An Equivalent Characterization of $CMO(\mathbb{R}^n)$ with $A_p$ Weights

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Abstract. Let $1 < p < \infty$ and $\omega \in A_p$. The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C^\infty_0(\mathbb{R}^n)$. In this paper, an equivalent characterization of $CMO(\mathbb{R}^n)$ with $A_p$ weights is established.

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1 Introduction

The goal of this paper is to provide an equivalent characterization of $CMO(\mathbb{R}^n)$, which is useful in the study of compactness of commutators of singular integral operator and fractional integral operator.

The space $BMO(\mathbb{R}^n)$ is defined by the set of functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$||f||_{BMO(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} M(f, Q) < \infty,$$

where

$$M(f, Q) := \frac{1}{|Q|} \int_Q |f(x) - f_Q|dx, \quad f_Q := \frac{1}{|Q|} \int_Q f(x)dx.$$

The space $CMO(\mathbb{R}^n)$ is the closure in $BMO(\mathbb{R}^n)$ of the set of $C^\infty_0(\mathbb{R}^n)$, which is a proper subspace of $BMO(\mathbb{R}^n)$.

In fact, it is known that $CMO(\mathbb{R}^n) = VMO_0(\mathbb{R}^n)$, where $VMO_0(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in $BMO(\mathbb{R}^n)$, see [2, 3, 9]. Here $C_0(\mathbb{R}^n)$ is the set of continuous functions on $\mathbb{R}^n$ which vanish at infinity. Neri [8] gave a characterization of $CMO(\mathbb{R}^n)$ by Riesz transforms. Meanwhile, Neri proposed the following characterization of $CMO(\mathbb{R}^n)$ and its proof was established by Uchiyama in his remarkable work [11].

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Theorem 1.1. Let $f \in \text{BMO}(\mathbb{R}^n)$. Then $f \in \text{CMO}(\mathbb{R}^n)$ if and only if $f$ satisfies the following three conditions

(a) $\lim_{a \to 0} \sup_{|Q|=a} M(f, Q) = 0$;
(b) $\lim_{a \to \infty} \sup_{|Q|=a} M(f, Q) = 0$;
(c) $\lim_{|x| \to \infty} M(f, Q+x) = 0$ for each cube $Q \subset \mathbb{R}^n$, where $Q+x := \{ y+x : y \in Q \}$.

Recently, Guo, Wu and Yang [6] established an equivalent characterization of space $\text{CMO}(\mathbb{R}^n)$ by local mean oscillations. Lots of works about space $\text{CMO}(\mathbb{R}^n)$ have been studied, see [4] for example. Muckenhoupt and Wheeden [7, Theorem 5] showed the norm of $\text{BMO}_\omega(\mathbb{R}^n)$ (see Definition 1.2) is equivalent to the norm of $\text{BMO}(\mathbb{R}^n)$, where the weight function $\omega$ is Muckenhoupt $A_p$ weight. So it is natural to consider equivalent characterizations of $\text{CMO}(\mathbb{R}^n)$ associated to $A_p$ weights.

To state our main results, we first recall some relevant notions and notations.

The following class of $A_p$ was introduced in [1, 5].

Definition 1.1. Let $\omega(x) \geq 0$ and $\omega(x) \in L^1_{\text{loc}}(\mathbb{R}^n)$. For $1 < p < \infty$, we say that $\omega(x) \in A_p$ if there exists a constant $C > 0$ such that for any cube $Q$,

$$
\left( \frac{1}{|Q|} \int_Q \omega(x)^p dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-\frac{p}{p-1}} dx \right)^{p-1} \leq C.
$$

Also, for $p=1$, we say that $\omega(x) \in A_1$ if there is a constant $C > 0$ such that

$$
M \omega(x) \leq C \omega(x),
$$

where $M$ is the Hardy-Littlewood maximal operator. For $p \geq 1$, the smallest constant appearing in (1.1) and (1.2) is called the $A_p$ characteristic constant of $\omega$ and is denoted by $[\omega]_{A_p}$.

