33}_{\lambda 3,j}(\xi)\}, \\ \mathcal{K}^{\lambda 3}(u) = -4\mu g^{\lambda\sigma} E_{3\sigma}(u), \\ \mathcal{K}^{\lambda\sigma}(u) = -(\mu g^{\lambda\sigma} + \lambda \mathrm{Div} u g^{\lambda\sigma} + 2\mu E^{\lambda\sigma}(u)), \\ M^{3\mu}_k(u) = m^{3\mu}_k(\xi) - \lambda \mathrm{Div} u \Pi^{3\mu}_k(\xi) - 2\mu E^{\lambda\sigma}(u)\Pi^{3\mu}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{3\mu}_{\lambda 3,k}(\xi), \\ M^{30}_k(u) = m^{30}_k(\xi) - \lambda \mathrm{Div} u \Pi^{30}_k(\xi) - 2\mu E^{\lambda\sigma}(u)\Pi^{30}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{30}_{\lambda 3,k}(\xi). \end{cases}$$

To sum up, nonlinearly elasticity operators can be written as

$$\begin{aligned} \mathcal{N}^i(u) = & (-(\lambda+\mu)\delta^i_3 + \mathcal{K}_0(u))\frac{\partial^2 u^i}{\partial \xi^2} + \mathcal{K}^i_j(u)\frac{\partial u^j}{\partial \xi} - (\lambda+\mu)\{g^{\alpha\gamma}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}} u + \frac{\partial u^3}{\partial \xi})\delta^i_\alpha \\ & + \overset{*}{\mathrm{div}}\frac{\partial u}{\partial \xi}\delta^i_3\} + \mathcal{K}^{\lambda 3}(u)\overset{*}{\nabla}_\lambda \frac{\partial u^i}{\partial \xi} + \mathcal{K}^{\lambda\sigma}(u)\overset{*}{\nabla}_\lambda \overset{*}{\nabla}_\sigma u^i + M^{i\mu}_k(u)\overset{*}{\nabla}_\mu u^k + M^{i0}_k(u)u^k \\ & - (\delta^i_\beta + \nabla_\beta u^i)\mathcal{D}^\beta(u) - (\delta^i_3 + \nabla_3 u^i)\mathcal{D}^3(u), \end{aligned} \tag{6.16}$$

where

$$\begin{cases} \mathcal{K}_0(u) = -\{\mu + \lambda \mathrm{Div} u + 2\mu E_{33}(u)\}, \\ \mathcal{K}^i_j(u) = -\{m^{i3}_j(\xi) + \lambda \mathrm{Div} u \Pi^{i3}_j(\xi) + 2\mu E_{33}(u) l^i_j - 2\mu E^{\lambda\sigma}(u) J_{\lambda\sigma}(\xi)\delta^i_j \\ \qquad\qquad + 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{i3}_{\lambda 3,j}(\xi)\}, \\ \mathcal{K}^{\lambda 3}(u) = -4\mu g^{\lambda\sigma} E_{3\sigma}(u), \\ \mathcal{K}^{\lambda\sigma}(u) = -((\mu + \lambda \mathrm{Div} u) g^{\lambda\sigma} + 2\mu E^{\lambda\sigma}(u)), \\ M^{i\mu}_k(u) = m^{i\mu}_k(\xi) - \lambda \mathrm{Div} u \Pi^{i\mu}_k(\xi) - 2\mu E^{\lambda\sigma}(u)\Pi^{i\mu}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{i\mu}_{\lambda 3,k}(\xi), \\ M^{i0}_k(u) = m^{i0}_k(\xi) - \lambda \mathrm{Div} u \Pi^{i0}_k(\xi) - 2\mu E^{\lambda\sigma}(u)\Pi^{i0}_{\lambda\sigma,k}(\xi) - 4\mu g^{\lambda\sigma} E_{3\sigma}(u)\Pi^{i0}_{\lambda 3,k}(\xi), \end{cases} \tag{6.17}$$

$$\begin{cases} \mathcal{D}^\alpha(u) := A^{\alpha\beta\lambda\sigma}(\overset{*}{\nabla}_\beta D_{\lambda\sigma}(u) + \Lambda^{\nu 3}_{\beta\lambda\sigma}(\xi) D_{\nu 3}(u) + \Lambda^{\nu\mu}_{\beta\lambda\sigma}(\xi) D_{\nu\mu}(u)) \\ \qquad\qquad + g^{\alpha\beta}(\lambda \overset{*}{\nabla}_\beta D_{33}(u) + 2\mu \frac{\partial}{\partial \xi} D_{3\beta}(u) - 2(\lambda+\mu) I^\lambda_\beta(\xi) D_{3\lambda}(u)) \\ \mathcal{D}^3(u) := 2\mu g^{\alpha\beta}\overset{*}{\nabla}_\alpha D_{3\beta}(u) + \lambda \frac{\partial}{\partial \xi}(g^{\alpha\beta} D_{\alpha\beta}(u)) + (\lambda + 2\mu)\frac{\partial}{\partial \xi} D_{33}(u) \\ \qquad\qquad - 2\mu g^{\alpha\beta}(\Phi^\lambda_{\alpha\beta}(\xi) D_{3\lambda}(u) + I^\lambda_\alpha D_{\lambda\beta}(u)) - 2\mu\theta^{-1}(2H - 2K\xi) D_{33}(u). \end{cases} \tag{6.18}$$

Hence we complete our proof. □

Theorem 6.1. *Assume that solution u of nonlinearly elastic operators defined by* (6.11) *and right hand term f can be made Taylor expansion with respect to ξ :*

$$\begin{cases} u(x,\xi) = u_0(x) + u_1(x)\xi + u_2(x)\xi^2 + \cdots, \\ f(x,\xi) = f_0(x) + f_1(x)\xi + f_2(x)\xi^2 + \cdots. \end{cases} \tag{6.19}$$

Then nonlinearly elastic operators defined by (6.11) *under the S-coordinates can be made Taylor expansion*

$$\left\{\begin{array}{l}\mathcal{N}^i(u):=\mathcal{N}_0^i(u_0)+\mathcal{N}_1^i(u_0,u_1)\xi+\mathcal{N}_2^i(u_0,u_1,u_2)\xi^2+\cdots,\\ \mathcal{N}_0^i(u_0)=\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_0^k+M_k^{i0}(0)u_0^k-(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(0)+\mathcal{F}_0^i(u_1,u_2),\\ \mathcal{N}_1^i(u_0,u_1)=\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_1^i+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_1^k+M_k^{i\mu}(1)\overset{*}{\nabla}_\mu u_0^k\\ \qquad +M_k^{i0}(0)u_1^k+M_k^{i0}(1)u_0^k-(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(1)-\gamma_{k1}^i(u_0,u_1)\mathcal{D}^k(0)+\mathcal{F}_1^i(u_1,u_2),\\ \mathcal{N}_2^i(u_0,u_1,u_2)=\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_2^i+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_1^i+\mathcal{K}^{\alpha\beta}(2)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_2^k\\ \qquad +M_k^{i\mu}(1)\overset{*}{\nabla}_\mu u_1^k+M_k^{i\mu}(2)\overset{*}{\nabla}_\mu u_0^k+M_k^{i0}(0)u_2^k+M_k^{i0}(1)u_1^k+M_k^{i0}(2)u_0^k\\ \qquad -(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(2)-\gamma_{k1}^i(u_0,u_1)\mathcal{D}^k(1)-\gamma_{k2}^i(u_0,u_1,u_2)\mathcal{D}^k(0)+\mathcal{F}_2^i(u_1,u_2),\end{array}\right. \tag{6.20}$$

and (u_0,u_1,u_2) *satisfy approximately following boundary value problems*

$$\left\{\begin{array}{l}\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_0^k+M_k^{i0}(0)u_0^k-(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(0)+\mathcal{F}_0^i(u_1,u_2)=f_0^i,\ in\ \omega,\\ \mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_1^i+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_1^k+M_k^{i\mu}(1)\overset{*}{\nabla}_\mu u_0^k+M_k^{i0}(0)u_1^k\\ \qquad +M_k^{i0}(1)u_0^k-(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(1)-\gamma_{k1}^i(u_0,u_1)\mathcal{D}^k(0)+\mathcal{F}_1^i(u_1,u_2)=f_1^i,\quad in\quad \omega,\\ \mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_2^i+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_1^i+\mathcal{K}^{\alpha\beta}(2)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u_0^i+M_k^{i\mu}(0)\overset{*}{\nabla}_\mu u_2^k+M_k^{i\mu}(1)\overset{*}{\nabla}_\mu u_1^k\\ \qquad +M_k^{i\mu}(2)\overset{*}{\nabla}_\mu u_0^k+M_k^{i0}(0)u_2^k+M_k^{i0}(1)u_1^k+M_k^{i0}(2)u_0^k-(\delta_k^i+\gamma_{k0}^i(u_0))\mathcal{D}^k(2)\\ \qquad -\gamma_{k1}^i(u_0,u_1)\mathcal{D}^k(1)-\gamma_{k2}^i(u_0,u_1,u_2)\mathcal{D}^k(0)+\mathcal{F}_2^i(u_1,u_2)=f_2^i,\quad in\quad \omega,\end{array}\right. \tag{6.21}$$

with boundary conditions

$$\left\{\begin{array}{l}u|_{\gamma_0}=0,\\ a_{\alpha\beta}n^\alpha\{(\delta_n^i u+\gamma_{\nu0}^i((u_0))(A_0^{\nu\beta\lambda\sigma}E_{\lambda\sigma}^0(u_0)+\lambda a^{\nu\beta}E_{33}^0(u_0))\\ \qquad +(\delta_3^i+\gamma_{30}^i(u_0))2\mu a^{\beta\mu}E_{3\mu}^0(u_0)\}=h^i,\quad on\quad \gamma_1,\end{array}\right. \tag{6.22}$$

where

$$\left\{\begin{array}{l}\mathcal{K}^{\alpha\beta}(0)=-\{a^{\alpha\beta}(\mu+\lambda)Div(0)+2\mu E^{\alpha\beta}(0)\},\\ \mathcal{K}^{\alpha\beta}(1)=-\{2b^{\alpha\beta}(\mu+\lambda)Div(0)+a^{\alpha\beta}\lambda Div(1)+2\mu E^{\alpha\beta}(1)\},\\ \mathcal{K}^{\alpha\beta}(2)=-\{3c^{\alpha\beta}(\mu+\lambda)Div(0)+2b^{\alpha\beta}\lambda Div(1)+a^{\alpha\beta}\lambda Div(2)+2\mu E^{\alpha\beta}(2)\},\\ Div(0)=a^{\alpha\beta}E_{\alpha\beta}^0(u_0,u_0)+E_{33}^0(u_0,u_0),\\ Div(1)=a^{\alpha\beta}E_{\alpha\beta}^1(u_0,u_1)+2b^{\alpha\beta}E_{\alpha\beta}^0(u_0,u_0)+E_{33}^1(u_0,u_1),\\ Div(2)=a^{\alpha\beta}E_{\alpha\beta}^2(u_0,u_1,u_2)+2b_1^{\alpha\beta}E_{\alpha\beta}^1(u_0,u_1)+3c^{\alpha\beta}E_{\alpha\beta}^0(u_0,u_0)+E_{33}^2(u_0,u_1,u_2),\\ E_{ij}^0(u_0)=\gamma_{ij}(u_0)+\varphi_{ij}(u_0,u_0),\quad E_{ij}^1(u_0,u_1)=\overset{1}{\gamma}_{ij}(u_1)+\psi_{ij}^1(u_0,u_1),\\ E_{ij}^2(u_0,u_1,u_2)=\overset{2}{\gamma}_{ij}(u_2)+\psi_{ij}^2(u_0,u_2),\end{array}\right. \tag{6.23}$$

where bilinear forms φ_{ij}^k *are defined in Lemma* 2.6 *and*

$$\begin{cases} \psi_{ij}^1(u_0,u_1)=2\varphi_{ij}(u_0,u_1)+\varphi_{ij}^1(u_0,u_0),\\ \psi_{ij}^2(u_0,u_1,u_2)=2\varphi_{ij}(u_1,u_0)+\varphi_{ij}(u_1,u_1)+2\varphi_{ij}^1(u_0,u_1)+\varphi_{ij}^2(u_0,u_0),\\ \psi_{ij}^3(u_0,u_1,u_2)=2\varphi_{ij}(u_1,u_2)+\varphi_{ij}^1(u_1,u_1)+2\varphi_{ij}^1(u_0,u_2)+2\varphi_{ij}^2(u_0,u_1), \end{cases}$$

