Stable Semi-Implicit Monolithic Scheme for Interaction Between Incompressible Neo-hookean Structure and Navier-Stokes Fluid

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Abstract. We present a monolithic algorithm for solving fluid-structure interaction. The Updated Lagrangian framework is used for the incompressible neo-hookean structure and Arbitrary Lagrangian Eulerian coordinate is employed for the Navier-Stokes equations. The algorithm uses a global mesh for the fluid-structure domain which is compatible with the fluid-structure interface. At each time step, a non-linear system is solved in a domain corresponding to the precedent time step. It is a semi-implicit algorithm in the sense that the velocity, the pressure are computed implicitly, but the domain is updated explicitly. Using one velocity field defined over the fluid-structure mesh, and globally continuous finite elements, the continuity of the velocity at the interface is automatically verified. The equation of the continuity of the stress at the interface does not appear in this formulation due to action and reaction principle. The stability in time is proved. A second algorithm is introduced where at each time step, only a linear system is solved in order to find the velocity and the pressure. Numerical experiments are presented.

1   Introduction

We can solve numerically fluid-structure interaction problems by partitioned procedures or monolithic algorithms and a large literature exists in this subject. In some monolithic formulations [12, 14], two non-overlapping meshes are used for fluid and structure domains and the boundary conditions at the fluid-structure interface appear as equations in the global system.

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Other monolithic formulations use Eulerian mesh which does not fit to the fluid-structure interface. In [6, 7, 22], an Eulerian approach is used for the fluid as well as for the structure and the interface is captured with Initial Point Set. Extended Finite Element Method (XFEM) was used in [9]. In [2, 3], fictitious domain method with Lagrange multiplier was employed where the structure is assumed to be visco-elastic. This assumption is used in [26], too. Explicit schemes for fluid-structure interaction problems using Nitsche's method and RobinRobin coupling are discussed in [4]. The stability is proved under a hyperbolic type CFL condition.

In [11, 16, 18-20, 24] one global mesh for fluid-structure domain which fits to the interface is used. In [11, 20], an Eulerian formulation derived from Cayley-Hamilton theorem is used for the incompressible Mooney-Rivlin structure. The fluid equations are solved by the Characteristics-Galerkin method. The fluid-structure equations are written in the unknown Eulerian domain and fixed-point iterations are performed at each time step. The authors prove the time stability of the scheme.

In [16] where the structure is linear elastic and in [18] where the structure is compressible neo-hookean, the Updated Lagrangian coordinates are used for the structure combined with the Arbitrary Lagrangian Eulerian framework for the fluid equations. Using one velocity field defined over the fluid-structure mesh, and globally continuous finite elements, the continuity of the velocity at the interface is automatically verified. The equation of the continuity of the stress at the interface does not appear in this formulation due to action and reaction principle. Another advantage of this approach is that the fluid-structure equations are written in the known domain obtained at the precedent time step. It is a semi-implicit algorithm in the sense that the fluid and structure unknowns are computed implicitly, but the time advancing scheme for the domain is explicitly.

We follow this idea in the present paper, but for incompressible neo-hookean structure with a different computation of the mesh velocity. The structure equations in Updated Lagrangian coordinates is well posed. The stability in time of the monolithic algorithm is proved. The system to be solved at each time step can be easely linearised. Numerical results are presented in the last section.

2   Fluid-structure interaction problem

Without restriction of generality, we consider the geometrical configuration of the benchmark FSI3 from [25]. The results presented in this paper, including the stability analysis, hold for different 2D geometrical configurations, for example the flow in a channel with elastic wall, [16]. For 3D configurations, for example blood flow in artery [19], the stability result for the structure remains true but the stabilization term added in the fluid scheme has to be adapted accordingly.

We consider a rectangular flexible structure of length ℓ and thickness h immersed in an incompressible fluid occupying the rectangular domain $(0,L) \times (0,H)$. The rectangular structure is attached to a fixed body of boundaries: the segment $[DE]$, which is the left

side of the flexible structure, and Σ_5 the circular curve of center (x_C, y_C), radius r, see Figure 1.

We denote by $\Sigma_1 = \{0\} \times [0, H]$, $\Sigma_3 = \{L\} \times [0, H]$ the left and the right vertical boundaries of the fluid domain and by $\Sigma_2 = [0, L] \times \{0\}$, $\Sigma_4 = [0, L] \times \{H\}$ the bottom and the top boundaries, respectively.

The interface between the fluid and the flexible structure is denoted by Γ_0 at the initial time and by Γ_t at time instant $t > 0$. To resume, the fluid domain denoted by Ω_t^F is bounded externally by $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$ and internally by $\Sigma_5 \cup \Gamma_t$ and the structure domain denoted by Ω_t^S is bounded by the $\Gamma_D \cup \Gamma_t$ where $\Gamma_D = [DE]$.
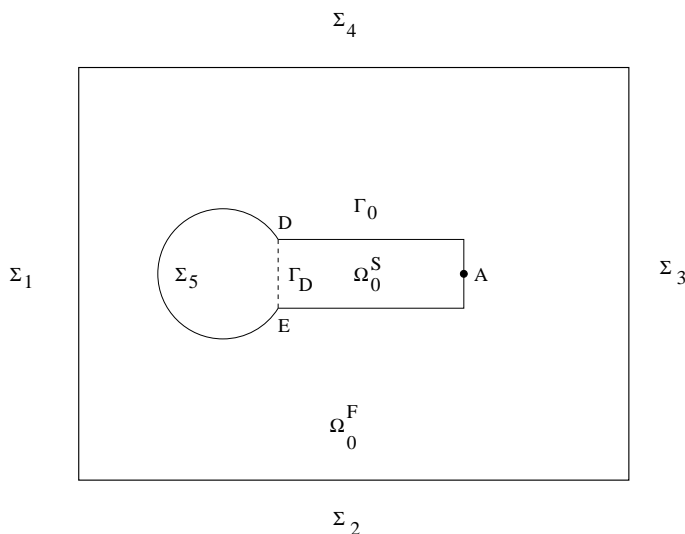

Figure 1: Initial geometrical configuration.

The displacement of the structure is denoted by $\mathbf{U}^S = (U_1^S, U_2^S) : \Omega_0^S \times [0, T] \rightarrow \mathbb{R}^2$ using Lagrangian coordinates. A particle of the structure of position $\mathbf{X}$ in the initial domain Ω_0^S, will moves to the position $\mathbf{x} = \mathbf{X} + \mathbf{U}^S(\mathbf{X}, t)$ in the deformed domain Ω_t^S. We use the notations $\mathbf{F}(\mathbf{X}, t) = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^S(\mathbf{X}, t)$ for the gradient of the deformation, where $\mathbf{I}$ is the unity matrix, $J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t)$ and

$$\nabla_{\mathbf{X}} \mathbf{U}^S(\mathbf{X}, t) = \begin{pmatrix} \frac{\partial U_1^S}{\partial X_1}(\mathbf{X}, t) & \frac{\partial U_1^S}{\partial X_2}(\mathbf{X}, t) \\ \frac{\partial U_2^S}{\partial X_1}(\mathbf{X}, t) & \frac{\partial U_2^S}{\partial X_2}(\mathbf{X}, t) \end{pmatrix}.$$

For a square matrix $\mathbf{A}$, we denote by $\det(\mathbf{A})$, $tr(\mathbf{A})$, $\mathbf{A}^{-1}$, $\mathbf{A}^T$, $\text{cof}(\mathbf{A})$ the determinant, the trace, the inverse, the transpose and the cofactor matrix of $\mathbf{A}$, respectively. We shall write $\mathbf{A}^{-T} = \left(\mathbf{A}^{-1}\right)^T$ and we have $\text{cof}(\mathbf{A}) = \det(\mathbf{A}) \left(\mathbf{A}^{-1}\right)^T$. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are square matrices, we have $\mathbf{AB} : \mathbf{C} = \mathbf{B} : \mathbf{A}^T \mathbf{C} = \mathbf{A} : \mathbf{CB}^T$ and $\mathbf{A} : \mathbf{B} = \mathbf{A}^T : \mathbf{B}^T$.

We denote by $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ the first and the second Piola-Kirchhoff stress tensors, respectively and we have the identity $\mathbf{\Pi} = \mathbf{F}\mathbf{\Sigma}$. The Cauchy stress tensor of the structure is σ^S and the following equality holds

$$\sigma^S(\mathbf{x},t) = \frac{1}{J(\mathbf{X},t)}\mathbf{F}(\mathbf{X},t)\mathbf{\Sigma}(\mathbf{X},t)\mathbf{F}^T(\mathbf{X},t).$$

We suppose that the material of the structure is incompressible neo-hookean. We have $J(\mathbf{X},t) = 1$ for all $\mathbf{X} \in \Omega_0^S$, $t \geq 0$ and

$$\sigma^S(\mathbf{x},t) = -p^S(\mathbf{x},t)\mathbf{I} + \mu^S\left(\mathbf{F}(\mathbf{X},t)\mathbf{F}^T(\mathbf{X},t) - \mathbf{I}\right),$$

where p^S is the structure pressure in the Eulerian coordinates and $\mu^S > 0$ is a constant. We can deduce

$$\mathbf{\Pi}(\mathbf{X},t) = (\mathbf{F}\mathbf{\Sigma})(\mathbf{X},t) = -P^S(\mathbf{X},t)\mathbf{F}^{-T}(\mathbf{X},t) + \mu^S\left(\mathbf{F}(\mathbf{X},t) - \mathbf{F}^{-T}(\mathbf{X},t)\right),$$

where P^S is the structure pressure in the Lagrangian coordinates and we have the identity $P^S(\mathbf{X},t) = p^S(\mathbf{x},t)$.