Definition 1.2. Let $\omega \in A_p$. For a cube $Q$ in $\mathbb{R}^n$, we say a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is in $\text{BMO}_\omega(\mathbb{R}^n)$ if $f$ satisfies

$$
\|f\|_{\text{BMO}_\omega(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} M(f, Q)_\omega < \infty,
$$

where

$$
m(f, Q)_\omega := \frac{1}{\omega(Q)} \int_Q f(x) \omega(x) dx,
$$

$$
M(f, Q)_\omega := \frac{1}{\omega(Q)} \int_Q |f(x) - m(f, Q)_\omega| \omega(x) dx.
$$
Let $\omega \in A_p (p \geq 1), q > 1, f \in L^1_{\text{loc}} (\mathbb{R}^n)$. Then $BMO_{\omega,q} (\mathbb{R}^n)$ is defined by
\[
\|f\|_{BMO_{\omega,q} (\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} M(f,Q)_{\omega,q} < \infty,
\]
where
\[
M(f,Q)_{\omega,q} := \left( \frac{1}{\omega(Q)} \int_Q |f(x) - m(f,Q)_{\omega}|^q \omega(x) dx \right)^{1/q}.
\]

Now, we can formulate our main results as follows.

**Theorem 1.2.** Let $p \geq 1, 1 < q < \infty$. Suppose $f \in BMO (\mathbb{R}^n)$ and $\omega \in A_p$. Then the following conditions are equivalent:

1. $f \in CMO (\mathbb{R}^n)$;
2. $f$ satisfies the following three conditions:
   (i) $\limsup_{a \to 0, |Q| = a} M(f,Q)_{\omega,q} = 0$,
   (ii) $\limsup_{a \to \infty, |Q| = a} M(f,Q)_{\omega,q} = 0$,
   (iii) $\lim_{|x| \to \infty} M(f,Q+x)_{\omega,q} = 0$ for each $Q \subset \mathbb{R}^n$.
3. $f$ satisfies the following three conditions:
   (i') $\limsup_{a \to 0, |Q| = a} M(f,Q)_{\omega} = 0$,
   (ii') $\limsup_{a \to \infty, |Q| = a} M(f,Q)_{\omega} = 0$,
   (iii') $\lim_{|x| \to \infty} M(f,Q+x)_{\omega} = 0$ for each $Q \subset \mathbb{R}^n$.

Throughout this paper, the letter $C$, will stand for positive constants, not necessarily the same one at each occurrence, but independent of the essential variables. If $f \leq C g$, we write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \sim g$. A dyadic cube $Q$ on $\mathbb{R}^n$ is a cube of the form
\[
\left\{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : k_i 2^j \leq x_i < (k_i + 1) 2^j, i = 1, \cdots, n, k_i \in \mathbb{Z}, j \in \mathbb{Z} \right\},
\]
$R_j$ means $\{ x \in \mathbb{R}^n : |x| < 2^j, i = 1, 2, \cdots, n \}$. For $\lambda > 0$, $\lambda Q$ denotes the cube with the same center as $Q$ and side-length $\lambda$ times the side-length of $Q$.

## 2 The proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. To do this, we firstly recall some auxiliary lemmas. Note that [7, Theorem 3] implies the following weighted John-Nirenberg inequalities, also see [1, 10].
Lemma 2.1. **(John-Nirenberg)** Let \( p \in [1, \infty) \), \( \omega \in A_p \) and \( f \in \text{BMO}_\omega(\mathbb{R}^n) \). For every \( \alpha > 0 \) and cube \( Q \), there exist constants \( c_1 \) and \( c_2 \) such that

\[
\omega(\{ x \in Q : |f(x) - f_Q| > \alpha \}) < c_1 e^{-\frac{\alpha}{c_2 \|f\|_{\text{BMO}_\omega(\mathbb{R}^n)}}} \omega(Q).
\]

Next, we recall some useful properties of \( A_p \) weights.

Lemma 2.2 ([5]). Let \( \omega \in A_p \) and \( 1 \leq p < \infty \).