$$\begin{cases} \mathcal{D}^\alpha(0):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)+A_0^{\alpha\nu 3}\varphi_{\nu 3}(u_0,u_0)+A_0^{\alpha\nu\mu}\varphi_{\nu\mu}(u_0,u_0)+a^{\alpha\beta}G_\beta(0),\\ \mathcal{D}^\alpha(1):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^1_{\lambda\sigma}(u_0,u_1)+A_1^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)+A_0^{\alpha\nu 3}\psi^1_{\nu 3}(u_0,u_1)\\ \qquad +A_1^{\alpha\nu 3}\varphi_{\nu 3}(u_0,u_0)+A_0^{\alpha\nu\mu}\psi^1_{\nu 3}(u_0,u_1)+A_1^{\alpha\nu\mu}\varphi_{\nu\mu}(u_0,u_0)+a^{\alpha\beta}G_\beta(1)+2b^{\alpha\beta}G_\beta(0),\\ \mathcal{D}^\alpha(2):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^2_{\lambda\sigma}(u_0,u_1,u_2)+A_1^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^1_{\lambda\sigma}(u_0,u_1)+A_2^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)\\ \qquad +A_0^{\alpha\nu 3}\psi^2_{\nu 3}(u_0,u_1,u_2)+A_1^{\alpha\nu 3}\psi^1_{\nu 3}(u_0,u_1)+A_2^{\alpha\nu 3}\varphi_{\nu 3}(u_0,u_0)\\ \qquad +A_0^{\alpha\nu\mu}\psi^2_{\nu 3}(u_0,u_1,u_2)+A_1^{\alpha\nu\mu}\psi^1_{\nu 3}(u_0,u_1)+A_2^{\alpha\nu\mu}\varphi_{\nu 3}(u_0,u_0)\\ \qquad +a^{\alpha\beta}G_\beta(2)+2b^{\alpha\beta}G_\beta(1)+3c^{\alpha\beta}G_\beta(0), \end{cases}$$

$$\begin{cases} \mathcal{D}^3(0)=G_0(1)+2\mu a^{\alpha\beta}G_{\alpha\beta}(0)+2\mu 2H\varphi_{33}(u_0,u_0),\\ \mathcal{D}^3(1)=2G_0(2)+2\mu(2b^{\alpha\beta}G_{\alpha\beta}(0+a^{\alpha\beta}G_{\alpha\beta}(1))\\ \qquad +2\mu(2H\psi^1_{33}(u_0,u_1)+(4H^2-2K)\varphi_{33}(u_0,u_0)),\\ \mathcal{D}^3(2)=3G_0(3)+2\mu(3c^{\alpha\beta}G_{\alpha\beta}(0)+2b^{\alpha\beta}G_{\alpha\beta}(1)+a^{\alpha\beta}G_\beta(2))\\ \qquad +2\mu(2H\psi^2_{33}(u_0,u_1,u_2)+(4H^2-2K)\psi^1_{33}(u_0,u_1)\\ \qquad +(8H^3-6HK)\varphi_{33}(u_0,u_0)), \end{cases}$$

$$\begin{cases} \mathcal{F}^i_0(u_1,u_2):=2\mathcal{K}_0(0)u_2^i+\mathcal{K}^i_j(0)u_1^j+\mathcal{K}^{\lambda 3}(0)\overset{*}{\nabla}_\lambda u_1^i\\ \qquad\qquad +a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_0+u_1^3)\delta^i_\alpha+\overset{*}{div}\,u_1\delta^i_3,\\ \mathcal{F}^i_1(u_0,u_1,u_2):=2\mathcal{K}_0(1)u_2^i+2\mathcal{K}^i_j(0)u_2^j+\mathcal{K}^i_j(1)u_1^j\\ \qquad\qquad +2\mathcal{K}^{\lambda 3}(0)\overset{*}{\nabla}_\lambda u_2^i+\mathcal{K}^{\lambda 3}(1)\overset{*}{\nabla}_\lambda u_1^i+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_0+u_1^3)\delta^i_\alpha\\ \qquad\qquad +2\overset{*}{div}\,u_2\delta^i_3+a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_1+2u_2^3)\delta^i_\alpha,\\ \mathcal{F}^i_2(u_0,u_1,u_2):=2\mathcal{K}_0(2)u_2^i+2\mathcal{K}^i_j(1)u_2^j+\mathcal{K}^i_j(2)u_1^j+2\mathcal{K}^{\lambda 3}(1)\overset{*}{\nabla}_\lambda u_2^i\\ \qquad\qquad +\mathcal{K}^{\lambda 3}(2)\overset{*}{\nabla}_\lambda u_1^i+a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_2)\delta^i_\alpha+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{div}\,u_1+2u_2^3)\delta^i_\alpha, \end{cases} \tag{6.24}$$

where $M^{im}_k(l)$, $\mathcal{K}^i_j(l)$, $\mathcal{K}^{\lambda 3}(l)$, G_0, G_β, $G_{\alpha\beta}$ *and* $A_l^{\alpha\beta m}$ *are defined by* (6.26) *and* (6.27).

Proof. As well known that the nonlinearly elastic operators are given by (6.11),

$$\begin{aligned} \mathcal{N}^i(u)=&\left(-(\lambda+\mu)\delta^i_3+\mathcal{K}_0(u)\right)\tfrac{\partial^2 u^i}{\partial\xi^2}+\mathcal{K}^i_j(u)\tfrac{\partial u^j}{\partial\xi}-(\lambda+\mu)\{g^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u+\tfrac{\partial u^3}{\partial\xi})\delta^i_\alpha\\ &+\overset{*}{\mathrm{div}}\,\tfrac{\partial u}{\partial\xi}\delta^i_3\}+\mathcal{K}^{\lambda 3}(u)\overset{*}{\nabla}_\lambda\tfrac{\partial u^i}{\partial\xi}+\mathcal{K}^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i+M^{i\mu}_k(u)\overset{*}{\nabla}_\mu u^k\\ &+M^{i0}_k(u)u^k-(\delta^i_\beta+\nabla_\beta u^i)\mathcal{D}^\beta(u)-(\delta^i_3+\nabla_3 u^i)\mathcal{D}^3(u). \end{aligned} \tag{6.25}$$

As well known all coefficients can be made Taylor expansions with respect to transverse variable ξ if taking (6.19) into account

$$\begin{cases} \mathcal{K}_0(u)=\mathcal{K}_0(0)+\mathcal{K}_0(1)\xi+\mathcal{K}_0(2)\xi^2+\cdots, \\ \mathcal{K}^i_j(u)=\mathcal{K}^i_j(0)+\mathcal{K}^i_j(1)\xi+\mathcal{K}^i_j(2)\xi^2+\cdots, \\ \mathcal{K}^{\lambda 3}(u)(u)=\mathcal{K}^{\lambda 3}(u)(0)+\mathcal{K}^{\lambda 3}(u)(1)\xi+\mathcal{K}^{\lambda 3}(2)\xi^2+\cdots, \\ \mathcal{K}^{\lambda\sigma}(u)=\mathcal{K}^{\lambda\sigma}(0)+\mathcal{K}^{\lambda\sigma}(1)\xi+\mathcal{K}^{\lambda\sigma}(2)\xi^2+\cdots, \\ M^{i\mu}_k(u)=M^{i\mu}_k(0)+M^{i\mu}_k(1)\xi+M^{i\mu}_k(2)\xi^2+\cdots, \\ M^{i0}_k(u)=M^{i0}_k(0)+M^{i0}_k(1)\xi+M^{i0}_k(2)\xi^2+\cdots. \end{cases} \tag{6.26}$$

Therefore

$$\begin{cases} \mathcal{K}_0(u)\frac{\partial^2 u^i}{\partial\xi^2}=2\mathcal{K}_0(0)u^i_2+2\mathcal{K}_0(1)u^i_2\xi+2\mathcal{K}_0(2)u^i_2\xi^2+\cdots, \\ \mathcal{K}^i_j(u)\frac{\partial u^j}{\partial\xi}=\mathcal{K}^i_j(0)u^j_1+(\mathcal{K}^i_j(0)2u^j_2+\mathcal{K}^i_j(1)u^j_1)\xi+(\mathcal{K}^i_j(1)2u^j_2+\mathcal{K}^i_j(2)u^j_1)\xi^2+\cdots, \\ \mathcal{K}^{\lambda 3}(u)\overset{*}{\nabla}_\lambda\frac{\partial u^i}{\partial\xi}=\mathcal{K}^{\lambda 3}(0)\overset{*}{\nabla}_\lambda u^i_1+(\mathcal{K}^{\lambda 3}(0)2\overset{*}{\nabla}_\lambda u^i_2+\mathcal{K}^{\lambda 3}(1)\overset{*}{\nabla}_\lambda u^i_1)\xi \\ \qquad +(\mathcal{K}^{\lambda 3}(2)\overset{*}{\nabla}_\lambda u^i_1+\mathcal{K}^{\lambda 3}(1)2\overset{*}{\nabla}_\lambda u^i_2)\xi^2+\cdots, \\ \mathcal{K}^{\lambda\sigma}(u)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i=\mathcal{K}^{\lambda\sigma}(0)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_0+(\mathcal{K}^{\lambda\sigma}(0)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_1+\mathcal{K}^{\lambda\sigma}(1)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_0)\xi \\ \qquad +(\mathcal{K}^{\lambda\sigma}(0)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_2+\mathcal{K}^{\lambda\sigma}(1)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_1+\mathcal{K}^{\lambda\sigma}(2)\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^i_0)\xi^2+\cdots, \\ M^{i\mu}_k(u)\overset{*}{\nabla}_\mu u^k=M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_0+(M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_1+M^{i\mu}_k(1)\overset{*}{\nabla}_\mu u^k_0)\xi \\ \qquad +(M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_2+M^{i\mu}_k(1)\overset{*}{\nabla}_\mu u^k_1M^{i\mu}_k(2)\overset{*}{\nabla}_\mu u^k_0)\xi^2+\cdots, \\ M^{i0}_k(u)u^k=M^{i\mu}_k(0)u^k_0+(M^{i\mu}_k(0)u^k_1+M^{i\mu}_k(1)u^k_0)\xi \\ \qquad +(M^{i\mu}_k(0)u^k_2+M^{i\mu}_k(1)u^k_1+M^{i\mu}_k(2)u^k_0)\xi^2+\cdots, \\ \{g^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u+\frac{\partial u^3}{\partial\xi})\delta^i_\alpha+\overset{*}{\operatorname{div}}\frac{\partial u}{\partial\xi}\delta^i_3\}=\{a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_0+u^3_1)\delta^i_\alpha+\overset{*}{\operatorname{div}}u_1\delta^i_3\} \\ \qquad +\{a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_1+2u^3_2)\delta^i_\alpha+2\overset{*}{\operatorname{div}}u_2\delta^i_3+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_0+u^3_1)\delta^i_\alpha\}\xi \\ \qquad +\{a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_2)\delta^i_\alpha+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_1+2u^3_2)\delta^i_\alpha\}\xi^2+\cdots. \end{cases} \tag{6.27}$$

Next we consider $\mathcal{D}^i$ defined by (6.18). To do that let remember bilinear and symmetric form $D(u)$ defined by (2.48)

$$D_{ij}(u)=\varphi_{ij}(u,u)+\varphi^1_{ij}(u,u)\xi+\varphi^2_{ij}(u,u)\xi^2.$$

Denote

$$\begin{cases} \psi^1_{ij}(u_0,u_1)=2\varphi_{ij}(u_0,u_1)+\varphi^1_{ij}(u_0,u_0), \\ \psi^2_{ij}(u_0,u_1,u_2)=2\varphi_{ij}(u_1,u_0)+\varphi_{ij}(u_1,u_1)+2\varphi^1_{ij}(u_0,u_1)+\varphi^2_{ij}(u_0,u_0), \\ \psi^3_{ij}(u_0,u_1,u_2)=2\varphi_{ij}(u_1,u_2)+\varphi^1_{ij}(u_1,u_1)+2\varphi^1_{ij}(u_0,u_2)+2\varphi^2_{ij}(u_0,u_1). \end{cases} \tag{6.28}$$