We assume that the fluid is governed by the Navier-Stokes equations, then the fluid stress tensor is $\sigma^F = -p^F\mathbf{I} + 2\mu^F\epsilon(\mathbf{v}^F)$, where

$$\epsilon\left(\mathbf{v}^F\right) = \frac{1}{2}\left(\nabla\mathbf{v}^F + \left(\nabla\mathbf{v}^F\right)^T\right)$$

is the fluid rate of strain tensor, $\mathbf{v}^F$ is the velocity, p^F is the pressure in the Eulerian coordinates and $\mu^F > 0$ is a constant. To simplify the notation, we write $\nabla\mathbf{v}^F$ in place of $\nabla_{\mathbf{x}}\mathbf{v}^F$, when the gradients are computed with respect to the Eulerian coordinates $\mathbf{x}$.

The fluid-structure interaction problem is: find the structure displacement $\mathbf{U}^S$ and pressure P^S, the fluid velocity $\mathbf{v}^F$ and pressure p^F, such that:

$$\rho_0^S(\mathbf{X})\frac{\partial^2\mathbf{U}^S}{\partial t^2}(\mathbf{X},t) - \nabla_{\mathbf{X}}\cdot(\mathbf{F}\mathbf{\Sigma})(\mathbf{X},t) = \rho_0^S(\mathbf{X})\mathbf{g}, \quad \text{in } \Omega_0^S\times(0,T), \tag{2.1}$$

$$\mathbf{U}^S(\mathbf{X},t) = 0, \quad \text{on } \Gamma_D\times(0,T), \tag{2.2}$$

$$\rho^F\left(\frac{\partial\mathbf{v}^F}{\partial t} + (\mathbf{v}^F\cdot\nabla)\mathbf{v}^F\right) - 2\mu^F\nabla\cdot\epsilon\left(\mathbf{v}^F\right) + \nabla p^F = \rho^F\mathbf{g}, \forall t\in(0,T), \quad \forall\mathbf{x}\in\Omega_t^F, \tag{2.3}$$

$$\nabla\cdot\mathbf{v}^F = 0, \forall t\in(0,T), \quad \forall\mathbf{x}\in\Omega_t^F, \tag{2.4}$$

$$\sigma^F\mathbf{n}^F = \mathbf{h}_{in}, \quad \text{on } \Sigma_1\times(0,T), \tag{2.5}$$

$$\sigma^F\mathbf{n}^F = \mathbf{h}_{out}, \quad \text{on } \Sigma_3\times(0,T), \tag{2.6}$$

$$\mathbf{v}^F = 0, \quad \text{on } \Sigma_2\cup\Sigma_4\cup\Sigma_5, \tag{2.7}$$

$$\tag{2.8}$$

$$\mathbf{v}^F\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right)=\frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},t),\quad\text{on }\Gamma_0\times(0,T), \tag{2.9}$$

$$\left(\sigma^F\mathbf{n}^F\right)_{(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t)}=-(\mathbf{F}\boldsymbol{\Sigma})(\mathbf{X},t)\mathbf{N}^S(\mathbf{X}),\quad\text{on }\Gamma_0\times(0,T), \tag{2.10}$$

$$\mathbf{U}^S(\mathbf{X},0)=\mathbf{U}^{S,0}(\mathbf{X}),\quad\text{in }\Omega_0^S, \tag{2.11}$$

$$\frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},0)=\mathbf{V}^{S,0}(\mathbf{X}),\quad\text{in }\Omega_0^S, \tag{2.12}$$

$$\mathbf{v}^F(\mathbf{X},0)=\mathbf{v}^{F,0}(\mathbf{X}),\quad\text{in }\Omega_0^F, \tag{2.13}$$

where $\rho_0^S:\Omega_0^S\to(0,\infty)$ is the mass density of the structure in the initial domain, $\mathbf{g}$ is the acceleration of gravity vector, here is constant, $\mathbf{N}^S$ is the unit outer normal vector to $\partial\Omega_0^S$, $\rho^F>0$ and $\mu^F>0$ are the mass density and the viscosity of the fluid which are constant, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$ are the inflow and outflow boundary stress, $\mathbf{n}^F$ is the unit outer normal vector to $\partial\Omega_t^F$. For the numerical tests, we use $\mathbf{v}=\mathbf{v}_{in}$ on Σ_1 as in the original paper [25], but the non-homogeneous boundary condition introduces complications in the proof of the stability.

3   Approximation of the structure using the Lagrangian coordinates

Introducing the structure velocity $\mathbf{V}^S$, the equation (2.1) is equivalent to

$$\rho_0^S(\mathbf{X})\frac{\partial\mathbf{V}^S}{\partial t}(\mathbf{X},t)-\nabla_{\mathbf{X}}\cdot(\mathbf{F}\boldsymbol{\Sigma})(\mathbf{X},t)=\rho_0^S(\mathbf{X})\mathbf{g},\quad\text{in }\Omega_0^S\times(0,T), \tag{3.1}$$

$$\frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},t)=\mathbf{V}^S(\mathbf{X},t),\quad\text{in }\Omega_0^S\times(0,T). \tag{3.2}$$

We denote by $N\in\mathbb{N}^*$ the number of time steps and by $\Delta t=T/N$ the time step and we set $t_n=n\Delta t$ for $n=0,1,\dots,N$. We consider $\mathbf{V}^{S,n}(\mathbf{X})$ and $\mathbf{U}^{S,n}(\mathbf{X})$ approximations of $\mathbf{V}^S(\mathbf{X},t_n)$ and $\mathbf{U}^S(\mathbf{X},t_n)$. We set also

$$\mathbf{F}^n=\mathbf{I}+\nabla_{\mathbf{X}}\mathbf{U}^{S,n},\quad\boldsymbol{\Sigma}^n=-P^{S,n}(\mathbf{F}^n)^{-1}(\mathbf{F}^n)^{-T}+\mu^S\left(\mathbf{I}-(\mathbf{F}^n)^{-1}(\mathbf{F}^n)^{-T}\right),\quad n\geq0.$$

The system (3.1)–(3.2) will be approached by the implicit Euler scheme

$$\rho_0^S(\mathbf{X})\frac{\mathbf{V}^{S,n+1}(\mathbf{X})-\mathbf{V}^{S,n}(\mathbf{X})}{\Delta t}-\nabla_{\mathbf{X}}\cdot\left(\mathbf{F}^{n+1}\boldsymbol{\Sigma}^{n+1}\right)(\mathbf{X})=\rho_0^S(\mathbf{X})\mathbf{g},\quad\text{in }\Omega_0^S, \tag{3.3}$$

$$\frac{\mathbf{U}^{S,n+1}(\mathbf{X})-\mathbf{U}^{S,n}(\mathbf{X})}{\Delta t}=\mathbf{V}^{S,n+1}(\mathbf{X}),\quad\text{in }\Omega_0^S. \tag{3.4}$$

From (3.4), we can eliminate the displacement $\mathbf{U}^{S,n+1}$ and we rewrite (3.3) in function of $\mathbf{V}^{S,n+1}$ and $P^{S,n+1}$.

By Green formula we can get the weak form of the equation (3.3): find $\mathbf{V}^{S,n+1}:\Omega_0^S\to\mathbb{R}^2$, $\mathbf{V}^{S,n+1}=0$ on Γ_D and $P^{S,n+1}:\Omega_0^S\to\mathbb{R}$, such that

$$\int_{\Omega_0^S}\rho_0^S\frac{\mathbf{V}^{S,n+1}-\mathbf{V}^{S,n}}{\Delta t}\cdot\mathbf{W}^S d\mathbf{X}+\int_{\Omega_0^S}\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}:\nabla_\mathbf{X}\mathbf{W}^S d\mathbf{X}$$
$$=\int_{\Omega_0^S}\rho_0^S\mathbf{g}\cdot\mathbf{W}^S d\mathbf{X}+\int_{\Gamma_0}\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\mathbf{N}^S\cdot\mathbf{W}^S dS \qquad (3.5)$$

for all $\mathbf{W}^S:\Omega_0^S\to\mathbb{R}^2$, $\mathbf{W}^S=0$ on Γ_D, subject to

$$\det\left(\mathbf{I}+\nabla_\mathbf{X}\mathbf{U}^{S,n}+\Delta t\nabla_\mathbf{X}\mathbf{V}^{S,n+1}\right)=1,\quad\text{in }\Omega_0^S. \qquad (3.6)$$

We have assumed that the forces $\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\mathbf{N}^S$ on the interface Γ_0 are known, for instant.

4 Approximation of the structure using the updated Lagrangian coordinates

We follow the approach as in [16], where the structure was a compressible neo-hookean material.

We set Ω_n^S the image of Ω_0^S via the map $\mathbf{X}\to\mathbf{X}+\mathbf{U}^{S,n}(\mathbf{X})$ and we define $\widehat{\Omega}^S=\Omega_n^S$ the computational domain for the structure.