1. There exist \( 0 < \delta < 1 \) and \( C > 0 \) that depending only on the dimension \( n \), \( p \), and \( [\omega]_{A_p} \) such that for any cube \( Q \) and any measurable subset \( S \) of \( Q \) we have

\[
\frac{\omega(S)}{\omega(Q)} \leq C \left( \frac{|S|}{|Q|} \right)^{\delta}.
\] (2.1)

2. There exist constants \( C \) and \( \gamma > 0 \) that depending only on the dimension \( n \), \( p \), and \( [\omega]_{A_p} \) such that for every cube \( Q \) we have

\[
\left( \frac{1}{|Q|} \int_Q \omega(x)^{1+\gamma} dx \right)^{\frac{1}{1+\gamma}} \leq C \int_Q \omega(x) dx.
\] (2.2)

3. For all \( \lambda > 1 \), and all cubes \( Q \),

\[
\omega(\lambda Q) \leq \lambda^{np} [\omega]_{A_p} \omega(Q).
\] (2.3)

Now, we are in position to prove the Theorem 1.2.

**Proof.** To prove (1) \( \Rightarrow \) (2) in Theorem 1.2. Assume that \( f \in \text{CMO}(\mathbb{R}^n) \). If \( f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \), then (i) \( - \) (iii) hold. It is obvious that (i) holds for uniformly continuous functions \( f \). Without loss of generality, we assume \( \text{supp}(f) \subset Q_0 \). Then for each \( Q \subset \mathbb{R}^n \), there exists \( h \in \mathbb{R}^n \), for \( |x| > |h| \), we have \( Q_0 \cap (Q + x) = \emptyset \), (iii) holds.

Note that

\[
\left( \frac{1}{\omega(Q)} \int_Q |f(x) - m(f,Q)_\omega|^q \omega(x) dx \right)^{1/q} \leq \left( \frac{1}{\omega(Q)} \int_{\mathbb{R}^n} |f(x) - m(f,Q)_\omega|^q \omega(x) dx \right)^{1/q}.
\]

For \( f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \), we have

\[
\left( \int_{\mathbb{R}^n} |f(x) - m(f,Q)_\omega|^q \omega(x) dx \right)^{1/q} < \infty.
\]

On the other hand, \( Q(0,r) \) denotes the closed cube centered at 0 with side-length \( r \). For any \( x_0 \in Q(0,r) \), there exists a cube \( Q \) centered at \( x_0 \) such that \( Q(0,r) \subset Q \), by (2.1), we get

\[
\frac{1}{\omega(Q)} \int_Q |f(x) - m(f,Q)_\omega|^q \omega(x) dx \lesssim \frac{1}{\omega(Q(0,r))} \left( \frac{|Q(0,r)|}{|Q|} \right)^{\delta},
\]
which tends to 0 as $|Q|$ tends to $+\infty$, (ii) holds.

If $f \in \text{CMO}(\mathbb{R}^n) \setminus \mathcal{C}^\infty_0(\mathbb{R}^n)$, for any given $\varepsilon > 0$, there exists $f_\varepsilon \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ satisfying (i)−(iii) and $\|f - f_\varepsilon\|_{\text{BMO}(\mathbb{R}^n)} < \varepsilon$. Then by Lemma 2.1 and (2.2), for $\omega \in A_p$, $1 < p < \infty$, it is easy to see

$$\|f - f_\varepsilon\|_{\text{BMO}_\omega(\mathbb{R}^n)} \lesssim \|f - f_\varepsilon\|_{\text{BMO}_\omega(\mathbb{R}^n)} \lesssim \|f - f_\varepsilon\|_{\text{BMO}(\mathbb{R}^n)} \lesssim \varepsilon. \quad (2.4)$$

The detailed proof of (2.4) also can be found in [1,7]. By (2.4) and the triangle inequality, we deduce that (i)−(iii) hold for $f$.

The proof of (2) $\Rightarrow$ (3). By the Hölder inequality, we get

$$\frac{1}{\omega(|Q|)} \int_Q |f(x) - m(f, Q)x| |\omega(x)| dx \leq \frac{1}{\omega(|Q|)} \left( \int_Q |f(x) - m(f, Q)x|^{q_0} |\omega(x)| dx \right)^{1/q_0} \left( \int_Q |\omega(x)| dx \right)^{1/q'} = \left( \frac{1}{|Q|} \right)^{1/q} \left( \int_Q |f(x) - m(f, Q)x|^{q_0} |\omega(x)| dx \right)^{1/q}, \quad (2.5)$$

where $1/q + 1/q' = 1$.