Then

$$\begin{cases} D_{ij}(u)=\varphi_{ij}(u_0,u_0)+\psi^1(u_0,u_1)\xi+\psi^2_{ij}(u_0,u_1,u_2)\xi^2+\psi^3(u_0,u_1,u_2)\xi^3+\cdots, \\ \frac{\partial}{\partial\xi}D_{ij}(u)=\psi^1(u_0,u_1)+2\psi^2_{ij}(u_0,u_1,u_2)\xi+3\psi^3(u_0,u_1,u_2)\xi^2, \\ \overset{*}{\nabla}_\beta D_{ij}(u)=\overset{*}{\nabla}_\beta\varphi_{ij}(u_0,u_0)+\overset{*}{\nabla}_\beta\psi^1(u_0,u_1)\xi+\overset{*}{\nabla}_\beta\psi^2_{ij}(u_0,u_1,u_2)\xi^2 \\ \qquad\qquad +\overset{*}{\nabla}_\beta\psi^3_{ij}(u_0,u_1,u_2)\xi^3+\cdots. \end{cases} \tag{6.29}$$

In addition, let

$$A^{\alpha\nu 3}(\xi):=A^{\alpha\beta\lambda\sigma}\Lambda^{\nu 3}_{\beta\lambda\sigma},\quad A^{\alpha\nu\mu}(\xi)=A^{\alpha\beta\lambda\sigma}\Lambda^{\nu\mu}_{\beta\lambda\sigma}(\xi). \tag{6.30}$$

Then

$$\left\{\begin{array}{l} A^{\alpha\nu 3}(\xi)=A^{\alpha\nu 3}_0+A^{\alpha\nu 3}_1\xi+A^{\alpha\nu 3}_2\xi^2+\cdots,\\ A^{\alpha\nu\mu}(\xi)=A^{\alpha\nu\mu}_0+A^{\alpha\nu\mu}_1\xi+A^{\alpha\nu\mu}_2\xi^2+\cdots,\\ A^{\alpha\nu 3}_0=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu 3}_{\beta\lambda\sigma}(0),\quad A^{\alpha\nu 3}_1=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu 3}_{\beta\lambda\sigma}(1)+A^{\alpha\beta\lambda\sigma}_1\Lambda^{\nu 3}_{\beta\lambda\sigma}(0),\\ A^{\alpha\nu 3}_2=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu 3}_{\beta\lambda\sigma}(2)+A^{\alpha\beta\lambda\sigma}_1\Lambda^{\nu 3}_{\beta\lambda\sigma}(1)+A^{\alpha\beta\lambda\sigma}_2\Lambda^{\nu 3}_{\beta\lambda\sigma}(0),\\ A^{\alpha\nu\mu}_0=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu\mu}_{\beta\lambda\sigma}(0),\quad A^{\alpha\nu\mu}_1=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu\mu}_{\beta\lambda\sigma}(1)+A^{\alpha\beta\lambda\sigma}_1\Lambda^{\nu\mu}_{\beta\lambda\sigma}(0),\\ A^{\alpha\nu\mu}_2=A^{\alpha\beta\lambda\sigma}_0\Lambda^{\nu\mu}_{\beta\lambda\sigma}(2)+A^{\alpha\beta\lambda\sigma}_1\Lambda^{\nu\mu}_{\beta\lambda\sigma}(1)+A^{\alpha\beta\lambda\sigma}_2\Lambda^{\nu\mu}_{\beta\lambda\sigma}(0). \end{array}\right. \tag{6.31}$$

Let us come back to (6.18). Note that

$$\left\{\begin{array}{l} \mathcal{D}^\alpha(u):=A^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta D_{\lambda\sigma}(u)+A^{\alpha\nu 3}_{\beta\lambda\sigma}(\xi)D_{\nu 3}(u)+A^{\alpha\nu\mu}_{\beta\lambda\sigma}(\xi)D_{\nu\mu}(u)+g^{\alpha\beta}G_\beta(u)\\ \qquad\quad =\mathcal{D}^\alpha(0)+\mathcal{D}^\alpha(1)\xi+\mathcal{D}^\alpha(2)\xi^2+\cdots,\\ \mathcal{D}^3(u):=\frac{\partial}{\partial\xi}G_0(u)+2\mu g^{\alpha\beta}G_{\alpha\beta}(u)+2\mu\theta^{-1}(-2H+2K\xi)D_{33}(u)\\ \qquad\quad =\mathcal{D}^3(0)+\mathcal{D}^3(1)\xi+\mathcal{D}^3(2)\xi^2+\cdots,\\ G_\beta(u):=\lambda\overset{*}{\nabla}_\beta D_{33}(u)+2\mu\frac{\partial}{\partial\xi}D_{3\beta}(u)-2(\lambda+\mu)I^\lambda_\beta(\xi)D_{3\lambda}(u)\\ \qquad\quad =G_\beta(0)+G_\beta(1)\xi+G_(2)\xi^2+\cdots,\\ G_{\alpha\beta}(u):=\Phi^\lambda_{\alpha\beta}(\xi)D_{3\lambda}(u)+I^\lambda_\alpha D_{\lambda\beta}(u)=G_{\alpha\beta}(0)+G_{\alpha\beta}(1)\xi+G_{\alpha\beta}(2)\xi^2+\cdots,\\ G_0(u)=\lambda g^{\alpha\beta}D_{\alpha\beta}(u)+(\lambda+2\mu)D_{33}(u)=G_0(0)+G_0(1)\xi+G_0(2)\xi^2+\cdots, \end{array}\right. \tag{6.32}$$

where

$$\left\{\begin{array}{l} G_\beta(0)=\lambda\overset{*}{\nabla}_\beta\varphi_{33}(u_0,u_0)+2\mu\psi^1_{3\beta}(u_0,u_0)+2(\lambda+\mu)b^\lambda_\beta\varphi_{3\lambda}(u_0,u_0),\\ G_\beta(1)=\lambda\overset{*}{\nabla}_\beta\psi^1_{33}(u_0,u_0)+2\mu\psi^2_{3\beta}(u_0,u_0)-2(\lambda+\mu)(-b^\lambda_\beta\psi^1_{3\lambda}(u_0,u_1)+K\varphi_{3\beta}(u_0,u_0)),\\ G_\beta(2)=\lambda\overset{*}{\nabla}_\beta\psi^2_{33}(u_0,u_1,u_2)+2\mu\psi^3_{3\beta}(u_0,u_0)-2(\lambda+\mu)(-b^\lambda_\beta\psi^2_{3\lambda}(u_0,u_1)+K\psi^1_{\beta 3}(u_0,u_1)),\\ G_{\alpha\beta}(0)=-b^\lambda_\alpha\varphi_{\lambda\beta}(u_0,u_0),\\ G_{\alpha\beta}(1)=-\overset{*}{\nabla}_\alpha b^\lambda_\beta\varphi_{3\lambda}(u_0,u_0)-b^\lambda_\alpha\psi^1_{\lambda\beta}(u_0,u_1)+K\varphi_{\alpha\beta}(u_0,u_0),\\ G_{\alpha\beta}(2)=-\overset{*}{\nabla}_\alpha b^\lambda_\beta\psi^1_{3\lambda}(u_0,u_1)+b^\lambda_\mu\overset{*}{\nabla}_\alpha b^\mu_\beta\varphi_{3\lambda}(u_0,u_0)-b^\lambda_\alpha\psi^2_{\lambda\beta}(u_0,u_1,u_2)+K\psi^1_{\alpha\beta}(u_0,u_0),\\ G_0(0)=\lambda a^{\alpha\beta}\varphi_{\alpha\beta}(u_0,u_0)+(\lambda+2\mu)\varphi_{33}(u_0,u_0),\\ G_0(1)=\lambda a^{\alpha\beta}\psi^1_{\alpha\beta}(u_0,u_1)+2\lambda b^{\alpha\gamma}\varphi_{\alpha\beta}(u_0,u_0)+(\lambda+2\mu)\psi^1_{33}(u_0,u_1),\\ G_0(2)=\lambda a^{\alpha\beta}\psi^2_{\alpha\beta}(u_0,u_1)+2\lambda b^{\alpha\gamma}\psi^1_{\alpha\beta}(u_0,u_1)+3\lambda c^{\alpha\beta}\varphi_{\alpha\beta}(u_0,u_0)+(\lambda+2\mu)\psi^2_{33}(u_0,u_1,u_2),\\ G_0(3)=\lambda(2b^{\alpha\beta}\psi^2_{\alpha\beta}(u_0,u_1,u_2)+3c^{\alpha\gamma}\psi^1_{\alpha\beta}(u_0,u_1))+(\lambda+2\mu)\psi^3_{33}(u_0,u_1,u_2), \end{array}\right. \tag{6.33}$$

$$\begin{cases} \mathcal{D}^\alpha(0):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)+A_0^{\alpha\nu3}\varphi_{\nu3}(u_0,u_0)+A_0^{\alpha\nu\mu}\varphi_{\nu\mu}(u_0,u_0)+a^{\alpha\beta}G_\beta(0),\\ \mathcal{D}^\alpha(1):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^1_{\lambda\sigma}(u_0,u_1)+A_1^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)+A_0^{\alpha\nu3}\psi^1_{\nu3}(u_0,u_1)\\ \qquad +A_1^{\alpha\nu3}\varphi_{\nu3}(u_0,u_0)+A_0^{\alpha\nu\mu}\psi^1_{\nu3}(u_0,u_1)+A_1^{\alpha\nu\mu}\varphi_{\nu\mu}(u_0,u_0)+a^{\alpha\beta}G_\beta(1)+2b^{\alpha\beta}G_\beta(0),\\ \mathcal{D}^\alpha(2):=A_0^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^2_{\lambda\sigma}(u_0,u_1,u_2)+A_1^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\psi^1_{\lambda\sigma}(u_0,u_1)+A_2^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta\varphi_{\lambda\sigma}(u_0,u_0)\\ \qquad +A_0^{\alpha\nu3}\psi^2_{\nu3}(u_0,u_1,u_2)+A_1^{\alpha\nu3}\psi^1_{\nu3}(u_0,u_1)+A_2^{\alpha\nu3}\varphi_{\nu3}(u_0,u_0)+A_0^{\alpha\nu\mu}\psi^2_{\nu3}(u_0,u_1,u_2)\\ \qquad +A_1^{\alpha\nu\mu}\psi^1_{\nu3}(u_0,u_1)+A_2^{\alpha\nu\mu}\varphi_{\nu3}(u_0,u_0)+a^{\alpha\beta}G_\beta(2)+2b^{\alpha\beta}G_\beta(1)+3c^{\alpha\beta}G_\beta(0), \end{cases} \tag{6.34}$$

$$\begin{cases} \mathcal{D}^3(0)=G_0(1)+2\mu a^{\alpha\beta}G_{\alpha\beta}(0)+2\mu 2H\varphi_{33}(u_0,u_0),\\ \mathcal{D}^3(1)=2G_0(2)+2\mu(2b^{\alpha\beta}G_{\alpha\beta}(0+a^{\alpha\beta}G_{\alpha\beta}(1))+2\mu(2H\psi^1_{33}(u_0,u_1)\\ \qquad +(4H^2-2K)\varphi_{33}(u_0,u_0)),\\ \mathcal{D}^3(2)=3G_0(3)+2\mu(3c^{\alpha\beta}G_{\alpha\beta}(0)+2b^{\alpha\beta}G_{\alpha\beta}(1)+a^{\alpha\beta}G_\beta(2))+2\mu(2H\psi^2_{33}(u_0,u_1,u_2)\\ \qquad +(4H^2-2K)\psi^1_{33}(u_0,u_1)+(8H^3-6HK)\varphi_{33}(u_0,u_0)). \end{cases} \tag{6.35}$$