The application from Ω_0^S to Ω_{n+1}^S given by $\mathbf{X}\to\mathbf{x}=\mathbf{X}+\mathbf{U}^{S,n+1}(\mathbf{X})$ is the composition of two maps: the application from Ω_0^S to $\widehat{\Omega}^S$ defined by $\mathbf{X}\to\widehat{\mathbf{x}}=\mathbf{X}+\mathbf{U}^{S,n}(\mathbf{X})$ with the application from $\widehat{\Omega}^S$ to Ω_{n+1}^S defined by

$$\widehat{\mathbf{x}}\to\mathbf{x}=\widehat{\mathbf{x}}+\mathbf{U}^{S,n+1}(\mathbf{X})-\mathbf{U}^{S,n}(\mathbf{X})=\widehat{\mathbf{x}}+\widehat{\mathbf{u}}(\widehat{\mathbf{x}}).$$

Using the notations $\widehat{\mathbf{F}}=\mathbf{I}+\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}}$ and $\widehat{J}=\det\widehat{\mathbf{F}}$, $J^n=\det\mathbf{F}^n$, we get

$$\mathbf{F}^{n+1}(\mathbf{X})=\widehat{\mathbf{F}}(\widehat{\mathbf{x}})\mathbf{F}^n(\mathbf{X}),\quad J^{n+1}(\mathbf{X})=\widehat{J}(\widehat{\mathbf{x}})J^n(\mathbf{X}). \qquad (4.1)$$

For an incompressible material, normally we have $J^n=J^{n+1}=\widehat{J}=1$. But, in the following, these constraints are not respected exactly, we will have only $J^n\approx1$, $J^{n+1}\approx1$, $\widehat{J}\approx1$.

The Cauchy stress tensor at the time instant t_{n+1} is

$$\sigma^{S,n+1}(\mathbf{x})=\left(\frac{1}{J^{n+1}}\mathbf{F^{n+1}}\mathbf{\Sigma}^{n+1}\left(\mathbf{F}^{n+1}\right)^T\right)(\mathbf{X}),\quad\mathbf{x}=\mathbf{X}+\mathbf{U}^{S,n+1}(\mathbf{X}).$$

We introduce $\widehat{\mathbf{v}}^{S,n+1}:\widehat{\Omega}^S\to\mathbb{R}^2$ and $\mathbf{v}^{S,n}:\widehat{\Omega}^S\to\mathbb{R}^2$ defined by $\widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}})=\mathbf{V}^{S,n+1}(\mathbf{X})$ and $\mathbf{v}^{S,n}(\widehat{\mathbf{x}})=\mathbf{V}^{S,n}(\mathbf{X})$. Also, for $\mathbf{W}^S:\Omega_0^S\to\mathbb{R}^2$, we define $\widehat{\mathbf{w}}^S:\widehat{\Omega}^S\to\mathbb{R}^2$ and $\mathbf{w}^S:\Omega_{n+1}^S\to\mathbb{R}^2$ by $\widehat{\mathbf{w}}^S(\widehat{\mathbf{x}})=\mathbf{w}^S(\mathbf{x})=\mathbf{W}^S(\mathbf{X})$.

We want to rewrite Eq. (3.5) over the domain $\widehat{\Omega}^S$. For the first term of (3.5), we get

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

and in a similar way, we get

$$\int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}.$$

Using [5], Chapter 1.2, we have $\left(\nabla \mathbf{w}^S(\mathbf{x}) \right) \mathbf{F}^{n+1}(\mathbf{X}) = \nabla_{\mathbf{X}} \mathbf{W}^S(\mathbf{X})$ and we get

$$\int_{\Omega_0^S} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X} = \int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S \, d\mathbf{x}.$$

Before to write the above integral over the domain $\widehat{\Omega}^S$, we introduce the tensor

$$\widehat{\mathbf{\Sigma}}(\widehat{\mathbf{x}}) = \widehat{J}(\widehat{\mathbf{x}}) \widehat{\mathbf{F}}^{-1}(\widehat{\mathbf{x}}) \sigma^{S,n+1}(\mathbf{x}) \widehat{\mathbf{F}}^{-T}(\widehat{\mathbf{x}}). \tag{4.2}$$

Since $\left(\nabla \mathbf{w}^S(\mathbf{x}) \right) \widehat{\mathbf{F}}(\widehat{\mathbf{x}}) = \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S(\widehat{\mathbf{x}})$, once again see [5], Chapter 1.2 and taking into account (4.2), we get

$$\int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S \, d\mathbf{x} = \int_{\widehat{\Omega}^S} \widehat{\mathbf{F}} \widehat{\mathbf{\Sigma}} : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}.$$

Now we present the updated Lagrangian version of (3.5). Knowing $\mathbf{U}^{S,n} : \Omega_0^S \to \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$, find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_D and $\widehat{p}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}$, such that

$$\int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{F}} \widehat{\mathbf{\Sigma}} : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

$$= \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS \tag{4.3}$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_D. The forces $\mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S$ on the interface Γ_0 are assumed known.

In the following, we derive the expression of $\widehat{\mathbf{F}} \widehat{\mathbf{\Sigma}}$ as function of $\widehat{\mathbf{v}}^{S,n+1}$ and $\widehat{p}^{S,n+1}$. From $\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X}) = \Delta t \mathbf{V}^{S,n+1}(\mathbf{X}) = \Delta t \widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}})$, we get

$$\widehat{\mathbf{F}} = \mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}. \tag{4.4}$$

From (4.1) and (4.2), it follows that

$$\widehat{\mathbf{\Sigma}} = \widehat{J} \widehat{\mathbf{F}}^{-1} \sigma^{S,n+1} \widehat{\mathbf{F}}^{-T} = \widehat{J} \frac{1}{\widehat{J} J^n} \widehat{\mathbf{F}}^{-1} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \left(\mathbf{F}^{n+1} \right)^T \widehat{\mathbf{F}}^{-T}$$

$$= \frac{1}{J^n} \mathbf{F}^n \mathbf{\Sigma}^{n+1} \left(\mathbf{F}^n \right)^T. \tag{4.5}$$

For the incompressible neo-hookean material, we have

$$\boldsymbol{\Sigma}^{n+1} = -P^{S,n+1}\left(\mathbf{F}^{n+1}\right)^{-1}\left(\mathbf{F}^{n+1}\right)^{-T} + \mu^S\left(\mathbf{I} - \left(\mathbf{F}^{n+1}\right)^{-1}\left(\mathbf{F}^{n+1}\right)^{-T}\right)$$

$$= -P^{S,n+1}\left(\mathbf{F}^n\right)^{-1}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\left(\mathbf{F}^n\right)^{-T} + \mu^S\left(\mathbf{I} - \left(\mathbf{F}^n\right)^{-1}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\left(\mathbf{F}^n\right)^{-T}\right).$$

Then

$$\widehat{\boldsymbol{\Sigma}} = \frac{1}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}^{n+1}\left(\mathbf{F}^n\right)^T$$

$$= -\frac{1}{J^n}\widehat{p}^{S,n+1}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T} + \frac{\mu^S}{J^n}\left(\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\right),$$

and finally

$$\widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} = -\frac{1}{J^n}\widehat{p}^{S,n+1}\widehat{\mathbf{F}}^{-T} + \frac{\mu^S}{J^n}\left(\widehat{\mathbf{F}}\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \widehat{\mathbf{F}}^{-T}\right)$$

with $\widehat{p}^{S,n+1}(\widehat{\mathbf{x}}) = P^{S,n+1}(\mathbf{X})$. Since $\det\widehat{\mathbf{F}} \approx 1$, we get that $\widehat{\mathbf{F}}^{-T} \approx \operatorname{cof}\left(\widehat{\mathbf{F}}\right)$, the cofactor matrix of $\widehat{\mathbf{F}}$. Then

$$\widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} \approx -\frac{1}{J^n}\widehat{p}^{S,n+1}\operatorname{cof}\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)$$

$$+\frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \operatorname{cof}\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\right)$$

$$= -\frac{1}{J^n}\widehat{p}^{S,n+1}\mathbf{I} - \frac{1}{J^n}\widehat{p}^{S,n+1}(\Delta t)\operatorname{cof}\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)$$

$$+\frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \mathbf{I} - (\Delta t)\operatorname{cof}\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\right). \qquad (4.6)$$

The exact incompressibility condition $\widehat{J} = 1$ gives

$$1 + (\Delta t)\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{S,n+1} + (\Delta t)^2\det(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}) = 1,$$

but it will approached by the condition

$$x\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{S,n+1} = 0, \quad \text{in } \widehat{\Omega}^S. \qquad (4.7)$$

First, we introduce

$$\widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1},\widehat{p}^{S,n+1}\right) = -\frac{1}{J^n}\widehat{p}^{S,n+1}\mathbf{I} + \frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \mathbf{I}\right). \qquad (4.8)$$

We remark that the expression (4.8) is first-order consistent in time with (4.6).