It follows from (2.5) that if $f$ satisfies (i)−(iii) then $f$ satisfies (i')−(iii').

The proof of (3) $\Rightarrow$ (1). Now we show that if $f$ satisfies (i')−(iii') then for all $\varepsilon > 0$, there exists $g_\varepsilon \in \text{BMO}(\mathbb{R}^n)$ such that

$$\inf_{h \in \mathcal{C}^\infty_0(\mathbb{R}^n)} \|g_\varepsilon - h\|_{\text{BMO}_\omega(\mathbb{R}^n)} < C_\varepsilon \varepsilon, \quad (2.6)$$
$$\|g_\varepsilon - f\|_{\text{BMO}_\omega(\mathbb{R}^n)} < C_\varepsilon \varepsilon. \quad (2.7)$$

We prove (2.6) and (2.7) by the following two steps.

**Step 1** By (i') and (ii'), there exist $i_\varepsilon$ and $k_\varepsilon$ such that

$$\sup \{M(f, Q)_\omega : |Q| \leq 2^{n(i_\varepsilon + 8)} \} < \varepsilon, \quad (2.8)$$
$$\sup \{M(f, Q)_\omega : |Q| \geq 2^{nk_\varepsilon} \} < \varepsilon. \quad (2.9)$$

By (iii'), there exists $j_\varepsilon > k_\varepsilon$ such that

$$\sup \{M(f, Q)_\omega : Q \cap R_{j_\varepsilon} = \emptyset \} < \varepsilon. \quad (2.10)$$

Now for each $x \in R_{j_\varepsilon}$, we take dyadic cube $Q_x$ with side-length $2^{j_\varepsilon}$ containing $x$; if $x \in R_m \setminus R_{m-1}$ ($j_\varepsilon < m$), $Q_x$ means a dyadic cube of side-length $2^{i_\varepsilon + m - j_\varepsilon}$. Set $g_\varepsilon'(x) = M(f, Q_x)_\omega$, by (ii'), there exists $m_\varepsilon > j_\varepsilon$ such that

$$\sup \{|g_\varepsilon'(x) - g_\varepsilon'(y)| : x, y \in R_{m_\varepsilon} \setminus R_{m_\varepsilon - 1} \} < \varepsilon. \quad (2.11)$$
For any \( Q \), since
\[
M(f, R_{m+1}) < \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)}
\]  
for some positive constant \( C_1 \).

To see this, by \( (ii') \), let \( m \geq j_e + k_e - i_e \) be large enough such that when \( \omega(R_{m}) \geq 2^{n(m+k_e-j_e)} \),
\[
M(f, R_{m+1}) < \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)}
\]
(2.12)

This together with (2.12) and (2.3) imply that
\[
2^{j\varepsilon - l\varepsilon} Q \subset R_{m+1} \subset 8 \cdot 2^{j\varepsilon - l\varepsilon} Q.
\]

For \( x \in R_{m_e} \setminus R_{m_e - 1} \), it is obvious that
\[
|f(x)| \geq \omega(R_{m_e}) \geq 2^{n(2^{m_e} + k_e - j_e)}.
\]

By (2.13), (2.14) and (2.12), we conclude that for any \( x \) \in \( R_{m_e} \setminus R_{m_e - 1} \), it is obvious that
\[
2^{j\varepsilon - l\varepsilon} Q \subset R_{m+1} \subset 8 \cdot 2^{j\varepsilon - l\varepsilon} Q.
\]

This together with (2.12) and (2.3) imply that
\[
|m(f, 2^{j\varepsilon - l\varepsilon} Q) - m(f, R_{m+1})| \leq \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)} \leq \frac{\varepsilon}{8}.
\]  
(2.13)

Since \( Q \subset R_{m_e} \setminus R_{m_e - 1} \), by (2.3) and (2.12), we have
\[
|m(f, R_{m+1}) - m(f, R_{m_e} \setminus R_{m_e - 1})| \leq \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)} \leq \frac{\varepsilon}{8}.
\]  
(2.14)