Futhermore, in view of (2.34) and (6.19) we claim

$$\begin{cases} \nabla_i u^j=\gamma^i_{i0}(u_0)+\gamma^i_{i1}(u_0,u_1)\xi+\gamma^i_{i2}(u_0,u_1,u_2)\xi^2+\cdots,\\ \gamma^\alpha_{\beta0}(u_0)=\overset{*}{\nabla}_\beta u_0^\alpha-b^\alpha_\beta u_0^3,\quad \gamma^\alpha_{\beta1}(u_0,u_1)=\overset{*}{\nabla}_\beta u_1^\alpha-b^\alpha_\beta u_1^3-c^\alpha_\beta u_0^3+\Phi^\alpha_{\beta\lambda}(1)u_0^\lambda,\\ \gamma^\alpha_{\beta2}(u_0,u_1,u_2)=\overset{*}{\nabla}_\beta u_2^\alpha-b^\alpha_\beta u_2^3-c^\alpha_\beta u_1^3+(Kb^\alpha_\beta-2Hc^\alpha_\beta)u_0^3+\Phi^\alpha_{\beta\lambda}(1)u_1^\lambda+\Phi^\alpha_{\beta\lambda}(2)u_0^\lambda,\\ \gamma^\alpha_{30}(u_0)=u_1^\alpha+b^\alpha_\beta u_0^\beta,\quad \gamma^\alpha_{31}(u_0,u_1)=u_2^\alpha+b^\alpha_\beta u_1^\beta-c^\alpha_\beta u_2^\beta,\\ \gamma^\alpha_{32}(u_0,u_1,u_2)=b^\alpha_\beta u_2^\beta-c^\alpha_\beta u_1^\beta+(Kb^\alpha_\beta-2Hc^\alpha_\beta)u_0^\beta,\\ \gamma^3_{\beta0}(u_0)=\overset{*}{\nabla}_\beta u_0^3+b_{\alpha\beta}u_0^\alpha,\quad \gamma^3_{\beta1}(u_0,u_1)=\overset{*}{\nabla}_\beta u_1^3+b_{\alpha\beta}u_1^\alpha-c_{\alpha\beta}u_0^\alpha,\\ \gamma^3_{\beta2}(u_0,u_1,u_2)=\overset{*}{\nabla}_\beta u_2^3+b_{\alpha\beta}u_2^\alpha-c_{\alpha\beta}u_1^\alpha,\quad \gamma^3_{30}=u_1^3,\quad \gamma^3_{31}=2u_2^3,\quad \gamma^3_{32}=0, \end{cases} \tag{6.36}$$

where

$$\Phi^\alpha_{\beta\lambda}(1)=-\overset{*}{\nabla}_\lambda b^\alpha_\beta;\quad \Phi^\alpha_{\beta\lambda}(2)=-b^\alpha_\mu\overset{*}{\nabla}_\lambda b^\mu_\beta. \tag{6.37}$$

Owing to (6.36) and (6.32) we assert

$$\begin{aligned} &(\delta^i_k+\nabla_k u^i)\mathcal{D}^k(u)\\ &=(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(0)+\{(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(1)\\ &-\gamma^i_{k1}(u_0,u_1)\mathcal{D}^k(0)\}\xi+\{(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(2)-\gamma^i_{k1}(u_0,u_1)\mathcal{D}^k(1)\\ &-\gamma^i_{k2}(u_0,u_1,u_2)\mathcal{D}^k(0)\}\xi^2. \end{aligned} \tag{6.38}$$

Substituting (6.38) and (6.26) into (6.25) leads to

$$\left\{\begin{array}{l}\mathcal{N}^i(u):=\mathcal{N}^i_0(u_0)+\mathcal{N}^i_1(u_0,u_1)\xi+\mathcal{N}^i_2(u_0,u_1,u_2)\xi^2+\cdots,\\ \mathcal{N}^i_0(u_0)=\mathcal{K}_0(0)2u^i_2+\mathcal{K}^i_j(0)u^j_1+\mathcal{K}^{\lambda 3}(0)\overset{*}{\nabla}_\lambda u^i_1+\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_0+M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_0\\ \qquad +M^{i0}_k(0)u^k_0+a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_0+u^3_1)\delta^i_\alpha+\overset{*}{\operatorname{div}}u_1\delta^i_3-(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(0),\\ \mathcal{N}^i_1(u_0,u_1)=\mathcal{K}_0(1)2u^i_2+\mathcal{K}^i_j(0)2u^j_2+\mathcal{K}^i_j(1)u^j_1+\mathcal{K}^{\lambda 3}(0)2\overset{*}{\nabla}_\lambda u^i_2+\mathcal{K}^{\lambda 3}(1)\overset{*}{\nabla}_\lambda u^i_1\\ \qquad +\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_1+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_0+M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_1+M^{i\mu}_k(1)\overset{*}{\nabla}_\mu u^k_0+M^{i0}_k(0)u^k_1\\ \qquad +M^{i0}_k(1)u^k_0+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_0+u^3_1)\delta^i_\alpha+2\overset{*}{\operatorname{div}}u_2\delta^i_3+a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_1+2u^3_2)\delta^i_\alpha\\ \qquad -(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(1)-\gamma^i_{k1}(u_0,u_1)\mathcal{D}^k(0),\\ \mathcal{N}^i_2(u_0,u_1,u_2)=\mathcal{K}_0(2)2u^i_2+\mathcal{K}^i_j(1)2u^j_2+\mathcal{K}^i_j(2)u^j_1+\mathcal{K}^{\lambda 3}(1)2\overset{*}{\nabla}_\lambda u^i_2+\mathcal{K}^{\lambda 3}(2)\overset{*}{\nabla}_\lambda u^i_1\\ \qquad +\mathcal{K}^{\alpha\beta}(0)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_2+\mathcal{K}^{\alpha\beta}(1)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_1+\mathcal{K}^{\alpha\beta}(2)\overset{*}{\nabla}_\alpha\overset{*}{\nabla}_\beta u^i_0+M^{i\mu}_k(0)\overset{*}{\nabla}_\mu u^k_2\\ \qquad +M^{i\mu}_k(1)\overset{*}{\nabla}_\mu u^k_1+M^{i\mu}_k(2)\overset{*}{\nabla}_\mu u^k_0+M^{i0}_k(0)u^k_2+M^{i0}_k(1)u^k_1+M^{i0}_k(2)u^k_0\\ \qquad +a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_2)\delta^i_\alpha+2b^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\operatorname{div}}u_1+2u^3_2)\delta^i_\alpha\\ \qquad -(\delta^i_k+\gamma^i_{k0}(u_0))\mathcal{D}^k(2)-\gamma^i_{k1}(u_0,u_1)\mathcal{D}^k(1)-\gamma^i_{k2}(u_0,u_1,u_2)\mathcal{D}^k(0).\end{array}\right. \tag{6.39}$$

This is (6.20).

Next we consider expansion for bilinearly symmetric form. Note Green St-vennen strain tensor $E_{ij}(u)$ of vector u is defined by (2.45)

$$E_{ij}(u)=e_{ij}(u)+D_{ij}(u)$$

and satisfies following formula (see Lemma 2.6)

$$\left\{\begin{array}{l}E_{ij}(u)=\overset{0}{E}_{ij}(u)+\overset{1}{E}_{ij}(u)\xi+\overset{2}{E}_{ij}(u)\xi^2,\\ \overset{k}{E}_{ij}(u)=\overset{k}{\gamma}_{ij}(u)+\varphi^k_{ij}(u,u),\end{array}\right.$$

which are polynomials of two degree with respect to transverse varialble ξ, and its coefficients do not contain three dimensional cavariant derivatives $\nabla_k u^i$ of displacement vector u but contain two dimensional derivatives $\overset{*}{\nabla}_k u^i$ on the surface. Therefore if displacement vector satisfies Taylor expansion (6.19), then we claim

$$\left\{\begin{array}{l}E_{ij}(u)=E^0_{ij}(u_0)+E^1_{ij}(u_0,u_1)\xi+E^2_{ij}(u_0,u_1,u_2)\xi^2+\cdots,\\ E^0_{ij}(u_0)=\gamma_{ij}(u_0)+\varphi_{ij}(u_0,u_0),\quad E^1_{ij}(u_0,u_1)=\overset{1}{\gamma}_{ij}(u_1)+\psi^1_{ij}(u_0,u_1),\\ E^2_{ij}(u_0,u_1,u_2)=\overset{2}{\gamma}_{ij}(u_2)+\psi^2_{ij}(u_0,u_2).\end{array}\right. \tag{6.40}$$

Let us denote

$$\begin{cases} E^{ij}(u)=g^{ik}g^{jm}E_{km}(u),\\ E^{\alpha\beta}(u)=g^{\alpha\lambda}g^{\beta\sigma}E_{\lambda\sigma}=E^{\alpha\beta}(0)+E^{\alpha\beta}(1)\xi+E^{\alpha\beta}(2)\xi^2+\cdots,\\ E^{33}(u)=g^{33}g^{33}E_{33}=E_{33}(u)=E^0_{33}(u_0)+E^1_{33}(u_0,u_1)\xi+E^2_{33}(u_0,u_1,u_2)\xi^2+\cdots,\\ E^{3\alpha}(u)=E^{\alpha 3}(u)=g^{\alpha\beta}g^{33}E_{3\beta}=g^{\alpha\beta}E_{3\beta}=(a^{\alpha\beta}+2b^{\alpha\beta}+3c^{\alpha\beta})E^0_{ij}(u_0)+E^1_{ij}(u_0,u_1)\xi\\ \qquad +E^2_{ij}(u_0,u_1,u_2)\xi^2+\cdots=E^{3\alpha}(0)+E^{3\alpha}(1)\xi+E^{3\alpha}(2)\xi^2+cdots, \end{cases} \tag{6.41}$$

where

$$\begin{cases} E^{\alpha\beta}(0)=a_0^{\alpha\lambda\beta\sigma}E^0_{\lambda\sigma}(u_0,u_0),\quad E^{\alpha\beta}(1)=a_0^{\alpha\lambda\beta\sigma}E^1_{\lambda\sigma}(u_0,u_1)+a_1^{\alpha\beta\lambda\sigma}E^0_{\lambda\sigma}(u_0,u_0),\\ E^{\alpha\beta}(2)=a_0^{\alpha\lambda\beta\sigma}E^2_{\lambda\sigma}(u_0,u_1,u_2)+a_1^{\alpha\beta\lambda\sigma}E^1_{\lambda\sigma}(u_0,u_1)+a_2^{\alpha\beta\lambda\sigma}E^0_{\lambda\sigma}(u_0,u_0),\\ E^{3\alpha}(0)=a_0^{\alpha\lambda}E^0_{3\lambda}(u_0,u_0),\qquad E^{3\alpha}(1)=a^{\alpha\beta}E^1_{3\beta}(u_0,u_1)+2b^{\alpha\beta}E^0_{3\beta}(u_0,u_0),\\ E^{3\alpha}(2)=a_0^{\alpha\beta}E^2_{3\beta}(u_0,u_1,u_2)+2b_1^{\alpha\beta}E^1_{3\beta}(u_0,u_1)+3c_2^{\alpha\beta}E^0_{3\beta}(u_0,u_0), \end{cases} \tag{6.42}$$

$$\begin{cases} g^{\alpha\lambda}g^{\beta\sigma}=a_0^{\alpha\beta\lambda\sigma}+a_1^{\alpha\beta\lambda\sigma}\xi+a_2^{\alpha\beta\lambda\sigma}\xi^2+\cdots,\\ a_0^{\alpha\beta\lambda\sigma}=a^{\alpha\lambda}a^{\beta\sigma},\quad a_1^{\alpha\beta\lambda\sigma}=2(a^{\alpha\lambda}b^{\beta\sigma}+b^{\alpha\lambda}a^{\beta\sigma}),\\ a_2^{\alpha\beta\lambda\sigma}=3(a^{\alpha\lambda}c^{\beta\sigma}+c^{\alpha\lambda}a^{\beta\sigma})+4b^{\alpha\lambda}b^{\sigma}, \end{cases} \tag{6.43}$$

$$\begin{cases} \mathrm{Div}u=g^{\alpha\beta}E_{\alpha\beta}(u)+E_{33}(u)=\mathrm{Div}(0)+\mathrm{Div}(1)\xi+\mathrm{Div}(2)\xi^2+\cdots\\ \mathrm{Div}(0)=a^{\alpha\beta}E^0_{\alpha\beta}(u_0,u_0)+E^0_{33}(u_0,u_0),\\ \mathrm{Div}(1)=a^{\alpha\beta}E^1_{\alpha\beta}(u_0,u_1)+2b^{\alpha\beta}E^0_{\alpha\beta}(u_0,u_0)+E^1_{33}(u_0,u_1),\\ \mathrm{Div}(2)=a^{\alpha\beta}E^2_{\alpha\beta}(u_0,u_1,u_2)+2b_1^{\alpha\beta}E^1_{\alpha\beta}(u_0,u_1)+3c^{\alpha\beta}E^0_{\alpha\beta}(u_0,u_0)+E^2_{33}(u_0,u_1,u_2). \end{cases} \tag{6.44}$$

Finally, the coefficients in (6.26) are given by using (6.12), (6.41), (6.42) and (6.44).