The first updated Lagrangian weak formulation of the structure is: knowing $\mathbf{U}^{S,n}$: $\Omega_0^S \to \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$, find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_D and $\widehat{p}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}$, such that

$$\int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{L}}_1 \left(\widehat{\mathbf{v}}^{S,n+1}, \widehat{p}^{S,n+1} \right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}}$$

$$= \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S dS \qquad (4.9)$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_D, subject to (4.7). For the new time step, using (3.4), we put

$$\mathbf{U}^{S,n+1}(\mathbf{X}) = \mathbf{U}^{S,n}(\mathbf{X}) + \Delta t \mathbf{V}^{S,n+1}(\mathbf{X}) = \mathbf{U}^{S,n}(\mathbf{X}) + \Delta t \widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}}). \qquad (4.10)$$

We have assumed that all expression before are well defined. In the follows, we precise the regularity of the data. We assume that $J^n \in L^\infty(\widehat{\Omega}^S)$ and there exits $\delta \in (0,1)$ such that $1 - \delta \leq J^n(\mathbf{X}) \leq 1 + \delta$, a.e. Also, we assume that each component of $\mathbf{F}^n$ is in $L^\infty(\widehat{\Omega}^S)$ and we introduce the spaces

$$\widehat{\mathbb{W}}^S = \left\{ \widehat{\mathbf{w}}^S \in \left(H^1\left(\widehat{\Omega}^S\right) \right)^2 ; \widehat{\mathbf{w}}^S = 0 \text{ on } \Gamma_D \right\}, \quad \widehat{\mathbb{Q}}^S = L^2\left(\widehat{\Omega}^S\right).$$

We can write the system (4.9), (4.7) as a mixed problem: find $\widehat{\mathbf{v}}^{S,n+1} \in \widehat{\mathbb{W}}^S$, $\widehat{p}^{S,n+1} \in \widehat{\mathbb{Q}}^S$ such that

$$\widehat{a}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{\mathbf{w}}^S) + \widehat{b}^S(\widehat{\mathbf{w}}^S, \widehat{p}^{S,n+1}) = \mathcal{L}_S(\widehat{\mathbf{w}}^S), \quad \forall \widehat{\mathbf{w}}^S \in \widehat{\mathbb{W}}^S, \qquad (4.11)$$

$$\widehat{b}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{q}^S) = 0, \quad \forall \widehat{q}^S \in \widehat{\mathbb{Q}}^S, \qquad (4.12)$$

where

$$\widehat{a}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{\mathbf{w}}^S) = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \frac{\mu^S}{J^n} \left(\Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right) \mathbf{F}^n (\mathbf{F}^n)^T : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}},$$

$$\widehat{b}^S(\widehat{\mathbf{w}}^S, \widehat{q}^S) = -\int_{\widehat{\Omega}^S} \frac{1}{J^n} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^S \right) \widehat{q}^S d\widehat{\mathbf{x}}.$$

Proposition 4.1. *The mixed problem* (4.11), (4.12) *has an unique solution.*

Proof. The continuous function $\widehat{\mathbf{w}}^S \to \widehat{a}^S(\widehat{\mathbf{w}}^S, \widehat{\mathbf{w}}^S)$ verifies for all $\widehat{\mathbf{w}}^S \in \widehat{\mathbb{W}}^S$, $\widehat{\mathbf{w}}^S \neq 0$

$$0 < \widehat{a}^S(\widehat{\mathbf{w}}^S, \widehat{\mathbf{w}}^S) = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{w}}^S}{\Delta t} \cdot \widehat{\mathbf{w}}^S d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \frac{\mu^S}{J^n} (\Delta t) \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \mathbf{F}^n : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \mathbf{F}^n d\widehat{\mathbf{x}}$$

and it has a minimum $\alpha^S > 0$ on the compact set $\{\widehat{\mathbf{w}}^S \in \widehat{\mathbb{W}}^S; \|\widehat{\mathbf{w}}^S\|_{1,\widehat{\Omega}^s} = 1\}$. Then, we get the ellipticity of $\widehat{a}^S$,

$$\alpha^S \left\| \widehat{\mathbf{w}}^S \right\|_{1,\widehat{\Omega}^s}^2 \leq \widehat{a}^S(\widehat{\mathbf{w}}^S, \widehat{\mathbf{w}}^S), \quad \forall \widehat{\mathbf{w}}^S \in \widehat{\mathbb{W}}^S.$$

Including J^n in $\widehat{q}^S$, we obtain that $\widehat{b}^S(\widehat{\mathbf{w}}^S, \widehat{q}^S)$ verifies the *inf-sup* condition in $\widehat{\mathbb{W}}^S \times \widehat{\mathbb{Q}}^S$, then we get the conclusion of the Proposition by Babuska-Brezzi theory, [1, 17]. □

Now, we will treat the condition (4.7) by penalization and we introduce

$$\widehat{\mathbf{L}}_2\left(\widehat{\mathbf{v}}^{S,n+1}\right) = \frac{1}{\varepsilon}\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{I}+\mu^S\left(\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right.$$

$$\left.-\text{cof}\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\right), \tag{4.13}$$

where $\varepsilon > 0$ is the penalization parameter. The second updated Lagrangian weak formulation of the structure using the penalization is: find $\widehat{\mathbf{v}}^{S,n+1}:\widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1}=0$ on Γ_D such that (4.9) holds, where $\widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1},\widehat{p}^{S,n+1}\right)$ is replaced by $\widehat{\mathbf{L}}_2\left(\widehat{\mathbf{v}}^{S,n+1}\right)$ and J^n replaced by 1. We point out that the condition (4.7) is not necessary and the structure pressure does not appear in this second formulation.

To justify the introduction of the second formulation, we precise that the pressure is not necessary continuous at the fluid-structure interface. Then using finite element function globally continuous like $\mathbb{P}_1$ in the fluid-structure domain is not appropriate.

5 Stability of the first updated Lagrangian algorithm for the structure

Theorem 5.1. *The time advancing scheme for the structure (4.9), (4.7) and (4.10) verifies*

$$\frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left|\mathbf{V}^{S,n+1}\right|^2 d\mathbf{X}+\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:\mathbf{F}^{n+1}d\mathbf{X}$$

$$\leq \frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left|\mathbf{V}^{S,n}\right|^2 d\mathbf{X}+\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^n:\mathbf{F}^n d\mathbf{X}$$

$$\leq \frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left|\mathbf{V}^{S,0}\right|^2 d\mathbf{X}+\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^0:\mathbf{F}^0 d\mathbf{X}, \tag{5.1}$$

if the right hand side of (4.9) is zero, where $\left|\mathbf{V}^{S,n}\right|^2 = \left(V_1^{S,n}\right)^2+\left(V_2^{S,n}\right)^2$.

Proof. We put $\widehat{\mathbf{w}}^S = (\Delta t)\widehat{\mathbf{v}}^{S,n+1}$ in (4.9). From the first term of (4.9), using $\widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}}) = \mathbf{V}^{S,n+1}(\mathbf{X})$, we get

$$\int_{\widehat{\Omega}^S}\frac{\rho_0^S}{J^n}\frac{\widehat{\mathbf{v}}^{S,n+1}-\mathbf{v}^{S,n}}{\Delta t}\cdot(\Delta t)\widehat{\mathbf{v}}^{S,n+1}d\widehat{\mathbf{x}} = \int_{\Omega_0^S}\rho_0^S\left(\mathbf{V}^{S,n+1}-\mathbf{V}^{S,n}\right)\cdot\mathbf{V}^{S,n+1}d\mathbf{X}.$$

Applying the inequality

$$\frac{a^2}{2}-\frac{b^2}{2}\leq (a-b)a, \quad \forall a,b\in\mathbb{R}, \tag{5.2}$$

we obtain

$$\frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left(\mathbf{V}^{S,n+1}\right)^2 d\mathbf{X}-\frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left(\mathbf{V}^{S,n}\right)^2 d\mathbf{X}$$

$$\leq \int_{\Omega_0^S}\rho_0^S\left(\mathbf{V}^{S,n+1}-\mathbf{V}^{S,n}\right)\cdot\mathbf{V}^{S,n+1}d\mathbf{X}.$$

From the second term of (4.9), using (4.7) we obtain

$$\widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1},\widehat{p}^{S,n+1}\right):(\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}=-\frac{1}{J^n}\widehat{p}^{S,n+1}(\Delta t)\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{S,n+1}\right)$$

$$+\frac{\mu^S}{J^n}\left(\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right):(\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}-\frac{\mu^S}{J^n}(\Delta t)\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{S,n+1}\right)$$

$$=\frac{\mu^S}{J^n}\left(\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right):(\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}.$$

Now, we change the domain of integration, from $\widehat{\Omega}^S=\Omega_n^S$ to Ω_{n+1}^S using $\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}(\widehat{\mathbf{x}})=(\nabla\mathbf{w}(\mathbf{x}))\widehat{\mathbf{F}}(\widehat{\mathbf{x}})$ and from Ω_{n+1}^S to Ω_0^S using $(\nabla\mathbf{w}(\mathbf{x}))\mathbf{F}^{n+1}(\mathbf{X})=\nabla_{\mathbf{X}}\mathbf{W}(\mathbf{X})$. We get

$$\int_{\widehat{\Omega}^S}\widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1},\widehat{p}^{S,n+1}\right):(\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}d\widehat{\mathbf{x}}$$

$$=\int_{\widehat{\Omega}^S}\frac{\mu^S}{J^n}\left(\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right):(\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}d\widehat{\mathbf{x}}$$

$$=\int_{\widehat{\Omega}^S}\frac{\mu^S}{J^n}\left(\widehat{\mathbf{F}}\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right):(\Delta t)\nabla\mathbf{v}^{S,n+1}\widehat{\mathbf{F}}d\widehat{\mathbf{x}}$$

$$=\int_{\widehat{\Omega}^S}\frac{\mu^S}{J^n}\left(\widehat{\mathbf{F}}\mathbf{F}^n\left(\mathbf{F}^n\right)^T\widehat{\mathbf{F}}^T\right):(\Delta t)\nabla\mathbf{v}^{S,n+1}d\widehat{\mathbf{x}}\int_{\Omega_{n+1}^S}\frac{\mu^S}{\widehat{J}J^n}\mathbf{F}^{n+1}\left(\mathbf{F}^{n+1}\right)^T:(\Delta t)\nabla\mathbf{v}^{S,n+1}d\mathbf{x}$$

$$=\int_{\Omega_{n+1}^S}\frac{\mu^S}{J^{n+1}}\mathbf{F}^{n+1}:(\Delta t)\left(\nabla\mathbf{v}^{S,n+1}\right)\mathbf{F}^{n+1}d\mathbf{x}\int_{\Omega_{n+1}^S}\frac{\mu^S}{J^{n+1}}\mathbf{F}^{n+1}:(\Delta t)\nabla_{\mathbf{X}}\mathbf{V}^{S,n+1}d\mathbf{x}$$

$$=\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:(\Delta t)\nabla_{\mathbf{X}}\mathbf{V}^{S,n+1}d\mathbf{X}=\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:\left(\mathbf{F}^{n+1}-\mathbf{F}^n\right)d\mathbf{X}.$$