By (2.13), (2.14) and (2.12), we conclude that for any \( Q \) with \( x \in R_{m_e} \setminus R_{m_e - 1} \),
\[
|m(f, Q) - m(f, R_{m_e} \setminus R_{m_e - 1})| \leq \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)} \leq \frac{\varepsilon}{8}.
\]  
(2.15)

For any \( Q, Q' \subset R_{m_e} \setminus R_{m_e - 1} \), by (2.15), we get
\[
|m(f, Q) - m(f, Q')| \leq \frac{\varepsilon}{C_1(j\varepsilon - l\varepsilon + 1)} \leq \frac{\varepsilon}{2}.
\]
Step II Define \( g_\varepsilon(x) = g_\varepsilon'(x) \) when \( x \in R_{m_i} \) and \( g_\varepsilon(x) = m(f, R_{m_i} \setminus R_{m_i-1})_\omega \) when \( x \in R_{m_i}^c \). Notice that

\[
\text{if } Q_x \cap Q_y \neq \emptyset, \quad \text{diam } Q_x \leq 2\text{diam } Q_y.
\]

By the definition of \( i_x, j_x \) and \( m_x \), if \( Q_x \cap Q_y \neq \emptyset \) or \( x, y \in R_{m_i-1}^c \), there exists \( C_2 > 0 \) such that

\[
|g_\varepsilon(x) - g_\varepsilon(y)| < C_2 \varepsilon.
\]

In fact, assume that \(|x| < |y|\). Firstly, we show that if \( x, y \in R_{m_i-1}^c \), then (2.17) holds. By noting that \( x, y \in R_{m_i}^c \), we get

\[
g_\varepsilon(x) = g_\varepsilon(y) = m(f, R_{m_i} \setminus R_{m_i-1})_\omega
\]

and (2.17) holds. Next, if \( x, y \in R_{m_i} \setminus R_{m_i-1} \), we deduce from (2.11) that

\[
|g_\varepsilon(x) - g_\varepsilon(y)| = |g_\varepsilon'(x) - g_\varepsilon'(y)| < \varepsilon.
\]

Thirdly, if \( x \in R_{m_i} \setminus R_{m_i-1} \) and \( y \in R_{m_i}^c \) (2.15) indicates that

\[
|g_\varepsilon(x) - g_\varepsilon(y)| = |m(f, Q_x)_\omega - m(f, R_{m_i} \setminus R_{m_i-1})_\omega| < \varepsilon.
\]

Now we show if \( Q_x \cap Q_y \neq \emptyset \), then (2.17) holds. We assume that \( Q_x \neq Q_y \) and let \( Q \) be the smallest cube containing \( Q_x \) and \( Q_y \), then \( Q \subset 4Q_x \). If \( x, y \in R_{j_x} \), then

\[
Q_x, Q_y \subset R_{j_x} \quad \text{and} \quad |Q| < 2^{n(i_x+4)},
\]

by (2.8), (2.17) holds. Similarly, if \( Q_x \subset R_{j_x} \), \( Q_y \subset R_{m_i-1} \) and \( Q_x \cap Q_y \neq \emptyset \), by (2.8), (2.17) also holds. If \( x, y \in R_{m_i-1}^c \), notice that \( Q \cap R_{j_x} = \emptyset \) and by (2.10),

\[
|g_\varepsilon'(x) - m(f, Q)_\omega| \lesssim \frac{\omega(Q)}{\omega(Q_x)} M(f, Q)_\omega \lesssim \varepsilon.
\]

Similarly, we have

\[
|g_\varepsilon'(y) - m(f, Q)_\omega| \lesssim \varepsilon.
\]

Hence

\[
|g_\varepsilon'(x) - g_\varepsilon'(y)| \lesssim \varepsilon + \varepsilon = 2\varepsilon.
\]

Combining these cases, (2.17) holds.

