J. Math. Study
doi: 10.4208/jms.v53n2.20.01

Vol. **53**, No. 2, pp. 125-142
June 2020

(Semi-)Nonrelativisitic Limit of the Nonlinear Dirac Equations

Yongyong Cai[1,2,*] and Yan Wang[3]

[1] *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China*
[2] *Beijing Computational Science Research Center, Beijing 100193, China*
[3] *School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China.*

Received Febuary 5, 2019; Accepted August 14, 2019;
published online April 20, 2020.

Dedicated to Professor Jie Shen on the Occasion of his 60th Birthday

Abstract. We consider the nonlinear Dirac equation (NLD) with time dependent external electro-magnetic potentials, involving a dimensionless parameter $\varepsilon \in (0,1]$ which is inversely proportional to the speed of light. In the nonrelativistic limit regime $\varepsilon \ll 1$ (speed of light tends to infinity), we decompose the solution into the eigenspaces associated with the 'free Dirac operator' and construct an approximation to the NLD with $O(\varepsilon^2)$ error. The NLD converges (with a phase factor) to a coupled nonlinear Schrödinger system (NLS) with external electric potential in the nonrelativistic limit as $\varepsilon \to 0^+$, and the error of the NLS approximation is first order $O(\varepsilon)$. The constructed $O(\varepsilon^2)$ approximation is well-suited for numerical purposes.

AMS subject classifications: 35Q41, 35Q55, 81Q05

Key words: Nonlinear Dirac equation, nonrelativistic limit, error estimates.

1 Introduction

In this paper, we consider the nonlinear Dirac equation (NLD) [3, 6, 14, 15, 23, 31] in the following dimensionless form:

$$\begin{cases} i\partial_t \psi(t,\mathbf{x}) = \left[-\frac{i}{\varepsilon}\sum_{j=1}^{3}\alpha_j\partial_j + \frac{1}{\varepsilon^2}\beta + V(t,\mathbf{x})I_4 - \sum_{j=1}^3 A_j(t,\mathbf{x})\alpha_j + F(\psi)\right]\psi(t,\mathbf{x}), \\ \psi(t=0,\mathbf{x}) = \psi_I^\varepsilon(\mathbf{x}), \qquad \mathbf{x}\in\mathbb{R}^3, \end{cases} \tag{1.1}$$

*Corresponding author. *Email addresses:* `yongyong.cai@bnu.edu.cn` (Y. Cai), `wang.yan@mail.ccnu.edu.cn` (Y. Wang)

where $i=\sqrt{-1}$, t is time, $\mathbf{x}=(x_1,x_2,x_3)^T\in\mathbb{R}^3$ is the spatial coordinate vector, $\partial_k=\frac{\partial}{\partial x_k}$ ($k=1,2,3$), $\psi:=\psi(t,\mathbf{x})=(\psi_1(t,\mathbf{x}),\psi_2(t,\mathbf{x}),\psi_3(t,\mathbf{x}),\psi_4(t,\mathbf{x}))^T\in\mathbb{C}^4$ is the complex-valued vector wave function of the "spinorfield", $V(t,\mathbf{x})$ and $\mathbf{A}=(A_1(t,\mathbf{x}),A_2(t,\mathbf{x}),A_3(t,\mathbf{x}))^T$ are the real-valued external electric potential and magnetic potential, respectively, $\varepsilon\in(0,1]$ is a dimensionless parameter inversely proportional to the speed of light. $F(\psi)\in\mathbb{C}^{4\times4}$ is the matrix nonlinearity and one common choice is of the following form

$$F(\psi)=\lambda(\psi^*\beta\psi)\beta+\gamma|\psi|^2I_4,\quad \lambda,\gamma\in\mathbb{R}, \tag{1.2}$$

where $\psi^*=\overline{\psi^T}$ denotes the conjugate transpose of ψ. The $\lambda\neq0,\gamma=0$ case is motivated from the famous Soler model [32], for which the solitary solutions and their dynamics have been widely studied in the literature [2,24,34]; The $\lambda=0,\gamma\neq0$ case is motivated from the Bose-Einstein condensates with a chiral confinement and /or spin-orbit coupling [12, 22,29]. The 4×4 matrices α_1, α_2, α_3 and β are defined as

$$\alpha_1=\begin{pmatrix}\mathbf{0}&\sigma_1\\ \sigma_1&\mathbf{0}\end{pmatrix},\qquad \alpha_2=\begin{pmatrix}\mathbf{0}&\sigma_2\\ \sigma_2&\mathbf{0}\end{pmatrix}, \tag{1.3a}$$