Using once again (5.2), we get

$$\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:\mathbf{F}^{n+1}d\mathbf{X}-\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^n:\mathbf{F}^nd\mathbf{X}\leq\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:\left(\mathbf{F}^{n+1}-\mathbf{F}^n\right)d\mathbf{X}.$$

It follows

$$\frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left|\mathbf{V}^{S,n+1}\right|^2d\mathbf{X}-\frac{1}{2}\int_{\Omega_0^S}\rho_0^S\left|\mathbf{V}^{S,n}\right|^2d\mathbf{X}$$

$$+\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^{n+1}:\mathbf{F}^{n+1}d\mathbf{X}-\frac{1}{2}\int_{\Omega_0^S}\mu^S\mathbf{F}^n:\mathbf{F}^nd\mathbf{X}\leq 0,$$

which is the first inequality of (5.1). If we write the above inequality for $n=0,1,\dots$, we get also the second inequality of (5.1). □

Remark 5.1. The proof of the structure stability exploits the incompressible neo-hookean model. A hyperelastic material is characterized by a positive energy density $\Psi(\mathbf{F})$ such that $\frac{\partial\Psi(\mathbf{F})}{\partial\mathbf{F}}=\mathbf{F}\boldsymbol{\Sigma}$. The convexity of the energy density as function of $\mathbf{F}$ can be used to prove the stability, as in [2]. More complicated incompressible materials as in [13] could be considered, but the energy density is convex as function of $\mathbf{E}=\frac{1}{2}\left(\mathbf{F}^T\mathbf{F}-\mathbf{I}\right)$, not of $\mathbf{F}$.

Remark 5.2. The midpoint scheme is of great interest in structure approximation because it is conservative. For quadratic energy density, a stability result for midpoint scheme is presented in [8]. For the incompressible neo-hookean material, the midpoint algorithm in the Updated Lagrangian framework will be studied in a future work.

6 Stability of an Arbitrary Lagrangian Eulerian scheme for fluid equations

The Arbitrary Eulerian Lagrangian (ALE) is a method for solving Navier-Stokes equations in a moving domain (see [21]).

We denote by Ω_n^F, $\mathbf{v}^{F,n}$, $p^{F,n}$ approximations for the fluid domain, velocity and pressure at time instant t_n. We follow the framework from [17] where the computational domain is $\widehat{\Omega}^F = \Omega_n^F$. We introduce the ALE map $\mathcal{A}_{n+1} : \Omega_n^F \to \mathbb{R}^2$ by

$$\mathcal{A}_{n+1}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + \Delta t \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}),$$

where $\widehat{\boldsymbol{\vartheta}}^{n+1}$ is so called mesh velocity. The boundary of Ω_n^F is composed by the fixed boundary $\bigcup_{i=1}^5 \Sigma_i$ and the moving boundary Γ_n which is the fluid-structure interface. We shall construct $\widehat{\boldsymbol{\vartheta}}^{n+1} : \Omega_n^F \to \mathbb{R}^2$ by harmonic extension such that the mesh velocity is zero on the fixed boundary and the mesh velocity is equal to the fluid velocity on the fluid-structure interface. The Jacobian of the ALE map is

$$\widehat{J}_{n+1}(\widehat{\mathbf{x}}) = \det(\nabla_{\widehat{\mathbf{x}}} \mathcal{A}_{n+1}(\widehat{\mathbf{x}})) = 1 + \Delta t \nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}) + (\Delta t)^2 \det(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})). \quad (6.1)$$

We note $\Omega_{n+1}^F = \mathcal{A}_{n+1}(\Omega_n^F)$, $\Gamma_{n+1} = \mathcal{A}_{n+1}(\Gamma_n)$ and we have $\mathcal{A}_{n+1}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}}$ on $\bigcup_{i=1}^5 \Sigma_i$. For $\mathbf{w}^F : \Omega_{n+1}^F \to \mathbb{R}^2$ and $q^F : \Omega_{n+1}^F \to \mathbb{R}$, we define $\widehat{\mathbf{w}}^F : \Omega_n^F \to \mathbb{R}^2$ and $\widehat{q}^F : \Omega_n^F \to \mathbb{R}$ by $\widehat{\mathbf{w}}^F(\widehat{\mathbf{x}}) = \mathbf{w}^F(\mathcal{A}_{n+1}(\widehat{\mathbf{x}}))$ and $\widehat{q}^F(\widehat{\mathbf{x}}) = q^F(\mathcal{A}_{n+1}(\widehat{\mathbf{x}}))$ respectively. We use the notation $\mathbf{x} = \mathcal{A}_{n+1}(\widehat{\mathbf{x}})$.

The time advancing scheme for fluid equations is: find $\widehat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{F,n+1} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, $\widehat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$, $\widehat{\boldsymbol{\vartheta}}^{n+1} : \Omega_n^F \to \mathbb{R}^2$, such that

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left(\left(\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{F,n+1} \right) \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}}$$

$$+ \frac{(\Delta t)}{2} \int_{\Omega_n^F} \rho^F \det\left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \widehat{\mathbf{v}}^{F,n+1} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{w}}^F \right) d\widehat{\mathbf{x}}$$

$$- \int_{\Omega_n^F} \widehat{p}^{F,n+1} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^F) d\widehat{\mathbf{x}} = \mathcal{L}_F(\widehat{\mathbf{w}}^F) + \int_{\Gamma_n} \left(\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1}) \mathbf{n}^F \right) \cdot \widehat{\mathbf{w}}^F ds, \quad (6.2)$$

$$- \int_{\Omega_n^F} \widehat{q}^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{F,n+1}) d\widehat{\mathbf{x}} = 0, \quad (6.3)$$

$$\Delta \widehat{\boldsymbol{\vartheta}}^{n+1} = 0 \text{ in } \Omega_n^F, \quad \widehat{\boldsymbol{\vartheta}}^{n+1} = \widehat{\mathbf{v}}^{F,n+1} \text{ on } \Gamma_n, \quad \widehat{\boldsymbol{\vartheta}}^{n+1} = 0 \text{ on } \partial\Omega_n^F \setminus \Gamma_n \quad (6.4)$$

for all $\widehat{\mathbf{w}}^F : \Omega_n^F \to \mathbb{R}^2$ such that $\widehat{\mathbf{w}}^F = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$ and for all $\widehat{q}^F : \Omega_n^F \to \mathbb{R}$, where

$$\mathcal{L}_F(\widehat{\mathbf{w}}^F) = \int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}}^F ds + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \widehat{\mathbf{w}}^F ds$$

and $\epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{w}}^F) = \frac{1}{2}\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}^F + (\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}^F)^T\right)$.

The term $\frac{(\Delta t)}{2}\int_{\Omega_n^F}\rho^F \det\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\vartheta}}^{n+1}\right)\widehat{\mathbf{v}}^{F,n+1}\cdot\widehat{\mathbf{w}}^F d\widehat{\mathbf{x}}$ in (6.2) was added to obtain the stability. This technique was introduced in [23] and also used in [17], Chapters 3 and 6.

Remark 6.1. The advantage of system (6.2)–(6.4) over to equations written over Ω_{n+1}^F which is unknown, is that we solve the system in the known domain Ω_n^F. This system is non-linear, but it can be easily linearized. We work with $\widehat{\boldsymbol{\vartheta}}^{n+1}$ for compatibility with the time advancing backward Euler scheme for the structure mesh. By replacing $\widehat{\boldsymbol{\vartheta}}^{n+1}$ with $\widehat{\boldsymbol{\vartheta}}^n$ in (6.2), we can decouple the equation (6.4) for solving $\widehat{\boldsymbol{\vartheta}}^{n+1}$, from the system (6.2) and (6.3) for solving $\widehat{\mathbf{v}}^{F,n+1}$, $\widehat{p}^{F,n+1}$.

Theorem 6.1. *The time advancing scheme for the fluid (6.2)–(6.4) verifies*

$$\frac{1}{2}\int_{\Omega_{n+1}^F}\rho^F\left|\mathbf{v}^{F,n+1}\right|^2 d\mathbf{x} + (\Delta t)\sum_{k=0}^{n}\int_{\Omega_k^F}2\mu^F\epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}):\epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1})d\widehat{\mathbf{x}}$$

$$\leq \frac{1}{2}\int_{\Omega_0^F}\rho^F\left|\mathbf{v}^{F,0}\right|^2 d\mathbf{X}, \tag{6.5}$$

if g, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$ and the forces acting on Γ_n are zero in the right-hand side of (6.2) and

$$\int_{\Sigma_1\cup\Sigma_3}(\widehat{\mathbf{v}}^{F,n+1}\cdot\mathbf{n}^F)|\widehat{\mathbf{v}}^{F,n+1}|^2 \geq 0,$$

where $|\widehat{\mathbf{v}}^{F,n+1}|^2 = (\widehat{\mathbf{v}}_1^{F,n+1})^2 + (\widehat{\mathbf{v}}_2^{F,n+1})^2$.