We turn to prove that \( g_\varepsilon \) satisfies (2.6). Set

\[
\tilde{h}_\varepsilon(x) := g_\varepsilon(x) - m(f, R_{m_i} \setminus R_{m_i-1})_\omega.
\]

By the definition of \( g_\varepsilon \), we get

\[
\tilde{h}_\varepsilon(x) = 0 \quad \text{for any } x \in R_{m_i}, \quad \|\tilde{h}_\varepsilon - g_\varepsilon\|_{BMO_\omega(\mathbb{R}^n)} = 0.
\]
Moreover, if \( \bar{Q}_x \cap \bar{Q}_y \neq \emptyset \) or \( x, y \in R_{m_i-1}^c \), by (2.17), we have

\[
|\tilde{h}_x(x) - \tilde{h}_y(y)| = |g_\epsilon(x) - g_\epsilon(y)| < C_2 \epsilon.
\]

Observe that \( \text{supp}(\tilde{h}_x) \subset R_{m_i} \). Take a positive valued function \( \varphi(x) \in C^\infty_c(\mathbb{R}^n) \) supported in \( B(0,1) \) and \( \int_{\mathbb{R}^n} \varphi(x) dx = 1 \). For \( t > 0 \), set

\[
\varphi_t(x) = \frac{1}{t^n} \varphi\left(\frac{x}{t}\right).
\]

Select \( t < 2^i \), then

\[
|\varphi_t \ast \tilde{h}_x(x) - \tilde{h}_x(x)| \lesssim \int_{\mathbb{R}^n} \varphi_t(y)|\tilde{h}_x(x-y) - \tilde{h}_x(x)| dy = \int_{\mathbb{R}^n} \varphi(u)|\tilde{h}_x(x-tu) - \tilde{h}_x(x)| du \lesssim \sup_{u \in \mathbb{R}^n} |\tilde{h}_x(x-tu) - \tilde{h}_x(x)|,
\]

where in the second inequality we make the change of variable \( y = ut \).

Since \( u \in B(0,1) \) and \( t < 2^i \), \( \forall x \in \mathbb{R}^n \),

\[
|(x-tu) - x| = |tu| < 2^i.
\]

By (2.17), if \( x, x-tu \in R_{m_i} \), \( \bar{Q}_x \cap \bar{Q}_{x-tu} \neq \emptyset \), hence

\[
|\tilde{h}_x(x-tu) - \tilde{h}_x(x)| < C_2 \epsilon.
\]

If one of \( x \) and \( x-tu \) in \( R_{m_i}^c \), the other must be in \( R_{m_i-1}^c \), we also have

\[
|\tilde{h}_x(x-tu) - \tilde{h}_x(x)| < C_2 \epsilon.
\]

Moreover, \( \varphi_t \ast \tilde{h}_x(x) \in C^\infty_c(\mathbb{R}^n) \) and

\[
\|\varphi_t \ast \tilde{h}_x - \tilde{h}_x\|_{\text{BMO}_c(\mathbb{R}^n)} \lesssim \|\varphi_t \ast \tilde{h}_x - \tilde{h}_x\|_{L^\infty(\mathbb{R}^n)} + \epsilon.
\]

Therefore

\[
\|\varphi_t \ast \tilde{h}_x - g_\epsilon\|_{\text{BMO}_c(\mathbb{R}^n)} \lesssim \|\varphi_t \ast \tilde{h}_x - \tilde{h}_x\|_{\text{BMO}_c(\mathbb{R}^n)} + \|\tilde{h}_x - g_\epsilon\|_{\text{BMO}_c(\mathbb{R}^n)} \lesssim \|\varphi_t \ast \tilde{h}_x - \tilde{h}_x\|_{L^\infty(\mathbb{R}^n)} + \epsilon.
\]

We obtain that (2.6) holds.
Now we prove (2.7). By the definition $i_{\varepsilon}$ and $j_{\varepsilon}$ again, we obtain that for any $x \in R_{m_{i}}$,

$$\int_{Q_{x}} |f(y) - g_{\varepsilon}(y)| \omega(y) dy \lesssim \omega(Q_{x}) \varepsilon. \quad (2.18)$$

Indeed,

$$\int_{Q_{x}} |f(y) - g_{\varepsilon}(y)| \omega(y) dy = \int_{Q_{x}} |f(y) - m(f, Q_{x}) \omega| \omega(y) dy.$$ 

If $Q_{x} \cap R_{j_{\varepsilon}} = \emptyset$, by (2.10), (2.18) holds. If $Q_{x} \cap R_{j_{\varepsilon}} \neq \emptyset$, using (2.8), (2.18) holds.