$$\begin{cases} \mathcal{K}_0(0)=-\{\mu+\lambda\mathrm{Div}(0)+2\mu E^0_{33}(u_0,u_0)\},\\ \mathcal{K}_0(1)=-\{\lambda\mathrm{Div}(1)+2\mu E^1_{33}(u_0,u_1)\},\\ \mathcal{K}_0(2)=-\{\lambda\mathrm{Div}(2)+2\mu E^2_{33}(u_0,u_1,u_2)\},\\ \mathcal{K}^i_j(0)=-\{m^{i3}_j(0)+\lambda\mathrm{Div}(0)\Pi^{i3}_j(0)+2\mu E^0_{33}(u_0,u_0)l^i_j(0)\\ \qquad -2\mu E^{\lambda\sigma}(0)b_{\lambda\sigma}\delta^i_j+4\mu E^0_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(0)\},\\ \mathcal{K}^i_j(1)=-\{m^{i3}_j(1)+\lambda(\mathrm{Div}(0)\Pi^{i3}_j(1)+\mathrm{Div}(1)\Pi^{i3}_j(0))\\ \qquad +2\mu(E^0_{33}(u_0,u_0)l^i_j(1)+E^1_{33}(u_0,u_1)l^i_j(0))+2\mu(E^{\lambda\sigma}(0)c_{\lambda\sigma}-E^{\lambda\sigma}(1)b_{\lambda\sigma})\delta^i_j\\ \qquad +4\mu(E^0_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(1)+E^1_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(0))\},\\ \mathcal{K}^i_j(2)=-\{m^{i3}_j(2)+\lambda(\mathrm{Div}(0)\Pi^{i3}_j(2)+\mathrm{Div}(1)\Pi^{i3}_j(1)+\mathrm{Div}(2)\Pi^{i3}_j(0))\\ \qquad +2\mu(E^0_{33}(u_0,u_0)l^i_j(2)+E^1_{33}(u_0,u_1)l^i_j(1)+E^2_{33}(u_0,u_1)l^i_j(0))-2\mu(-E^{\lambda\sigma}(1)c_{\lambda\sigma}\\ \qquad +E^{\lambda\sigma}(2)b_{\lambda\sigma})\delta^i_j+4\mu(E^0_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(2)+E^1_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(1)+E^2_{3\sigma}(u_0,u_0)\Pi^{i3}_{\lambda 3,j}(0))\},\\ \mathcal{K}^{\lambda 3}(0)=-4\mu a^{\lambda\sigma}E^0_{3\sigma}(u_0,u_0),\\ \mathcal{K}^{\lambda 3}(1)=-4\mu(a^{\lambda\sigma}E^1_{3\sigma}(u_0,u_1)+2b^{\lambda\sigma}E^0_{3\sigma}(u_0,u_0)),\\ \mathcal{K}^{\lambda 3}(2)=-4\mu(a^{\lambda\sigma}E^2_{3\sigma}(u_0,u_1,u_2)+2b^{\lambda\sigma}E^1_{3\sigma}(u_0,u_1))+3c^{\lambda\sigma}E^0_{3\sigma}(u_0,u_0), \end{cases} \tag{6.45}$$

$$\begin{cases} \mathcal{K}^{\alpha\beta}(0) = -\{a^{\alpha\beta}(\mu+\lambda\text{Div}(0)) + 2\mu E^{\alpha\beta}(0)\}, \\ \mathcal{K}^{\alpha\beta}(1) = -\{2b^{\alpha\beta}(\mu+\lambda\text{Div}(0)) + a^{\alpha\beta}\lambda\text{Div}(1) + 2\mu E^{\alpha\beta}(1)\}, \\ \mathcal{K}^{\alpha\beta}(2) = -\{3c^{\alpha\beta}(\mu+\lambda\text{Div}(0)) + 2b^{\alpha\beta}\lambda\text{Div}(1) + a^{\alpha\beta}\lambda\text{Div}(2) + 2\mu E^{\alpha\beta}(2)\}, \end{cases} \tag{6.46}$$

$$\begin{cases} M_k^{i\mu}(0) = m_k^{i\mu}(0) - \lambda\text{Div}(0)\Pi_k^{i\mu}(0) - 2\mu E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i\mu}(0) - 4\mu a^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(0), \\ M_k^{i\mu}(1) = m_k^{i\mu}(1) - \lambda(\text{Div}(0)\Pi_k^{i\mu}(1) + \text{Div}(1)\Pi_k^{i\mu}(0)) \\ \qquad -2\mu(E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i\mu}(1) + E^{\lambda\sigma}(1)\Pi_{\lambda\sigma,k}^{i\mu}(0)) \\ \qquad -4\mu\{a^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(1) + E_{3\sigma}^1(u_0,u_1)\Pi_{\lambda 3}^{i\mu}(0)) + 2b^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(0)\}, \\ M_k^{i\mu}(2) = m_k^{i\mu}(2) - \lambda(\text{Div}(0)\Pi_k^{i\mu}(2) + \text{Div}(1)\Pi_k^{i\mu}(1) + \text{Div}(2)\Pi_k^{i\mu}(0)) \\ \qquad -2\mu(E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i\mu}(2) + E^{\lambda\sigma}(1)\Pi_{\lambda\sigma,k}^{i\mu}(1) + E^{\lambda\sigma}(2)\Pi_{\lambda\sigma,k}^{i\mu}(0)) \\ \qquad -4\mu\{a^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(2) + E_{3\sigma}^1(u_0,u_1)\Pi_{\lambda 3}^{i\mu}(1) + E_{3\sigma}^2(u_0,u_1,u_2)\Pi_{\lambda 3}^{i\mu}(0)) \\ \qquad +2b^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(1) + E_{3\sigma}^1(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(0)) + 3c^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i\mu}(0)\}, \\ M_k^{i0}(0) = m_k^{i0}(0) - \lambda\text{Div}(0)\Pi_k^{i0}(0) - 2\mu E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i0}(0) - 4\mu a^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(0), \\ M_k^{i\mu}(1) = m_k^{i\mu}(1) - \lambda(\text{Div}(0)\Pi_k^{i}(1) + \text{Div}(1)\Pi_k^{i}(0)) \\ \qquad -2\mu(E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i0}(1) + E^{\lambda\sigma}(1)\Pi_{\lambda\sigma,k}^{i0}(0)) \\ \qquad -4\mu\{a^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(1) + E_{3\sigma}^1(u_0,u_1)\Pi_{\lambda 3}^{i0}(0)) + 2b^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(0)\}, \\ M_k^{i\mu}(2) = m_k^{i\mu}(2) - \lambda(\text{Div}(0)\Pi_k^{i0}(2) + \text{Div}(1)\Pi_k^{i0}(1) + \text{Div}(2)\Pi_k^{i0}(0)) \\ \qquad -2\mu(E^{\lambda\sigma}(0)\Pi_{\lambda\sigma,k}^{i0}(2) + E^{\lambda\sigma}(1)\Pi_{\lambda\sigma,k}^{i0}(1) + E^{\lambda\sigma}(2)\Pi_{\lambda\sigma,k}^{i0}(0)) \\ \qquad -4\mu\{a^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(2) + E_{3\sigma}^1(u_0,u_1)\Pi_{\lambda 3}^{i0}(1) + E_{3\sigma}^2(u_0,u_1,u_2)\Pi_{\lambda 3}^{i0}(0)) \\ \qquad +2b^{\lambda\sigma}(E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(1) + E_{3\sigma}^1(u_0,u_0)\Pi_{\lambda 3}^{i0}(0)) + 3c^{\lambda\sigma}E_{3\sigma}^0(u_0,u_0)\Pi_{\lambda 3}^{i0}(0)\}. \end{cases} \tag{6.47}$$

Next we have to give boundary conditions

$$u_k|_{\gamma_0} = 0, \quad \sigma n|_{\gamma_1} = h,$$

where σ are the first Piola-Kirchhoff stress defined by (6.2)

$$\sigma^{ij} = (\alpha_k^i + \nabla_k u^i)A^{kjml}E_{ml}(u).$$

The normal vector n on γ_1 is $n - n_\alpha e^\alpha$,

$$\begin{aligned} \sigma n|_{\gamma_1} &= a_{\alpha\beta}\sigma^{i\beta}n^\alpha = a_{\alpha\beta}n^\alpha(\delta_k^i + \gamma_{k0}^i(u_0))A^{k\beta ml}E_{ml}(u)|_{\xi=0} \\ &= a_{\alpha\beta}n^\alpha\{(\delta_n^i u + \gamma_{\nu 0}^i((u_0))(A_0^{\nu\beta\lambda\sigma}E\gamma_{\lambda\sigma}^0(u_0) + A^{\nu\beta 33}E_{33}^0(u_0)) \\ &\quad + (\delta_3^i + \gamma_{30}^i(u_0))(A^{3\beta 3\mu}E_{3\mu}^0(u_0) + A^{3\beta\mu 3}E_{\mu 3}^0(u_0))\} \quad (\text{see}(2.57)) \\ &= a_{\alpha\beta}n^\alpha\{(\delta_n^i u + \gamma_{\nu 0}^i((u_0))(A_0^{\nu\beta\lambda\sigma}E_{\lambda\sigma}^0(u_0) + \lambda a^{\nu\beta}E_{33}^0(u_0)) + (\delta_3^i + \gamma_{30}^i(u_0))2\mu a^{\beta\mu}E_{3\mu}^0(u_0)\}. \end{aligned}$$

Hence boundary conditions are given by

$$\begin{cases} u|_{\gamma_0}=0, \\ a_{\alpha\beta}n^\alpha\{(\delta^i_n u+\gamma^i_{\nu 0}((u_0))(A_0^{\nu\beta\lambda\sigma}E^0_{\lambda\sigma}(u_0)+\lambda a^{\nu\beta}E^0_{33}(u_0)) \\ +(\delta^i_3+\gamma^i_{30}(u_0))2\mu a^{\beta\mu}E^0_{3\mu}(u_0)\}=h^i, \quad \text{on } \gamma_1. \end{cases} \tag{6.48}$$

The proof is completed. □

Theorem 6.2. *Assume that solution u of nonlinearly elastic operators defined by* (6.11) *and right hand term f can be made Taylor expansion with respect to* ξ (6.19)*. Then the first Piola-Kirchhoff stress*

$$\sigma^{ij}(u)=(\delta^i_k+\nabla_k u^i)A^{kjlm}E_{lm}(u)$$

defined by (6.2) *on* $\Im$ *under the S-coordinates can be made Taylor expansion*

$$\sigma^{ij}(u)=\sigma^{ij}_0(u_0)+\sigma^{ij}_1(u_0,u_1)\xi+\sigma^{ij}_2(u_0,u_1,u_2)\xi^2+\cdots, \tag{6.49}$$

where

$$\begin{cases} \sigma^{\alpha\beta}_0(u_0)=(\delta^\alpha_\lambda+\gamma^\alpha_{\lambda 0}(u_0))\{A_0^{\lambda\beta\nu\mu}E^0_{\nu\mu}(u_0)+\lambda a^{\lambda\beta}E^0_{33}(u_0)\}(\delta^\alpha_3+\gamma^\alpha_{30}(u_0))2\mu a^{\beta\mu}E^0_{3\mu}(u_0), \\ \sigma^{\alpha 3}_0(u_0)=2\mu(\delta^\alpha_\lambda+\gamma^\alpha_{\lambda 0}(u_0))a^{\lambda\sigma}E^0_{3\sigma}(u_0)+\lambda(\delta^\alpha_3+\gamma^\alpha_{30}(u_0))a^{\nu\mu}E^0_{\nu\mu}(u_0), \\ \sigma^{33}_0(u_0)=2\mu(\delta^3_\lambda+\gamma^3_{\lambda 0}(u_0))a^{\lambda\mu}E^0_{3\mu}(u_0)+\lambda(1+u^3_1)E^0_{33}(u_0), \end{cases} \tag{6.50}$$

$$\begin{cases} \sigma^{\alpha\beta}_1(u_0,u_1)=(\delta^\alpha_\lambda+\gamma^\alpha_{\lambda 0}(u_0))\{A_0^{\lambda\beta\nu\mu}E^1_{\nu\mu}(u_0,u_1)+A_1^{\lambda\beta\nu\mu}E^0_{\nu\mu}(u_0) \\ \quad +\lambda(a^{\lambda\beta}E^1_{33}(u_0,u_1)+2b^{\lambda\beta}E^0_{33}(u_0))\}+\gamma^\alpha_{\lambda 1}(u_0,u_1)\{A_0^{\lambda\beta\nu\mu}E^0_{\nu\mu}(u_0)+\lambda a^{\lambda\beta}E^0_{33}(u_0)\} \\ \quad +2\mu(\delta^\alpha_3+\gamma^\alpha_{30}(u_0))(a^{\beta\mu}E^1_{3\mu}(u_0)+2b^{\beta\mu}E^0_{3\mu}(u_0))+2\mu\gamma^\alpha_{31}(u_0,u_1)a^{\beta\mu}E^0_{3\mu}(u_0), \\ \sigma^{\alpha 3}_1(u_0,u_1)=2\mu(\delta^\alpha_\lambda+\gamma^\alpha_{\lambda 0}(u_0))(a^{\lambda\sigma}E^1_{3\sigma}(u_0,u_1)+2b^{\lambda\sigma}E^0_{3\sigma}(u_0))+2\mu\gamma^\alpha_{\lambda 1}(u_0,u_1)a^{\lambda\sigma}E^0_{3\sigma}(u_0) \\ \quad +\lambda(\delta^\alpha_3+\gamma^\alpha_{30}(u_0))(a^{\nu\mu}E^1_{\nu\mu}(u_0,u_1)+2b^{\nu\mu}E^0_{\nu\mu}(u_0))+\lambda\gamma^\alpha_{31}(u_0,u_1)a^{\nu\mu}E^0_{\nu\mu}(u_0), \\ \sigma^{33}_1(u_0,u_1)=2\mu(\delta^3_\lambda+\gamma^3_{\lambda 0}(u_0))\{a^{\lambda\mu}E^1_{3\mu}(u_0,u_1)+2b^{\mu\mu}E^0_{3\mu}(u_0)) \\ \quad +2\mu\gamma^3_{\lambda 1}(u_0,u_1)a^{\lambda\mu}E^0_{3\mu}(u_0)+\lambda(1+u^3_1)E^1_{33}(u_0,u_1)+2u^3_2E^0_{33}(u_0), \end{cases} \tag{6.51}$$