Proof. We put $\widehat{\mathbf{w}}^F = (\Delta t)\widehat{\mathbf{v}}^{F,n+1}$ in (6.2) and the terme containg $\widehat{p}^{F,n+1}$ will disappear, using (6.3). We obtain

$$\int_{\Omega_n^F}\rho^F\left(\widehat{\mathbf{v}}^{F,n+1}-\mathbf{v}^{F,n}\right)\cdot\widehat{\mathbf{v}}^{F,n+1}d\widehat{\mathbf{x}}$$

$$+(\Delta t)\int_{\Omega_n^F}\rho^F\left(\left(\left(\widehat{\mathbf{v}}^{F,n+1}-\widehat{\boldsymbol{\vartheta}}^{n+1}\right)\cdot\nabla_{\widehat{\mathbf{x}}}\right)\widehat{\mathbf{v}}^{F,n+1}\right)\cdot\widehat{\mathbf{v}}^{F,n+1}d\widehat{\mathbf{x}}$$

$$+\frac{(\Delta t)^2}{2}\int_{\Omega_n^F}\rho^F\det\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\vartheta}}^{n+1}\right)\widehat{\mathbf{v}}^{F,n+1}\cdot\widehat{\mathbf{v}}^{F,n+1}d\widehat{\mathbf{x}}$$

$$+(\Delta t)\int_{\Omega_n^F}2\mu^F\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{F,n+1}\right):\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{F,n+1}\right)d\widehat{\mathbf{x}}=0.$$

By using $[(\widehat{\mathbf{w}} \cdot \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}] \cdot \widehat{\mathbf{v}} = \frac{1}{2} \widehat{\mathbf{w}} \cdot (\nabla_{\widehat{\mathbf{x}}} |\widehat{\mathbf{v}}|^2)$, we get

$$\int_{\Omega_n^F} [((\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot \nabla_{\widehat{\mathbf{x}}}) \widehat{\mathbf{v}}^{F,n+1}] \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}}$$

$$= \frac{1}{2} \int_{\Omega_n^F} (\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot (\nabla_{\widehat{\mathbf{x}}} |\widehat{\mathbf{v}}^{F,n+1}|^2) d\widehat{\mathbf{x}}$$

$$= \frac{1}{2} \int_{\partial\Omega_n^F} (\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot \mathbf{n}^F |\widehat{\mathbf{v}}^{F,n+1}|^2 d\mathbf{s} - \frac{1}{2} \int_{\Omega_n^F} \nabla_{\widehat{\mathbf{x}}} \cdot (\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$= \frac{1}{2} \int_{\Sigma_1 \cup \Sigma_3} \widehat{\mathbf{v}}^{F,n+1} \cdot \mathbf{n}^F |\widehat{\mathbf{v}}^{F,n+1}|^2 d\mathbf{s} + \frac{1}{2} \int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}.$$

For the last equality, we have used the fact that $\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{F,n+1} = 0$ in Ω_n^F, and the boundary conditions: $\widehat{\mathbf{v}}^{F,n+1} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, $\widehat{\mathbf{v}}^{F,n+1} = \boldsymbol{\vartheta}^{n+1}$ on Γ_n and $\boldsymbol{\vartheta}^{n+1} = 0$ on $\Sigma_1 \cup \Sigma_3$.

Using the assumption $\int_{\Sigma_1 \cup \Sigma_3} (\widehat{\mathbf{v}}^{F,n+1} \cdot \mathbf{n}^F) |\widehat{\mathbf{v}}^{F,n+1}|^2 \geq 0$, it follows

$$\int_{\Omega_n^F} \rho^F \left(\widehat{\mathbf{v}}^{F,n+1} - \mathbf{v}^{F,n} \right) \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}} + \frac{\Delta t}{2} \int_{\Omega_n^F} \rho^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$+ \frac{(\Delta t)^2}{2} \int_{\Omega_n^F} \rho^F \det \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$+ (\Delta t) \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) d\widehat{\mathbf{x}} \leq 0.$$

Using (5.2), we obtain

$$\frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \widehat{\mathbf{v}}^{F,n+1} \right|^2 d\widehat{\mathbf{x}} - \frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \mathbf{v}^{F,n} \right|^2 d\widehat{\mathbf{x}} \leq \int_{\Omega_n^F} \rho^F \left(\widehat{\mathbf{v}}^{F,n+1} - \mathbf{v}^{F,n} \right) \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}}.$$

Consequently,

$$\frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \widehat{\mathbf{v}}^{F,n+1} \right|^2 d\widehat{\mathbf{x}} + \frac{\Delta t}{2} \int_{\Omega_n^F} \rho^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$+ \frac{(\Delta t)^2}{2} \int_{\Omega_n^F} \rho^F \det \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$+ (\Delta t) \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) d\widehat{\mathbf{x}} \leq \frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \mathbf{v}^{F,n} \right|^2 d\widehat{\mathbf{x}}.$$

From (6.1), we have that

$$\frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \widehat{\mathbf{v}}^{F,n+1} \right|^2 d\widehat{\mathbf{x}} + \frac{\Delta t}{2} \int_{\Omega_n^F} \rho^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$+ \frac{(\Delta t)^2}{2} \int_{\Omega_n^F} \rho^F \det \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}$$

$$= \frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \widehat{\mathbf{v}}^{F,n+1} \right|^2 \widehat{J}_{n+1} d\widehat{\mathbf{x}} = \frac{1}{2} \int_{\Omega_{n+1}^F} \rho^F \left| \mathbf{v}^{F,n+1} \right|^2 d\mathbf{x}.$$

Finally, we obtain

$$\frac{1}{2}\int_{\Omega_{n+1}^F}\rho^F\left|\mathbf{v}^{F,n+1}\right|^2 d\mathbf{x}+(\Delta t)\int_{\Omega_n^F}2\mu^F\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{F,n+1}\right):\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{F,n+1}\right)d\widehat{\mathbf{x}}$$

$$\leq\frac{1}{2}\int_{\Omega_n^F}\rho^F\left|\mathbf{v}^{F,n}\right|^2 d\widehat{\mathbf{x}}.$$

We replace in the above inequality n by k and summing for $0\leq k\leq n$, we get (6.5). □

7 Stable monolithic algorithm for fluid-structure interaction

We follow the approach as in [16, 18, 24] and we introduce globally fields defined in the whole fluid-structure domain. The global mesh is compatible with the fluid-structure interface.

At time instant t_n, we have $\partial\Omega_n^S=\Gamma_D\cup\Gamma_n$, where Γ_n is the fluid-structure interface and $\partial\Omega_n^F=\left(\cup_{i=1}^5\Sigma_i\right)\cup\Gamma_n$. We introduce $\Omega_n=\Omega_n^S\cup\Gamma_n\cup\Omega_n^F$. Let us introduce the global velocity, pressure and test function $\widehat{\mathbf{v}}^{n+1}:\Omega_n\to\mathbb{R}^2$, $\widehat{p}^{n+1}:\Omega_n\to\mathbb{R}$, $\widehat{\mathbf{w}}:\Omega_n\to\mathbb{R}^2$ as follow

$$\widehat{\mathbf{v}}^{n+1}=\left\{\begin{array}{l}\widehat{\mathbf{v}}^{F,n+1}\text{ in }\Omega_n^F\\\widehat{\mathbf{v}}^{S,n+1}\text{ in }\Omega_n^S\end{array}\right.,\quad\widehat{p}^{n+1}=\left\{\begin{array}{l}\widehat{p}^{F,n+1}\text{ in }\Omega_n^F\\\widehat{p}^{S,n+1}\text{ in }\Omega_n^S\end{array}\right.,\quad\widehat{\mathbf{w}}=\left\{\begin{array}{l}\widehat{\mathbf{w}}^F\text{ in }\Omega_n^F\\\widehat{\mathbf{w}}^S\text{ in }\Omega_n^S\end{array}\right..$$

For the velocity and the test function, we impose to be from $\left(H^1(\Omega_n)\right)^2$, then

$$\widehat{\mathbf{v}}^{F,n+1}=\widehat{\mathbf{v}}^{S,n+1}\text{ on }\Gamma_n,\quad\widehat{\mathbf{w}}^F=\widehat{\mathbf{w}}^S\text{ on }\Gamma_n.$$

For the pressure, we can use $L^2(\Omega_n)$.

Combining the equations (4.9), (4.7) with (6.2)–(6.4), we obtain the monolithic system for the fluid-structure interaction problem: find

1) the velocity $\widehat{\mathbf{v}}^{n+1}\in\left(H^1(\Omega_n)\right)^2$, $\widehat{\mathbf{v}}^{n+1}=0$ on $\Sigma_2\cup\Sigma_4\cup\Sigma_5$,

2) the pressure $\widehat{p}^{n+1}\in L^2(\Omega_n)$,

3) the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1}\in\left(H^1(\Omega_n^F)\right)^2$, $\widehat{\boldsymbol{\vartheta}}^{n+1}=\widehat{\mathbf{v}}^{F,n+1}$ on Γ_n, $\widehat{\boldsymbol{\vartheta}}^{n+1}=0$ on $\cup_{i=1}^5\Sigma_i$,

   such that:

$$\int_{\Omega_n^F}\rho^F\frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t}\cdot\widehat{\mathbf{w}}d\widehat{\mathbf{x}}+\int_{\Omega_n^F}\rho^F\left(\left(\left(\widehat{\mathbf{v}}^{n+1}-\widehat{\boldsymbol{\vartheta}}^{n+1}\right)\cdot\nabla_{\widehat{\mathbf{x}}}\right)\widehat{\mathbf{v}}^{n+1}\right)\cdot\widehat{\mathbf{w}}d\widehat{\mathbf{x}}$$

$$+\frac{(\Delta t)}{2}\int_{\Omega_n^F}\rho^F\det\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\vartheta}}^{n+1}\right)\widehat{\mathbf{v}}^{n+1}\cdot\widehat{\mathbf{w}}d\widehat{\mathbf{x}}+\int_{\Omega_n^F}2\mu^F\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{n+1}\right):\epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{w}})d\widehat{\mathbf{x}}$$

$$-\int_{\Omega_n}\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{w}}\right)\widehat{p}^{n+1}d\widehat{\mathbf{x}}+\int_{\Omega_n^S}\frac{\rho_0^S}{J^n}\frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t}\cdot\widehat{\mathbf{w}}d\widehat{\mathbf{x}}+\int_{\Omega_n^S}\widehat{\mathbf{L}}_3\left(\widehat{\mathbf{v}}^{n+1}\right):\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}d\widehat{\mathbf{x}}$$

$$= \int_{\Omega_n^F} \rho^F \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}}$$

$$+ \int_{\Omega_n^S} \frac{\rho_0^S}{J^n} \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}}, \tag{7.1}$$

$$- \int_{\Omega_n} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1}) \widehat{q} d\widehat{\mathbf{x}} = 0, \tag{7.2}$$