Let $Q$ be a arbitrary cube in $\mathbb{R}^{n}$. In order to prove (2.7) holds, it suffices to show

$$M(f - g_{\varepsilon}, Q)_{\omega} < \varepsilon. \quad (2.19)$$

We consider the following four cases:

Case(i): $Q \subset R_{m_{i}}$ and $\max \{\text{diam } Q_{x} : Q_{x} \cap Q \neq \emptyset\} > 4 \text{diam } Q$, by (2.16), the number of $Q_{x} \cap Q \neq \emptyset$ is finite. If $Q_{x_{i}} \cap Q \neq \emptyset$ and $Q_{x_{j}} \cap Q \neq \emptyset$, $Q_{x_{i}} \cap Q_{x_{j}} \neq \emptyset$, by (2.17),

$$M(g_{\varepsilon}, Q)_{\omega} \lesssim \frac{1}{\omega(Q)} \sum_{Q_{x_{i}} \cap Q \neq \emptyset} \int_{Q_{x_{i}}} |g_{\varepsilon}(x) - m(g_{\varepsilon}, Q)_{\omega}| \omega(x) dx \lesssim \frac{1}{\omega(Q)} \sum_{Q_{x_{i}} \cap Q \neq \emptyset} \int_{Q_{x_{i}}} |g_{\varepsilon}(x) - g_{\varepsilon}(y)| \omega(y) dy \omega(x) dx \lesssim \varepsilon.$$ 

Moreover, if $Q \cap R_{j_{\varepsilon}} \neq \emptyset$, then $|Q| \leq 2^{n(i_{\varepsilon} + 1)}$, by (2.8), we have $M(f, Q)_{\omega} < \varepsilon$; if $Q \cap R_{j_{\varepsilon}} = \emptyset$, by (2.10), we also obtain $M(f, Q)_{\omega} < \varepsilon$. Hence

$$M(f - g_{\varepsilon}, Q)_{\omega} \lesssim M(f, Q)_{\omega} + M(g_{\varepsilon}, Q)_{\omega} \lesssim \varepsilon.$$ 

Case(ii): $Q \subset R_{m_{i}}$ and $\max \{\text{diam } Q_{x} : Q_{x} \cap Q \neq \emptyset\} \leq 4 \text{diam } Q$, we have

$$\bigcup_{Q_{x_{i}} \cap Q \neq \emptyset} Q_{x_{i}} \supset Q, \quad \sum_{Q_{x_{i}} \cap Q \neq \emptyset} \omega(Q_{x_{i}}) \sim \omega(Q).$$

Invoking (2.18), we get

$$M(f - g_{\varepsilon}, Q)_{\omega} \lesssim \frac{2}{\omega(Q)} \sum_{Q_{x_{i}} \cap Q \neq \emptyset} \int_{Q_{x_{i}}} |f(y) - g_{\varepsilon}(y)| \omega(y) dy \lesssim \frac{2}{\omega(Q)} \sum_{Q_{x_{i}} \cap Q \neq \emptyset} \omega(Q_{x_{i}}) \varepsilon \lesssim \varepsilon.$$ 

Case(iii): $Q \subset R_{m_{i} - 1}$, then $Q \cap R_{j_{\varepsilon}} = \emptyset$ and $M(f, Q)_{\omega} < \varepsilon$. Using (2.17),

$$M(g_{\varepsilon}, Q)_{\omega} \lesssim \frac{1}{\omega(Q)} \int_{Q} \frac{1}{\omega(Q)} \int_{Q} |g_{\varepsilon}(x) - g_{\varepsilon}(y)| \omega(y) dy \omega(x) dx < \varepsilon.$$
Hence

\[ M(f - g_\varepsilon, Q)_\omega \lesssim M(f, Q)_\omega + M(g_\varepsilon, Q)_\omega < \varepsilon. \]

Case (iv): \( Q \cap R^c_{m_1} \neq \emptyset \) and \( Q \cap R_{m_1 - 1} \neq \emptyset \). Let \( P_Q \) be a smallest positive number such that \( Q \subset R_{P_Q} \). Then

\[ M(f, Q)_\omega \lesssim M(f, R_{P_Q})_\omega. \]