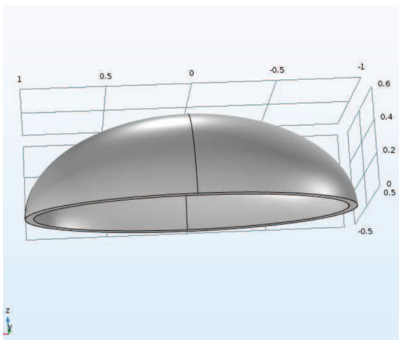
$$
\left\{
\begin{aligned}
&\sigma_2^{\alpha\beta}(u_0,u_1,u_2)=(\delta_\lambda^\alpha+\gamma_{\lambda 0}^\alpha(u_0))\{A_0^{\lambda\beta\nu\mu}E_{\nu\mu}^2(u_0,u_1,u_2)+A_1^{\lambda\beta\nu\mu}E_{\nu\mu}^1(u_0,u_1)\\
&\quad +A_2^{\lambda\beta\nu\mu}E_{\nu\mu}^0(u_0)+\lambda(a^{\lambda\beta}E_{33}^2(u_0,u_1,u_2)+2b^{\lambda\beta}E_{33}^1(u_0,u_1))+3c^{\lambda\beta}E_{33}^0(u_0))\}\\
&\quad +\gamma_{\lambda 1}^\alpha(u_0,u_1)\{A_0^{\lambda\beta\nu\mu}E_{\nu\mu}^1(u_0,u1)+A_1^{\lambda\beta\nu\mu}E_{\nu\mu}^0(u_0)+\lambda(a^{\lambda\beta}E_{33}^1(u_0,u_1)\\
&\quad +2b^{\lambda\beta}E_{33}^0(u_0))\}+\gamma_{\lambda 2}^\alpha(u_0,u_1,u_2)\{A_0^{\lambda\beta\nu\mu}E_{\nu\mu}^0(u_0)+\lambda a^{\lambda\beta}E_{33}^0(u_0)\}\\
&\quad +2\mu(\delta_3^\alpha+\gamma_{30}^\alpha(u_0))(a^{\beta\mu}E_{3\mu}^2(u_0)+2b^{\beta\mu}E_{3\mu}^1(u_0,u_1)+3c^{\beta\mu}E_{3\mu}^0(u_0))\\
&\quad +2\mu\gamma_{31}^\alpha(u_0,u_1)(a^{\beta\mu}E_{3\mu}^1(u_0,u_1)+2b^{\beta\mu}E_{3\mu}^0(u_0))+2\mu\gamma_{32}^\alpha(u_0,u_1)a^{\beta\mu}E_{3\mu}^0(u_0),\\
&\sigma_2^{\alpha 3}(u_0,u_1,u_2)=2\mu(\delta_\lambda^\alpha+\gamma_{\lambda 0}^\alpha(u_0))\{a^{\lambda\sigma}E_{3\sigma}^2(u_0,u_1,u_2)+2b^{\lambda\sigma}E_{3\sigma}^1(u_0,u_1)+3c^{\lambda\sigma}E_{3\sigma}^0(u_0)\}\\
&\quad +2\mu\gamma_{\lambda 1}^\alpha(u_0,u_1)\{a^{\lambda\sigma}E_{3\sigma}^1(u_0,u_1)+2b^{\lambda\sigma}E_{3\sigma}^0(u_0)\}+2\mu\gamma_{\lambda 2}^\alpha(u_0,u_1,u_2)a^{\lambda\sigma}E_{3\sigma}^0(u_0)\\
&\quad +\lambda(\delta_3^\alpha+\gamma_{30}^\alpha(u_0))(a^{\nu\mu}E_{\nu\mu}^2(u_0,u_1,u_2)+2b^{\nu\mu}E_{\nu\mu}^1(u_0,u_1)+3c^{\nu\mu}E_{\nu\mu}^0(u_0))\\
&\quad +\lambda\gamma_{31}^\alpha(u_0,u_1)(a^{\nu\mu}E_{\nu\mu}^1(u_0,u_1)+2b^{\nu\mu}E_{\nu\mu}^0(u_0))+\lambda\gamma_{32}^\alpha(u_0,u_1,u_2)(a^{\nu\mu}E_{\nu\mu}^0(u_0)),\\
&\sigma_2^{33}(u_0,u_1,u_2)=2\mu(\delta_\lambda^3+\gamma_{\lambda 0}^3(u_0))\{a^{\lambda\mu}E_{3\mu}^2(u_0,u_1,u_2)+2b^{\mu\mu}E_{3\mu}^1(u_0,u_1)+3c^{\mu\mu}E_{3\mu}^0(u_0)\}\\
&\quad +2\mu\gamma_{\lambda 1}^3(u_0,u_1)\{a^{\lambda\mu}E_{3\mu}^1(u_0,u_1)+2b^{\lambda\mu}E_{3\mu}^0(u_0)\}+2\mu\gamma_{\lambda 2}^3(u_0,u_1,u_2)a^{\lambda\mu}E_{3\mu}^0(u_0)\\
&\quad +\lambda(1+u_1^3)E_{33}^2(u_0,u_1)+2u_2^3E_{33}^1(u_0).
\end{aligned}
\right.
\tag{6.52}
$$

Proof. As well known that according to (6.2) the first Piola Kirchhoff stress tensor are given by

$$
\begin{aligned}
&\sigma^{ij}(u)=(\delta_k^i+\nabla_k u^i)A^{kjlm}E_{lm}(u),\\
&\sigma^{\alpha\beta}(u)=(\delta_k^\alpha+\nabla_k u^\alpha)A^{k\beta lm}E_{lm}(u)=(\delta_\lambda^\alpha+\nabla_\lambda u^\alpha)A^{\lambda\beta lm}E_{lm}(u)+(\delta_3^\alpha+\nabla_3 u^\alpha)A^{3\beta lm}E_{lm}(u)\\
&\qquad =(\delta_\lambda^\alpha+\nabla_\lambda u^\alpha)\{A^{\lambda\beta\lambda\sigma}E_{\lambda\sigma}(u)+A^{\lambda\beta 33}E_{33}(u)+(\delta_3^\alpha+\nabla_3 u^\alpha)\{A^{3\beta 3\sigma}E_{3\sigma}(u)\\
&\qquad\qquad +A^{3\beta\sigma 3}E_{\sigma 3}(u)\}\\
&\qquad =(\delta_\lambda^\alpha+\nabla_\lambda u^\alpha)\{A^{\lambda\beta\lambda\sigma}E_{\lambda\sigma}(u)+\lambda g^{\lambda\beta}E_{33}(u)+(\delta_3^\alpha+\nabla_3 u^\alpha)2\mu g^{\beta\sigma}E_{\sigma 3}(u)\}.
\end{aligned}
$$

Then using (2.57), (6.51) and (6.52) we can obtain Taylor expansion for σ^{ij}. By similar manner we also can obtain for other expansions. The proof is completed. ☐

7 An Example: Hemi-ellipsoid shell



Let us consider the hemi ellipsoid shell. As well known that parametric equation of the ellipsoid be given by

$$\begin{cases} \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \\ x = \alpha\cos\varphi\cos\theta, \quad y = \beta\sin\varphi\cos\theta, \quad z = \gamma\sin\theta, \\ 0 < \gamma < \beta < \alpha, \qquad \alpha,\beta,\gamma = \text{constants}, \\ \omega := \{0 \le \varphi \le 2\pi, 0 \le \theta \le \pi/2\}, \end{cases} \tag{7.1}$$

where $(\mathbf{i},\mathbf{j},\mathbf{k})$ are Cartesian basis, (φ,θ) are the parameters and $(x^1 = \varphi, x^2 = \theta)$ are called Guassian coordinates of ellipsoid. The base vectors on the ellipsoid

$$\begin{cases} \mathbf{e}_1 = \partial_\varphi \mathbf{r} = -\alpha\sin\varphi\cos\theta\mathbf{i} + \beta\cos\varphi\cos\theta\mathbf{j}, \\ \mathbf{e}_2 = \partial_\theta \mathbf{r} = -\alpha\cos\varphi\sin\theta\mathbf{i} - \beta\sin\varphi\sin\theta\mathbf{j} + \gamma\cos\theta\mathbf{k}, \\ \mathbf{n} = \frac{1}{\sqrt{a}}\mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{\sqrt{a}}[\beta\gamma\cos\varphi\cos^2\theta\mathbf{i} + \alpha\gamma\sin\varphi\cos^2\theta\mathbf{j} + \alpha\beta\sin\theta\cos\theta\mathbf{k}]. \end{cases} \tag{7.2}$$

The metric tensor of the ellipsoid is given by

$$\begin{cases} a_{\alpha\beta} = \mathbf{e}_\alpha \mathbf{e}_\beta, \\ a_{11} = (\alpha^2\sin^2\varphi + \beta^2\cos^2\varphi)\cos^2\theta, \\ a_{22} = (\alpha^2\cos^2\varphi + \beta^2\sin^2\varphi)\sin^2\theta + \gamma^2\cos^2\theta, \\ a_{12} = \frac{\alpha^2+\beta^2}{4}\sin 2\varphi\sin 2\theta, \\ a = \det(a_{\alpha\beta}) = \alpha^2\beta^2(\sin^4\varphi - \frac{1}{8}\sin^2 2\varphi\sin^2 2\theta + \cos^4\varphi) \\ \qquad + \gamma^2(\alpha^2\sin^2\varphi + \beta^2\cos^2\varphi)\cos^4\theta, \quad a \neq 0, \ \forall(\varphi,\theta) \in \omega, \\ a^{11} = \frac{a_{22}}{a}, \quad a^{22} = \frac{a_{11}}{a}, \quad a^{12} = a^{21} = -\frac{a_{12}}{a}. \end{cases} \tag{7.3}$$

Owing to

$$\begin{cases} \partial_\varphi\partial_\varphi \mathbf{r} = -\alpha\cos\varphi\cos\theta\mathbf{i} - \beta\sin\varphi\cos\theta\mathbf{j}, \\ \partial_\varphi\partial_\theta \mathbf{r} = \alpha\sin\varphi\sin\theta\mathbf{i} - \beta\cos\varphi\sin\theta\mathbf{j}, \\ \partial_\theta\partial_\theta \mathbf{r} = -\alpha\cos\varphi\cos\theta\mathbf{i} - \beta\sin\varphi\cos\theta\mathbf{j} - \gamma\sin\theta\mathbf{k}, \end{cases} \tag{7.4}$$

the coefficients of second fundamental form, i.e. curvature tensor of ellipsoid

$$\begin{cases} b_{11} = \frac{1}{\sqrt{a}} \begin{vmatrix} x_{\varphi\varphi} & y_{\varphi\varphi} & z_{\varphi\varphi} \\ x_\varphi & y_\varphi & z_\varphi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = -\frac{\alpha\beta\gamma}{\sqrt{a}}\cos^3\theta, \\ b_{12} = b_{21} = \frac{1}{\sqrt{a}} \begin{vmatrix} x_{\varphi\theta} & y_{\varphi\theta} & z_{\varphi\theta} \\ x_\varphi & y_\varphi & z_\varphi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = 0, \\ b_{22} = \frac{1}{\sqrt{a}} \begin{vmatrix} x_{\theta\theta} & y_{\theta\theta} & z_{\theta\theta} \\ x_\varphi & y_\varphi & z_\varphi \\ x_\theta & y_\theta & z_\theta \end{vmatrix} = -\frac{\alpha\beta\gamma}{\sqrt{a}}\cos\theta. \end{cases} \tag{7.5}$$

