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# A Note on Discrete Einstein Metrics

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**Abstract.** In this note, we prove that the space of all admissible piecewise linear metrics parameterized by the square of length on a triangulated manifold is a convex cone. We further study Regge's Einstein-Hilbert action and give a more reasonable definition of discrete Einstein metric than the former version. Finally, we introduce a discrete Ricci flow for three dimensional triangulated manifolds, which is closely related to the existence of discrete Einstein metrics.

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**Key words**: Discrete Einstein metric, Discrete Ricci flow.

## 1 The space of piecewise linear metrics

Consider an $n$ dimensional compact manifold $M$ with a triangulation $\mathcal{T}$. The triangulation is written as $\mathcal{T}=\{\mathcal{T}_0,\mathcal{T}_1,\cdots,\mathcal{T}_n\}$, where $\mathcal{T}_i$ $(0\le i\le n)$ represents the set of all $i$ dimensional simplices. A piecewise linear metric is a map $l:\mathcal{T}_1\to(0,+\infty)$ making each simplex an Euclidean simplex.

There are two disadvantages to think of $l$ as the analogue of smooth Riemannian metric tensor $g$. For one thing, we know that $\mathfrak{M}_{\mathcal{T}}$, the space of all admissible piecewise linear metrics, is not convex (although it is a simply connected open set). For another, the scaling property of $l$ is not good enough. If the smooth Riemannian metric tensor $g$ scales to $cg$ in the smooth manifold $M^n$, then the length $l(\gamma)$ of a curve $\gamma:[0,1]\to M$ scales to $\sqrt{c}l(\gamma)$.

If we take $l^2$ as the direct analogue of metric tensor $g$, both the above two disadvantages can be overcome. The idea of considering the square of $l$, not $l$ itself, as an analogue of smooth Riemannian metric tensor comes naturally from the former work by the first author and Xu [4], where the idea has been used for piecewise linear manifolds with circle or sphere packing metrics. Firstly, we have

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**Theorem 1.1.** *For a manifold $M^n$ with triangulation $\mathcal{T}$, denote $g_{ij}=l_{ij}^2$ for each adjacent edge $i\sim j$. Then $\mathfrak{M}_{\mathcal{T}}^2$, the space of all admissible piecewise linear metrics parameterized by $g_{ij}$, is a nonempty connected open convex cone.*

*Proof.* Rivin [11] first observed this fact for a single simplex case. Gu *et al.* [8] proved this fact for $n=2$ by direct calculation. The proof here follows from Rivin's idea. For an $n$-simplex $\Delta$ embedded in the Euclidean space, we label all vertices as $v_0,v_1,\cdots,v_n$ and all $\frac{n(n+1)}{2}$ edges as $l_{01},\cdots,l_{n-1n}$. For brevity, let $n^*=\frac{n(n+1)}{2}$, then we need to show

$$\mathfrak{M}_\Delta^2=\left\{\left(l_{01}^2,\cdots,l_{n-1n}^2\right)\in\mathbb{R}^{n^*}\,\middle|\,l_{01},\cdots,l_{n-1n}\text{ are edges of some Euclidean }n\text{-simplex}\right\}$$

is convex. Construct a map from $\mathfrak{M}_\Delta^2$ to the set of all symmetric $n\times n$ matrices, which transforms $(l_{01}^2,\cdots,l_{n-1n}^2)$ to

$$\frac{1}{2}\begin{pmatrix} 2l_{01}^2 & l_{01}^2+l_{02}^2-l_{12}^2 & l_{01}^2+l_{03}^2-l_{13} & \cdots & l_{01}^2+l_{0n}^2-l_{1n}^2 \\ * & 2l_{02}^2 & l_{02}^2+l_{03}^2-l_{23}^2 & \cdots & l_{02}^2+l_{0n}^2-l_{2n}^2 \\ * & * & 2l_{03}^2 & \cdots & l_{03}^2+l_{0n}^2-l_{3n}^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & 2l_{0n}^2 \end{pmatrix}.$$

The above matrix is the Gram matrix of $n$ linear independent vectors $\vec{01},\vec{02},\cdots,\vec{0n}$ and hence is positive definite. Obviously, the map is injective and surjective. Note that the set of all positive definite $n\times n$ matrices is a nonempty open convex subset of $\mathbb{R}^{n^*}$. Thus $\mathfrak{M}_\Delta^2$ is also a nonempty open convex subset of $\mathbb{R}^{n^*}$.

Next we prove the theorem for general triangulations. Assuming all edges are labeled in turn as $e_1,\cdots,e_m$, where $m=|\mathcal{T}_1|$. Then for any $n$-simplex $\Delta=(v_0,\cdots,v_n)$ with edges $e_{i_1},\ldots,e_{i_{n^*}}$, $(i_1,\cdots,i_{n^*}\in\{1,2,\cdots,m\})$, denote

$$\widetilde{\mathfrak{M}_\Delta^2}=\left\{\left(\cdots,l_{i_1}^2,\cdots,l_{i_2}^2,\cdots,l_{i_{n^*}}^2,\cdots\right)\,\middle|\,\left(l_{i_1}^2,\cdots,l_{i_{n^*}}^2\right)\in\mathfrak{M}_\Delta^2\right\}=\mathfrak{M}_\Delta^2\times\mathbb{R}^{m-n^*},$$

we have

$$\mathfrak{M}_{\mathcal{T}}^2=\bigcap_{\Delta\in\mathcal{T}_n}\widetilde{\mathfrak{M}_\Delta^2}.$$

This implies that $\mathfrak{M}_{\mathcal{T}}^2$ is a nonempty connected open convex cone of $\mathbb{R}^m$. □

## 2 An interpretation of Regge's Einstein-Hilbert action

Now we deal with three dimensional case. We give an interpretation to three dimensional Regge's Einstein-Hilbert action. The idea here is natural when taking $l^2$ as the analog of