Moreover,

\[
M(f - g_\varepsilon, R_{P_Q})_\omega \omega(R_{P_Q}) \lesssim \int_{R_{P_Q}} |(f - g_\varepsilon)(x) - m(f - g_\varepsilon, R_{P_Q} \setminus R_{m_1})_\omega| \omega(x) dx
\]

\[
\lesssim \int_{R_{P_Q}} |f(x) - m(f, R_{P_Q} \setminus R_{m_1})_\omega| \omega(x) dx + \int_{R_{P_Q}} |g_\varepsilon(x) - m(g_\varepsilon, R_{P_Q} \setminus R_{m_1})_\omega| \omega(x) dx.
\]

On the one hand, by (2.18), we have

\[
\int_{R_{P_Q}} |f(x) - m(f, R_{P_Q} \setminus R_{m_1})_\omega| \omega(x) dx
\]

\[
\lesssim \int_{R_{P_Q}} |f(x) - m(f, R_{P_Q})_\omega| \omega(x) dx + |m(f, R_{P_Q})_\omega - m(f, R_{P_Q} \setminus R_{m_1})_\omega| \omega(R_{P_Q})
\]

\[
\lesssim \int_{R_{P_Q}} |f(x) - m(f, R_{P_Q})_\omega| \omega(x) dx
\]

\[
\lesssim \omega(R_{P_Q}) \varepsilon.
\]

On the other hand, it is easy to prove that

\[
\sum_{i: Q_i \subset R_{m_1}} \omega(Q_{x_i}) \sim \omega(R_{m_1}).
\]

Combining with (2.9) and (2.18) and the fact that \( g_\varepsilon(x) = g_\varepsilon(y) \) for any \( x, y \in R^c_{m_1} \), we obtain

\[
\int_{R_{P_Q}} |g_\varepsilon(x) - m(g_\varepsilon, R_{P_Q} \setminus R_{m_1})_\omega| \omega(x) dx
\]

\[
\lesssim \frac{1}{\omega(R_{P_Q} \setminus R_{m_1})} \int_{R_{P_Q}} \int_{R_{P_Q} \setminus R_{m_1}} |g_\varepsilon(x) - g_\varepsilon(y)| \omega(y) dy \omega(x) dx
\]

\[
= \frac{1}{\omega(R_{P_Q} \setminus R_{m_1})} \int_{R_{m_1}} \int_{R_{P_Q} \setminus R_{m_1}} |g_\varepsilon(x) - g_\varepsilon(y)| \omega(y) dy \omega(x) dx
\]

\[
\lesssim \frac{1}{\omega(R_{P_Q} \setminus R_{m_1})} \int_{R_{m_1}} \int_{R_{P_Q} \setminus R_{m_1}} \left[ |g_\varepsilon(x) - f(x)| + |f(x) - m(f, R_{m_1} \setminus R_{m_1 - 1})| \omega(y) dy \omega(x) dx
\]

\[
\lesssim \frac{1}{\omega(R_{P_Q} \setminus R_{m_1})} \int_{R_{P_Q} \setminus R_{m_1}} \sum_{i: Q_i \subset R_{m_1}} \int_{Q_{x_i}} |g_\varepsilon(x) - f(x)| \omega(x) dx \omega(y)
\]
\[
\begin{align*}
+ \int_{R_m} & \left[ |f(x) - m(f, R_m)\omega_x| + |m(f, R_m)\omega_x - m(f, R_m \setminus R_{m_k-1})\omega_x| \right] \omega(x) dx \\
\leq & \frac{1}{\omega(R_{P_Q} \setminus R_m)} \int_{R_{P_Q} \setminus R_m} \varepsilon \sum_{Q_{x_i} \subset R_m} \omega(Q_{x_i}) \omega(y) dy \\
+ & \int_{R_m} |f(x) - m(f, R_m)\omega_x| \omega(x) dx \\
\leq & \omega(R_m) \varepsilon + \omega(R_{P_Q}) \varepsilon.
\end{align*}
\]

This implies (2.7) and completes the proof of Theorem 1.2.

References