$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot (\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\psi}}) d\widehat{\mathbf{x}} = 0, \tag{7.3}$$

for all $\widehat{\mathbf{w}} \in \left(H^1(\Omega_n) \right)^2$, $\widehat{\mathbf{w}} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, for all $\widehat{q} \in L^2(\Omega_n)$ and for all $\widehat{\boldsymbol{\psi}} \in \left(H_0^1 \left(\Omega_n^F \right) \right)^2$, where $\widehat{\mathbf{L}}_3$ is obtained from $\widehat{\mathbf{L}}_1$ by deleting the term with the pressure, i.e.

$$\widehat{\mathbf{L}}_3 \left(\widehat{\mathbf{v}}^{n+1} \right) = \frac{\mu^S}{J^n} \left(\left(\mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right) \mathbf{F}^n \left(\mathbf{F}^n \right)^T - \mathbf{I} \right).$$

The equation (7.1) was obtained by adding (4.9) with (6.2). Using the boundary condition (2.9), the sum of the integrals over the fluid-structure interface in the right-hand side of (4.9) and (6.2) vanishes. The sum of the terms containing pressure in (4.9) and (6.2) is

$$- \int_{\Omega_n^S} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^S \right) \frac{1}{J^n} \widehat{p}^{S,n+1} d\widehat{\mathbf{x}} - \int_{\Omega_n^F} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^F \right) \widehat{p}^{F,n+1} d\widehat{\mathbf{x}}.$$

Only in the first term, we replace J^n by 1, to obtain $- \int_{\Omega_n} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}) \widehat{p}^{n+1} d\widehat{\mathbf{x}}$. For the other terms concerning the structure, we keep J^n in order to apply Theorem 5.1. The equation (7.2) combines (4.7) and (6.3). The equation (7.3) is the weak version of (6.4).

Algorithm 7.1.

Time advancing scheme from n to $n+1$.
We assume that we know Ω_n, Γ_n, $\mathbf{v}^n$.

Step 1: Solve the non-linear system (7.1)-(7.3) written in Ω_n and get the velocity $\widehat{\mathbf{v}}^{n+1}$, the pressure $\widehat{p}^{n+1}$ and the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1}$.

Step 2: Define the map $\mathbb{T}_n : \overline{\Omega}_n \to \mathbb{R}^2$ by:

$$\mathbb{T}_n(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + (\Delta t) \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}) \chi_{\Omega_n^F}(\widehat{\mathbf{x}}) + (\Delta t) \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}) \chi_{\Omega_n^S}(\widehat{\mathbf{x}}),$$

where $\chi_{\Omega_n^F}$ and $\chi_{\Omega_n^S}$ are the characteristic functions of fluid and structure domains.

Step 3: We set $\Omega_{n+1} = \mathbb{T}_n(\Omega_n)$, $\Gamma_{n+1} = \mathbb{T}_n(\Gamma_n)$. We define $\mathbf{v}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$, $p^{n+1} : \Omega_{n+1} \to \mathbb{R}$ by:

$$\mathbf{v}^{n+1}(\mathbf{x}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}), \ p^{n+1}(\mathbf{x}) = \widehat{p}^{n+1}(\widehat{\mathbf{x}}), \ \forall \widehat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}})$$

and $\boldsymbol{\vartheta}^{n+1} : \Omega_{n+1}^F \to \mathbb{R}^2$ by

$$\boldsymbol{\vartheta}^{n+1}(\mathbf{x}) = \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}), \ \forall \widehat{\mathbf{x}} \in \Omega_n^F \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}}).$$

The Lagrangian structure displacement and velocity are defined by

$$\mathbf{U}^{S,n+1}(\mathbf{X}) = \mathbf{U}^{S,n}(\mathbf{X}) + \Delta t \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}), \quad \mathbf{V}^{S,n+1}(\mathbf{X}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}})$$

for all $\mathbf{X} \in \Omega_0^S$ and $\widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X})$.

Theorem 7.1. *The **Algorithm 7.1** verifies*

$$\frac{1}{2} \int_{\Omega_{n+1}^F} \rho^F \left| \mathbf{v}^{F,n+1} \right|^2 d\mathbf{x} + (\Delta t) \sum_{k=0}^{n} \int_{\Omega_k^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) : \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) d\widehat{\mathbf{x}}$$

$$+ \frac{1}{2} \int_{\Omega_0^S} \rho_0^S \left| \mathbf{V}^{S,n+1} \right|^2 d\mathbf{X} + \frac{1}{2} \int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \mathbf{F}^{n+1} d\mathbf{X}$$

$$\leq \frac{1}{2} \int_{\Omega_0^F} \rho^F \left| \mathbf{v}^{F,0} \right|^2 d\mathbf{X} + \frac{1}{2} \int_{\Omega_0^S} \rho_0^S \left| \mathbf{V}^{S,0} \right|^2 d\mathbf{X} + \frac{1}{2} \int_{\Omega_0^S} \mu^S \mathbf{F}^0 : \mathbf{F}^0 d\mathbf{X} \qquad (7.4)$$

if g, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$ are zero and $\int_{\Sigma_1 \cup \Sigma_3} (\widehat{\mathbf{v}}^{F,n+1} \cdot \mathbf{n}^F) |\widehat{\mathbf{v}}^{F,n+1}|^2 \geq 0$.

Proof. We put $\widehat{\mathbf{w}} = (\Delta t)\widehat{\mathbf{v}}^{n+1}$ in (7.1) and we follow the proofs of Theorems 5.1 and 6.1. We obtain

$$\frac{1}{2} \int_{\Omega_{n+1}^F} \rho^F \left| \mathbf{v}^{F,n+1} \right|^2 d\mathbf{x} + (\Delta t) \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,n+1}) : \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,n+1}) d\widehat{\mathbf{x}}$$

$$+ \frac{1}{2} \int_{\Omega_0^S} \rho_0^S \left| \mathbf{V}^{S,n+1} \right|^2 d\mathbf{X} + \frac{1}{2} \int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \mathbf{F}^{n+1} d\mathbf{X}$$

$$\leq \frac{1}{2} \int_{\Omega_n^F} \rho^F \left| \mathbf{v}^{F,n} \right|^2 d\widehat{\mathbf{x}} + \frac{1}{2} \int_{\Omega_0^S} \rho_0^S \left| \mathbf{V}^{S,n} \right|^2 d\mathbf{X} + \frac{1}{2} \int_{\Omega_0^S} \mu^S \mathbf{F}^n : \mathbf{F}^n d\mathbf{X}.$$

As before, by replacing in the above inequality n by k and summing for $0 \leq k \leq n$, we get (7.4). $\square$

Remark 7.1. For arbitrary g, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$, the conclusion of the before Theorem keeps true. We can adapt the proofs from [17], Chapter 6, for example.

Compared to the explicit methods [4], here there is no specific condition on the time step.

The system (7.1)–(7.3) is non-linear. In the following, we introduce a linear approximation of this system. Also, we shall decouple the equation for solving $\widehat{\boldsymbol{\vartheta}}^{n+1}$, from the system for solving $\widehat{\mathbf{v}}^{n+1}$, $\widehat{p}^{n+1}$. The incompressibility in the structure domain will be treated by penalization.

Algorithm 7.2.

We assume that we know Ω_n, Γ_n, $\mathbf{v}^n$, $\boldsymbol{\vartheta}^n$.

Step 1-1: Find the velocity $\widehat{\mathbf{v}}^{n+1} \in \left(H^1(\Omega_n)\right)^2$, $\widehat{\mathbf{v}}^{n+1}=0$ on $\Sigma_2\cup\Sigma_4\cup\Sigma_5$, the pressure $\widehat{p}^{F,n+1}\in L^2(\Omega_n^F)$, such that:

$$\int_{\Omega_n^F}\rho^F\frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Omega_n^F}\rho^F\left(((\mathbf{v}^n-\boldsymbol{\vartheta}^n)\cdot\nabla_{\widehat{\mathbf{x}}})\widehat{\mathbf{v}}^{n+1}\right)\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}$$

$$+\int_{\Omega_n^F}2\mu^F\epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{n+1}\right):\epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{w}})\,d\widehat{\mathbf{x}}$$

$$-\int_{\Omega_n^F}(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{w}})\widehat{p}^{F,n+1}\,d\widehat{\mathbf{x}}+\int_{\Omega_n^S}\frac{1}{\varepsilon}\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{n+1}\right)\left(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{w}}\right)d\widehat{\mathbf{x}}$$

$$+\int_{\Omega_n^S}\rho_0^S\frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Omega_n^S}\widehat{\mathbf{L}}_4\left(\widehat{\mathbf{v}}^{n+1}\right):\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}$$

$$=\int_{\Omega_n^F}\rho^F\frac{\mathbf{v}^n}{\Delta t}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Omega_n^F}\rho^F\mathbf{g}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Sigma_1}\mathbf{h}_{in}^{n+1}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Sigma_3}\mathbf{h}_{out}^{n+1}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}$$

$$+\int_{\Omega_n^S}\rho_0^S\frac{\mathbf{v}^n}{\Delta t}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}}+\int_{\Omega_n^S}\rho_0^S\mathbf{g}\cdot\widehat{\mathbf{w}}\,d\widehat{\mathbf{x}},\tag{7.5}$$

$$-\int_{\Omega_n^F}(\nabla_{\widehat{\mathbf{x}}}\cdot\widehat{\mathbf{v}}^{n+1})\widehat{q}^F\,d\widehat{\mathbf{x}}=0,\tag{7.6}$$

for all $\widehat{\mathbf{w}}\in\left(H^1(\Omega_n)\right)^2$, $\widehat{\mathbf{w}}=0$ on $\Sigma_2\cup\Sigma_4\cup\Sigma_5$, for all $\widehat{q}^F\in L^2(\Omega_n^F)$, where $\widehat{\mathbf{L}}_4$ is obtained from $\widehat{\mathbf{L}}_2$ by deleting the penalization term, i.e.