Consequently, curvature tensor, mean curvature and Gaussian curvature are given by

$$\begin{cases} b=\det b_{\alpha\beta}=\frac{1}{a}(\alpha\beta\gamma)^2\cos^4\theta, \\ K=\frac{b}{a}=(\frac{\alpha\beta\gamma}{a})^2\cos^4\theta, \\ H=a^{\alpha\beta}b_{\alpha\beta}=\frac{1}{a}(a_{22}b_{11}+a_{11}b_{22})=-\frac{\alpha\beta\gamma}{a\sqrt{a}}\cos^3\theta\{\alpha^2(\sin^2\theta\cos^2\varphi+\sin^2\varphi) \\ \qquad +\beta^2(\sin^2\theta\sin^2\varphi+\cos^2\varphi)+\gamma^2\cos^2\theta\}. \end{cases} \tag{7.6}$$

Semi-Geodesic coordinate system based on ellipsoid $\Im$

Note that

$$x^2=\varphi, \quad x^1=\theta, \quad x^3=\xi.$$

The radial vector at any point in $\Re^3$

$$\mathbf{R}=\mathbf{r}+\xi\mathbf{n}.$$

We remainder to give the covariant derivatives of the velocity field, Laplace-Betrami operator and trac–Laplace operator. To do this we have to give the first and second kind of Christoffel symbols on the ellipsoid $\Im$ as a two dimensional manifolds

$$\overset{*}{\Gamma}_{\alpha\beta,\lambda}=\mathbf{r}_{\alpha\beta}\mathbf{r}_{\lambda}, \quad \overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}=a^{\lambda\sigma}\overset{*}{\Gamma}_{\alpha\beta,\lambda},$$

$$\begin{aligned} &\overset{*}{\Gamma}_{11,1}=\tfrac{1}{2}(\alpha^2-\beta^2)\sin^2\varphi\cos^2\theta, \\ &\overset{*}{\Gamma}_{11,2}=\tfrac{1}{2}(\alpha^2\cos^2\varphi+\beta^2\sin^2\varphi)\sin 2\theta, \\ &\overset{*}{\Gamma}_{12,1}=-\tfrac{1}{2}(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)\sin 2\theta, \quad \overset{*}{\Gamma}_{12,2}=\tfrac{\beta^2-\alpha^2}{2}\sin 2\varphi\sin^2\theta, \\ &\overset{*}{\Gamma}_{22,1}=\tfrac{1}{2}(\beta-\alpha^2)\sin 2\varphi\cos^2\theta, \quad \overset{*}{\Gamma}_{22,2}=\tfrac{1}{2}(\alpha^2 cos^2\varphi+\beta^2 sin^2\varphi-\gamma^2)\sin 2\theta, \end{aligned} \tag{7.7}$$

$$\begin{aligned} &\overset{*}{\Gamma}{}^1{}_{11}=\tfrac{1}{2}\tfrac{1}{a}\{\gamma^2(\alpha^2-\beta^2)\cos^2\theta-2\beta^2(\alpha^2\cos^2\varphi+\beta^2\sin^2\varphi)\sin^2\theta\}\sin 2\varphi\cos\theta, \\ &\overset{*}{\Gamma}{}^2{}_{11}=\tfrac{1}{a}\tfrac{1}{2}\sin 2\theta\cos^2\theta\{\tfrac{1}{2}\beta^4\sin^2 2\varphi+\alpha^2\beta^2(\sin^4\varphi\varphi+\cos^4\varphi)\}, \\ &\overset{*}{\Gamma}{}^1{}_{12}=\tfrac{\sin 2\theta}{2a}\{\tfrac{\alpha^4-\beta^4}{4}\sin^2 2\varphi\cos 2\theta+\alpha^2\beta^2(\sin^4\varphi+\cos^4\varphi)\sin^2\theta \\ &\qquad -\gamma^2(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)\cos^2\theta\}, \\ &\overset{*}{\Gamma}{}^2{}_{12}=\beta^2\tfrac{\sin^2 2\theta}{2a}\sin 2\varphi(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi), \\ &\overset{*}{\Gamma}{}^1{}_{22}=\tfrac{\sin 2\varphi\cos^2\theta}{2a}\{\gamma^2(\beta^2-\alpha^2\cos 2\theta)-2\alpha^2\sin^2\theta(\alpha^2\cos^2\varphi+\beta^3\sin^2\varphi)\}, \\ &\overset{*}{\Gamma}{}^2{}_{22}=\tfrac{\sin 2\theta\cos^2\theta}{2a}\{\tfrac{\alpha^4-\beta^4}{4}\sin^2 2\varphi+(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)(\alpha^2\cos^2\varphi+\beta^2\sin^2\varphi-\gamma^2)\}. \end{aligned} \tag{7.8}$$

Then covariant derivatives of vector $\mathbf{u}=u^\alpha\mathbf{e}_\alpha+u^3\mathbf{n}$ on the two dimensional manifold $\Im$

$$\overset{*}{\nabla}_\alpha u^\beta=\partial_\alpha u^\beta+\overset{*}{\Gamma}{}^\beta_{\alpha\lambda}u^\lambda,$$

$$\begin{cases}\overset{*}{\nabla}_1 u^1=\partial_\varphi u^1+\gamma^1_{11}u^1+\gamma^1_{12}u^2, \quad \overset{*}{\nabla}_1 u^2=\partial_\varphi u^2+\gamma^2_{11}u^1+\gamma^2_{12}u^2,\\ \overset{*}{\nabla}_2 u^1=\partial_\theta u^1+\gamma^1_{21}u^1+\gamma^1_{22}u^2, \quad \overset{*}{\nabla}_2 u^2=\partial_\varphi u^2+\gamma^2_{21}u^1+\gamma^2_{22}u^2,\\ \gamma^1_{11}:=\frac{\sin2\varphi\cos^2\theta}{2a}\{\gamma^2(\alpha^2-\beta^2)\cos^2\theta-2\beta^2(\alpha^2\cos^2\varphi+\beta^2\sin^2\varphi)\sin^2\theta\},\\ \gamma^1_{12}:=\frac{\sin2\theta}{2a}\{\frac{\alpha^4-\beta^4}{4}\sin^22\varphi\cos2\theta+\alpha^2\beta^2(\sin^4\varphi+\cos^4\varphi)\sin^2\theta\\ \qquad -\gamma^2(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)\cos^2\theta\},\\ \gamma^2_{11}:=\frac{1}{a}\frac{1}{2}\sin2\theta\cos^2\theta\{\frac{1}{2}\beta^4\sin^22\varphi+\alpha^2\beta^2(\sin^4\varphi\varphi+\cos^4\varphi)\},\\ \gamma^2_{12}:=\beta^2\frac{\sin^22\theta}{2a}\sin2\varphi(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi),\\ \gamma^1_{21}:=\frac{\sin2\theta}{2a}\{\frac{\alpha^4-\beta^4}{4}\sin^22\varphi\cos2\theta+\alpha^2\beta^2(\sin^4\varphi+\cos^4\varphi)\sin^2\theta\\ \qquad -\gamma^2(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)\cos^2\theta\},\\ \gamma^1_{22}=\frac{\sin2\varphi\cos^2\theta}{2a}\{\gamma^2(\beta^2-\alpha^2\cos2\theta)-2\alpha^2\sin^2\theta(\alpha^2\cos^2\varphi+\beta^3\sin^2\varphi)\},\\ \gamma^2_{21}:=\beta^2\frac{\sin^22\theta}{2a}\sin2\varphi(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi),\\ \gamma^2_{22}:=\frac{\sin2\theta\cos^2\theta}{2a}\{\frac{\alpha^4-\beta^4}{4}\sin^22\varphi+(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)(\alpha^2\cos^2\varphi+\beta^2\sin^2\varphi-\gamma^2)\},\end{cases} \tag{7.9}$$

$$\begin{cases}\overset{*}{\operatorname{div}}\,\mathbf{u}=\partial_\theta u^2+\partial_\varphi u^1+\frac{\partial\ln\sqrt{a}}{\partial\varphi}u^1+\frac{\partial\ln\sqrt{a}}{\partial\theta}u^2\\ \qquad =\partial_\theta u^2+\partial_\varphi u^1+d_1u^1+d_2u^2,\\ d_1=\frac{\partial\ln\sqrt{a}}{\partial\varphi}=\frac{\sin2\varphi}{2a}\{\gamma^2(\alpha^2-\beta^2)\cos^4\theta-2\alpha^2\beta^2\cos2\varphi(1+\frac{1}{4}\sin^22\varphi)\},\\ d_2=\frac{\partial\ln\sqrt{a}}{\partial\theta}=-\frac{\sin2\theta}{2a}\{\frac{1}{4}\alpha^2\beta^2\sin^22\varphi\cos2\theta+2\gamma^2(\alpha^2\sin^2\varphi+\beta^2\cos^2\varphi)\cos^2\theta.\end{cases} \tag{7.10}$$

As well known that displacement vector $u=(u^1,u^2,u^3)$ on middle surface of shell has three components, third component u^3 looked as scale function, $\overset{*}{\Delta}u^3$ is a Laplace-Betrami operator on $\Im$ which is given by

$$\overset{*}{\Delta}u^3=a^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^3=\frac{1}{\sqrt{a}}\frac{\partial}{\partial x^\lambda}(\sqrt{a}a^{\lambda\sigma}\frac{\partial u^3}{\partial x^\sigma}). \tag{7.11}$$

The trace- Laplace operatoe on $\Im$ is given by

$$\begin{aligned}&\overset{*}{\Delta}u^\alpha=a^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma u^\alpha=a^{\lambda\sigma}[\partial_\lambda\overset{*}{\nabla}_\sigma u^\alpha+\overset{*}{\Gamma}{}^\alpha_{\lambda\nu}\overset{*}{\nabla}_\sigma u^\nu-\overset{*}{\Gamma}{}^\nu_{\lambda\sigma}\overset{*}{\nabla}_\nu u^\alpha],\\ &\overset{*}{\Delta}u^\alpha=a^{\lambda\sigma}\frac{\partial^2u^\alpha}{\partial x^\lambda\partial x^\sigma}+A^{\alpha\tau}_\mu\frac{\partial u^\mu}{\partial x^\tau}+A^\alpha_\mu u^\mu,\\ &A^\alpha_\mu=a^{\lambda\sigma}(\partial_\lambda\overset{*}{\Gamma}{}^\alpha_{\sigma\mu}+\overset{*}{\Gamma}{}^\alpha_{\lambda\nu}\overset{*}{\Gamma}{}^\nu_{\mu\sigma}-\overset{*}{\Gamma}{}^\alpha_{\mu\nu}\overset{*}{\Gamma}{}^\nu_{\lambda\sigma}),\\ &A^{\alpha\tau}_\mu=a^{\lambda\sigma}[\overset{*}{\Gamma}{}^\alpha_{\sigma\mu}\delta^\tau_\lambda+\overset{*}{\Gamma}{}^\alpha_{\lambda\mu}\delta^\tau_\sigma-\overset{*}{\Gamma}{}^\tau_{\lambda\sigma}\delta^\alpha_\mu].\end{aligned} \tag{7.12}$$

Next let return to stationary equations (5.6) with boundary value

$$(u_0,u_1,u_2)|_{\partial\omega}=0,$$

and consider associate variational formulations for (u,u_1), find $(u_0^i,u_1^i,i=1,2,3)\in H_0^1(\omega)^3\times H_0^1(\omega)^3$, such that

$$\begin{cases} (\mathcal{K}_0^i(u_0),a_{ij}v^j)+(L_0^i(u_1),a_{ij}v^j)=0, & \forall v\in H_0^1(\omega)^3,\\ (\mathcal{K}_0^i(u_1),a_{ij}v^j)+(\mathcal{K}_1^i(u_0),a_{ij}v^j)+(L_1^i(u_1),a_{ij}v^j)=0, & \forall v\in H_0^1(\omega)^3, \end{cases} \tag{7.13}$$

where $H_0^1(\omega)$ is a Sobolev space. Let denote

$$\{a_{ij}\}=\{a_{\alpha\beta},\quad a_{\alpha 3}=a_{3\alpha}=0,\quad a_{33}=1\}. \tag{7.14}$$