$$\widehat{\mathbf{L}}_4\left(\widehat{\mathbf{v}}^{n+1}\right)=\mu^S\left(\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T-\mathrm{cof}\left(\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\right).$$

Step 1-2: Find the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1}\in\left(H^1(\Omega_n^F)\right)^2$, $\widehat{\boldsymbol{\vartheta}}^{n+1}=\widehat{\mathbf{v}}^{F,n+1}$ on Γ_n, $\widehat{\boldsymbol{\vartheta}}^{n+1}=0$ on $\bigcup_{i=1}^5\Sigma_i$ such that
$$\int_{\Omega_n^F}(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\vartheta}}^{n+1})\cdot(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\psi}})\,d\widehat{\mathbf{x}}=0$$

for all $\widehat{\boldsymbol{\psi}}\in\left(H_0^1(\Omega_n^F)\right)^2$.

The **Steps 2, 3** are the same as in **Algorithm 7.1**.

Remark 7.2. Starting from the non-linear system (7.1)-(7.3), we have obtained the linear system (7.5)-(7.6), just by treated the convection term semi-implicitly and by deleting the term containing $\det\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\boldsymbol{\vartheta}}^{n+1}\right)$.

8   A numerical test

The numerical tests have been produced using *FreeFem++* (see [10]). We have tested the benchmark FSI3 from [25], the geometric configuration is presented in Figure 1.

The fluid domain has the length $L=2.5m$ and the width $H=0.41m$. The fluid dynamic viscosity is $\mu^F=1Kg/(ms)$ and the mass density is $\rho^F=1000Kg/m^3$.

The rectangular flexible structure is of length $\ell=0.35m$, thickness $h=0.02m$, mass density $\rho^S=1000Kg/m^3$ and $\mu^S=2\times10^6Kg/(ms^2)$. It is attached to the fixed cylinder of center $(x_C,y_C)=(0.2,0.2)m$ and radius $r=0.5m$. The point A is at the middle of the right side of the flexible structure.

The boundary condition at the inflow Σ_1 is $\mathbf{v}=\mathbf{v}_{in}$, with

$$\mathbf{v}_{in}(x_1,x_2,t)=\begin{cases}\left(1.5\overline{U}\frac{x_2(H-x_2)}{(H/2)^2}\frac{(1-\cos(\pi t/2))}{2},0\right), & (x_1,x_2)\in\Sigma_1,0\le t\le 2\\\left(1.5\overline{U}\frac{x_2(H-x_2)}{(H/2)^2},0\right), & (x_1,x_2)\in\Sigma_1,2\le t\le T=12\end{cases}$$

and $\overline{U}=1.8$. We have imposed the no-slip boundary condition $\mathbf{v}=\mathbf{0}$ at Σ_2, Σ_4, Σ_5. At the outflow Σ_3, we set the traction free $\sigma^F(\mathbf{v},p)\mathbf{n}^F=0$. The fluid and the structure are at rest, initially.
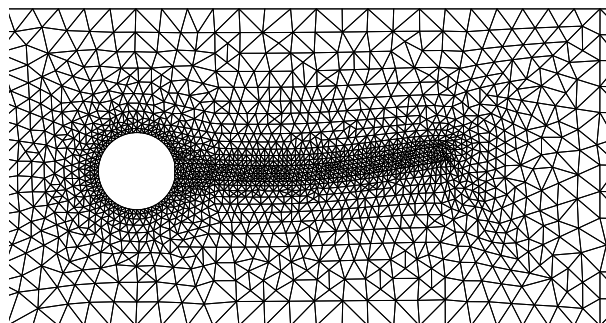

Figure 2: Details of the fluid-structure mesh at time instant $t=9.05$.

We use a mesh of 2604 vertices and 4964 triangles, we can see a zoom in Figure 2. For the time step Δt, we have employed 0.005, 0.002 and 0.001. We have used the **Algorithm 7.2**, the structure is written in the updated Lagrangian formulation and the imcompressibility condition is treated by penalization, where $\varepsilon=10^{-4}$. We have employed the triangular finite element $\mathbb{P}_1+bubble$ for the approximation of the fluid-structure velocity and
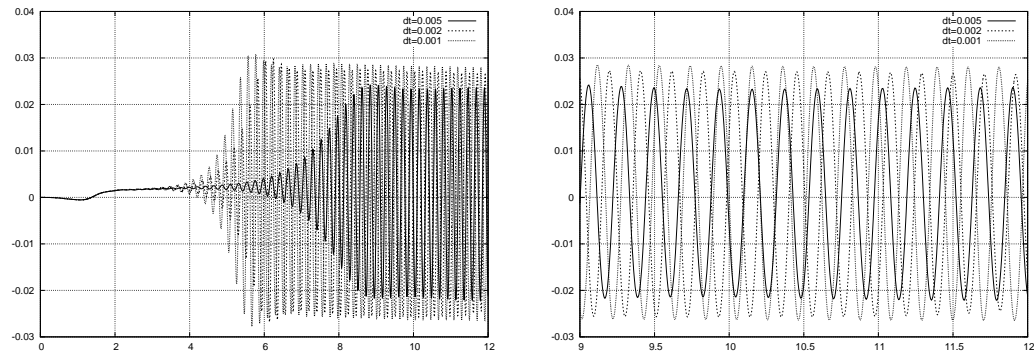
Figure 3: Time history of the vertical displacement of the point A for $\Delta t = 0.005, 0.002, 0.001$ (left) and a detail in the time interval $[9,12]$ (right).
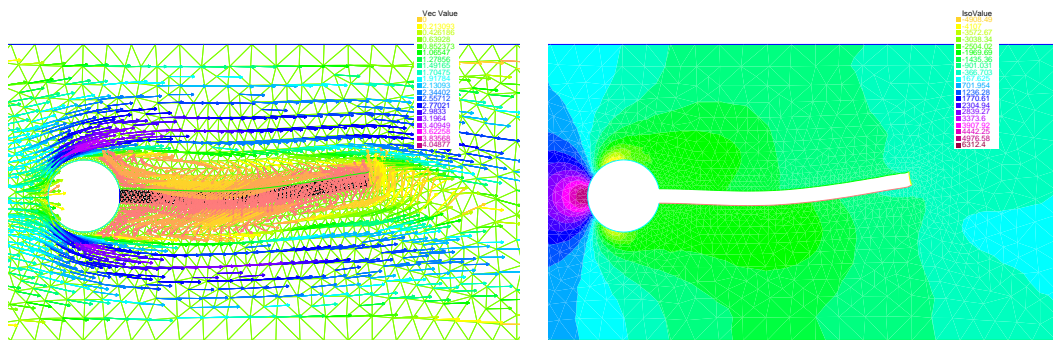

Figure 4: Fluid-structure velocity (left) and fluid pressure (right) at $t = 9.05$.

$\mathbb{P}_1$ for the pressure. We point out that, we have to use for the solid part the same finite element as for the fluid part.

After a transient period, the structure oscillates periodically, see Figure 3. For $\Delta t = 0.005$, the amplitude is 0.022 m, the frequency 4.72 Hz, for $\Delta t = 0.002$, the amplitude is 0.025 m, the frequency 4.77 Hz, for $\Delta t = 0.001$, the amplitude is 0.027 m, the frequency 5 Hz. We observe that for smaller time step, the oscillations start before. In [16], for $\Delta t = 0.002$, the amplitude is 0.03 m, the frequency 5 Hz in the case of a compressible neo-hookean structure.

For the boundary condition at the inflow Σ_1, we have used $\overline{U} = 1.8$ which is smaller that $\overline{U} = 2$ employed in [6, 11]. The reason is that for $\overline{U} = 2$, in the zone of the right end of the structure, some triangles of the ALE mesh became flat. To avoid this, it is possible to use ALE framework with remeshing as in [15], but we shall treat this in a forcoming paper.

For $\Delta t = 0.005$, in [6] the amplitude is 0.016 m, the frequency 6.86 Hz and in [11] the amplitude is 0.03 m, the frequency 5.4 Hz. In [6], the Eulerian mesh is adapted at each time step to the structure position and in [11], the fluid mesh is generated at each time step in order to fit to the fluid-structure interface.

In Figure 4, we can see the velocity field for the fluid-structure and the fluid pressure. We observe that, there is no pressure field in the structure domain. If we want, we can interpret $-\frac{1}{\varepsilon}\left(\nabla\cdot\mathbf{v}^{n+1}\right)$ as the structure pressure.

Algorithm 7.2 is fast, about 3000 iterations by hour on a PC.

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