Note that (5.13) shows that

$$\begin{cases} \mathcal{K}_0^\alpha(u_0)=-\mu\overset{*}{\Delta}u_0^\alpha-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0)-m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha 0}(0)u_0^k,\\ \mathcal{K}_1^\alpha(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^\alpha-2(\lambda+\mu)b^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\mathrm{div}}\,u_0-m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{\alpha 0}(1)u_0^k,\\ \mathcal{K}_0^3(u_0)=-\mu\overset{*}{\Delta}u_0^3+m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_0^k,\\ \mathcal{K}_1^3(u_0)=-2\mu b^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma u_0^3+m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{30}(1)u_0^k. \end{cases} \tag{7.15}$$

Note that integrals by applying the Gaussian theorem become

$$\begin{aligned}(\mathcal{K}_0^i(u_0),a_{ij}v^j)&=\int_\omega\mathcal{K}_0^i(u_0)a_{ij}v^j\sqrt{a}d\theta d\varphi=\int_\omega\{-\mu\overset{*}{\Delta}u_0^ia_{ij}v^j\\ &\quad-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta(\overset{*}{\mathrm{div}}\,u_0)a_{\alpha\lambda}v^\lambda\}\sqrt{d}d\theta d\varphi+\mu\int_\omega\{m_k^{i\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{i0}(0)u_0^k\}a_{ij}v^j\sqrt{a}d\theta d\varphi\\ &=\int_\omega\mu\{\mu a^{\lambda\sigma}a_{ij}\overset{*}{\nabla}_\sigma u_0^i\overset{*}{\nabla}_\lambda v^j+(\lambda+\mu)\overset{*}{\mathrm{div}}\,u_0\overset{*}{\mathrm{div}}\,v\}\sqrt{a}d\theta d\varphi-\int_\omega\{m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k\\ &\quad+m_k^{\alpha 0}(0)u_0^k\}a_{\alpha\beta}v^\beta\sqrt{a}d\theta d\varphi+\int_\omega\{m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_0^k\}v^3\sqrt{a}d\theta d\varphi.\end{aligned}$$

Let us denote

$$a(u_0,v):=\int_\omega\{\mu a^{\lambda\sigma}a_{ij}\overset{*}{\nabla}_\sigma u_0^i\overset{*}{\nabla}_\lambda v^j+(\lambda+\mu)\overset{*}{\mathrm{div}}\,u_0\overset{*}{\mathrm{div}}\,v\}\sqrt{a}d\theta d\varphi, \tag{7.16a}$$

$$\begin{aligned}c(u_0,v):=&-\int_\omega\{m_k^{\alpha\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{\alpha 0}(0)u_0^k\}a_{\alpha\lambda}v^\lambda\sqrt{a}d\theta d\varphi\\ &+\int_\omega\{m_k^{3\beta}(0)\overset{*}{\nabla}_\beta u_0^k+m_k^{30}(0)u_0^k\}v^3\sqrt{a}d\theta d\varphi.\end{aligned} \tag{7.16b}$$

By similar manner

$$\begin{aligned}&(\mathcal{K}_1^i(u_0),a_{ij}v^j)=\int_\omega \mathcal{K}_1^i(u_0)a_{ij}v^j\sqrt{a}d\theta d\varphi\\ =&2\mu(a_{\alpha\lambda}\overset{*}{\nabla}_\sigma u_0^\alpha,\overset{*}{\nabla}_\lambda(b^{\beta\sigma}v^\lambda))+\mu(a^{\sigma\sigma}\overset{*}{\nabla}_\sigma u_0^3,\overset{*}{\nabla}_\lambda v^3)\\ &+2(\lambda+\mu)(\overset{*}{\mathrm{div}}\,u_0,\overset{*}{\nabla}_\beta(b_\lambda^\beta v^\lambda))-(m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k+m_k^{\alpha0}(0)u_0^k,a_{\alpha\lambda}v^\lambda)\\ &+(m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{30}(1)u_0^k,v^3).\end{aligned}\tag{7.17}$$

In addition,

$$\left\{\begin{array}{l}a_1(u_0,v)=2\mu(a_{\alpha\beta}\overset{*}{\nabla}_\sigma u_0^\alpha,\overset{*}{\nabla}_\lambda(b^{\lambda\sigma}v^\beta))+\mu(a^{\sigma\sigma}\overset{*}{\nabla}_\sigma u_0^3,\overset{*}{\nabla}_\lambda v^3)+2(\lambda+\mu)(\overset{*}{\mathrm{div}}\,u_0,\overset{*}{\nabla}_\beta(b_\lambda^\beta v^\lambda)),\\ c_1(u_0,v)=-(m_k^{\alpha\beta}(1)\overset{*}{\nabla}_\beta u_0^k+m_k^{\alpha0}(0)u_0^k,a_{\alpha\lambda}v^\lambda)+(m_k^{3\beta}(1)\overset{*}{\nabla}_\beta u_0^k-m_k^{30}(1)u_0^k,v^3).\end{array}\right.\tag{7.18}$$

Therefore, we assert

$$\left\{\begin{array}{l}(\mathcal{K}_0^i(u_0),a_{ij}v^j)=a(u_0,v)+c(u_0,v),\\ (\mathcal{K}_1^i(u_0),a_{ij}v^j)=a_1(u_0,v)+c_1(u_0,v),\end{array}\right.\tag{7.19}$$

$$\left\{\begin{array}{ll}L_0^\alpha(u_1)=-m_\beta^{\alpha3}(0)u_1^\beta-(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3, & L_1^\alpha(u_1)=-m_\beta^{\alpha3}(1)u_1^\beta+2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3,\\ L_0^3(u_1)=m_k^{33}(0)u_1^k-(\lambda+\mu)\overset{*}{\mathrm{div}}\,u_1, & L_1^3(u_1)=m_k^{33}(1)u_1^k.\end{array}\right.\tag{7.20}$$

The variational problem (7.13) can be rewritten as, find $(u_0^i,u_1^i,i=1,2,3)\in H_0^1(\omega)^3\times H_0^1(\omega)^3$ such that

$$\left\{\begin{array}{ll}a(u_0,v)+c(u_0,v)+(L_0^i(u_1),a_{ij}v^j)=0, & \forall v\in H_0^1(\omega)^3,\\ a_1(u_0,v)+c_1(u_0,v)+(L_1^i(u_1),a_{ij}v^j)=0, & \forall v\in H_0^1(\omega)^3.\end{array}\right.\tag{7.21}$$

Taking (7.5), (7.15) and Remark 5.1 into account, simple calculations show that

$$\left\{\begin{array}{l}m_\nu^{\alpha\beta}(0)=0,\quad m_3^{\alpha\beta}(o)=2\mu b^{\alpha\beta}+2(\lambda+\mu)Ha^{\alpha\beta},\\ m_\nu^{\alpha0}(0)=\mu K\delta_\nu^\alpha,\quad m_3^{\alpha0}(0)=(\lambda+3\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta H,\\ m_\nu^{3\beta}(0)=-2\mu b_\nu^\beta,\quad m_3^{3\beta}(0)=0,\\ m_\nu^{30}(0)=2\lambda\overset{*}{\nabla}_\nu H,\quad m_3^{30}(0)=4\lambda H^2+2\mu(4H^2-2K),\\ m_\nu^{\alpha\beta}(1)=\mu\overset{*}{\nabla}_\nu b^{\alpha\beta}+((\lambda+\mu)a^{\alpha\beta}\delta_\nu^\lambda-2\mu a^{\beta\lambda}\delta_\nu^\alpha)\overset{*}{\nabla}_\lambda H,\quad m_3^{\alpha\beta}(1)=2(\lambda+4\mu)c^{\alpha\beta},\\ m_\nu^{\alpha0}(1)=\mu(\overset{*}{\Delta}b_\nu^\alpha-2HK\delta_\nu^\alpha)+(\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\nu H),\\ m_3^{\alpha0}(1)=\lambda(2a^{\alpha\lambda}\overset{*}{\nabla}_\lambda(2H^2-K)+a^{\lambda\beta}\overset{*}{\nabla}_\lambda c_\beta^\alpha+8Ha^{\alpha\lambda}\overset{*}{\nabla}_\lambda H)\\ \qquad\qquad+(\lambda+3\mu)\{4(b^{\alpha\beta}-2Ha^{\alpha\beta})\overset{*}{\nabla}_\beta H+2a^{\alpha\beta}\overset{*}{\nabla}_\beta K\},\\ m_\nu^{3\beta}(1)=-2\mu c_\nu^\beta,\quad m_3^{3\beta}(1)=-2\mu a^{\beta\lambda}\overset{*}{\nabla}_\lambda H,\\ m_\nu^{30}(1)=2(\lambda+\mu)\overset{*}{\nabla}_\nu(K-2H^2),\quad m_3^{30}(1)=(4\lambda+6\mu)H(4H^2-3K),\\ m_\nu^{33}(0)=0,\quad m_3^{33}(0)=(\lambda+2\mu)2H,\\ m_\nu^{33}(1)=(\lambda+\mu)\overset{*}{\nabla}_\nu H,\quad m_3^{33}(1)=(\lambda+2\mu)(4H^2-2K)-2\mu K,\\ m_\nu^{\alpha3}(0)=-2\mu b_\nu^\alpha,\quad m_3^{\alpha3}(0)=m_3^{\alpha3}(1)=0,\quad m_\nu^{\alpha3}(1)=2\mu(K\delta_\nu^\alpha-2Hb_\nu^\alpha).\end{array}\right.\tag{7.22}$$

Substituting (7.22) into (7.18) and (7.20) leads to

$$\begin{cases} c(u_0,v) = -\int_\omega \{\{((2\mu b_\lambda^\beta + 2(\lambda+\mu)H\delta_\lambda^\beta)\overset{*}{\nabla}_\beta u_0^3 + \mu K a_{\alpha\lambda} u_0^\alpha \\ \qquad + (\lambda+3\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta H u_0^3)v^\lambda + (2\mu b_\lambda^\beta \overset{*}{\nabla}_\beta u_0^\lambda - 2\lambda \overset{*}{\nabla}_\lambda H u_0^\lambda \\ \qquad + (2\mu K - (2\mu+4\lambda)H^2)u_0^3)v^3\}\sqrt{a}d\theta d\varphi, \\ L_0^\alpha(u_1) = 2\mu b_\beta^\alpha u_1^\beta - (\lambda+\mu)a^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3, \\ L_0^3(u_1) = 2(\lambda+2\mu)H u_1^3 - (\lambda+\mu)\overset{*}{\mathrm{div}}\, u_1, \end{cases} \tag{7.23}$$

$$\begin{cases} c_1(u_0,v) = \int_\omega \{\{-(\mu \overset{*}{\nabla}_\nu b_\sigma^\beta + ((\lambda+\mu)\delta_\sigma^\beta \delta_\nu^\lambda - 2\mu a^{\beta\lambda} a_{\nu\sigma})\overset{*}{\nabla}_\lambda H)\overset{*}{\nabla}_\beta u_0^\nu \\ \quad -2(\lambda+\mu)c_\sigma^\beta \overset{*}{\nabla}_\beta u_0^3 - (\mu(\overset{*}{\Delta} b_{\sigma\nu} - 2HK a_{\sigma\nu} + (\lambda+\mu)\overset{*}{\nabla}_\sigma \overset{*}{\nabla}_\nu H)u_0^\nu \\ \quad + (\lambda(2\overset{*}{\nabla}_\sigma(2H^2-K) + \overset{*}{\nabla}_\sigma c_\alpha^\alpha + 8H\overset{*}{\nabla}_\sigma H) + (\lambda+3\mu)\{4(b_\sigma^\beta - 2H\delta_\sigma^\beta)\overset{*}{\nabla}_\beta H \\ \quad + 2\overset{*}{\nabla}_\sigma K\})u_0^3\}v^\sigma + \{-2\mu c_\nu^\beta \overset{*}{\nabla}_\beta u_0^\nu - 2\mu a^{\beta\lambda} H \overset{*}{\nabla}_\beta u_0^3 \\ \quad + 2(\lambda+\mu)\overset{*}{\nabla}_\nu (K-2H^2)u_0^\nu + (4\lambda+6\mu)H(4H^2-3K)u_0^3\}v^3\}\sqrt{a}d\theta d\varphi, \\ L_1^\alpha(u_1) = 2\mu((K\delta_\beta^\alpha - 2Hb_\beta^\alpha)u_1^\beta + 2b^{\alpha\beta}\overset{*}{\nabla}_\beta u_1^3. \\ L_1^3(u_1) = (\lambda+\mu)\overset{*}{\nabla}_\beta H u_1^\beta + (\lambda+2\mu)(4H^2-2K) - 2\mu K)u_1^3. \end{cases} \tag{7.24}$$

The bilinear form of the variational problem (7.21) is given.

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