$$\alpha_3=\begin{pmatrix}\mathbf{0}&\sigma_3\\ \sigma_3&\mathbf{0}\end{pmatrix},\qquad \beta=\begin{pmatrix}I_2&\mathbf{0}\\ \mathbf{0}&-I_2\end{pmatrix}, \tag{1.3b}$$

where σ_1, σ_2, σ_3 are the Pauli matrices given by

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{1.4}$$

The Dirac equation (1.1) conserves the total mass

$$\|\psi(t,\cdot)\|^2:=\int_{\mathbb{R}^3}|\psi(t,\mathbf{x})|^2d\mathbf{x}=\int_{\mathbb{R}^3}\sum_{j=1}^4|\psi_j(t,\mathbf{x})|^2d\mathbf{x}\equiv\|\psi(0,\cdot)\|^2=\|\psi_I^\varepsilon\|^2,\qquad t\geq0. \tag{1.5}$$

In addition, if the external electromagnetic potentials are time independent, i.e., $V(t,\mathbf{x})=V(\mathbf{x})$ and $A_j(t,\mathbf{x})=A_j(\mathbf{x})$ ($j=1,2,3$), the NLD (1.1) conserves the energy

$$\begin{aligned}E(t):=&\int_{\mathbb{R}^3}\left(-\frac{i}{\varepsilon}\sum_{j=1}^3\psi^*\alpha_j\partial_j\psi+\frac{1}{\varepsilon^2}\psi^*\beta\psi+V(\mathbf{x})|\psi|^2-\sum_{j=1}^3A_j(\mathbf{x})\psi^*\alpha_j\psi+\frac{\lambda}{2}|\psi^*\beta\psi|^2+\frac{\gamma}{2}|\psi|^4\right)d\mathbf{x}\\ \equiv&E(0),\quad t\geq0.\end{aligned} \tag{1.6}$$

There have been many studies on the Dirac equations [1,8,13,16–19,21] including the well-posedness, dynamics of wave packets, etc. The purpose of this paper is to analyze the nonrelativistic limit of the nonlinear Dirac equation (1.1), when $\varepsilon\to0^+$.

For the linear case, the nonrelativistic limit has been investigated thoroughly in [7, 11,20,23,25,28,30,31,33]. It has been shown that the Dirac equation is a perturbation of the Schrödinger equation as $\varepsilon \to 0^+$ and Pauli equation is another approximation of the Dirac equation in the nonrelativistic limit regime [23, 25, 30, 31]. For the nonlinear case, Najman [30] proved that the nonrelativistic limit of the NLD (1.1) with $V = A_j = 0$ $(j=1,2,3)$ and $\gamma = 0$ is the nonlinear Schrödinger equation for the $(H^2(\mathbb{R}^3))^4$ initial data. Later, Matsuyama [27] studied the nonrelativistic limit in the weighted Sobolev space and the rapidly decreasing function space. Machihara *et al.* [24] proved small global solution of NLD (1.1) and the nonrelativistic limit with $(H^s(\mathbb{R}^3))^4$ $(s>1)$ initial data. In [24,27,30], the nonrelativistic limit was analyzed by separating the 'spinorfield' ψ into the upper 'spinor' part $(\psi_1,\psi_2)^T$ and the lower 'spinor' part $(\psi_3,\psi_4)^T$. Formally, let $\psi := \psi(t,\mathbf{x})$ be the solution of NLD (1.1) and $\phi = e^{it\beta/\varepsilon^2}\psi$ (upper and lower 'spinors' have different phases), then the modulated function ϕ satisfies ($V = A_j = 0$ $(j=1,2,3)$ and $\gamma = 0$),

$$\begin{cases} i\partial_t\phi = e^{2it\beta/\varepsilon^2}\left[-\dfrac{i}{\varepsilon}\sum\limits_{j=1}^{3}\alpha_j\partial_j\right]\phi + \lambda(\phi^*\beta\phi)\beta\phi, \\ \phi(t=0) = \phi_I^\varepsilon, \end{cases} \tag{1.7}$$

and if ϕ_I^ε converges to ϕ_I as $\varepsilon \to 0^+$, the solution $\phi^\varepsilon(t,\mathbf{x})$ of the above Cauchy problem converges to the solution of the following coupled nonlinear Schrödinger system

$$\begin{cases} i\partial_t\varphi(t,\mathbf{x}) = -\dfrac{1}{2}\beta\Delta\varphi(t,\mathbf{x}) + \lambda(\varphi^*\beta\varphi)\beta\varphi, \\ \varphi(t=0) = \phi_I. \end{cases} \tag{1.8}$$

Similar nonlinear Schrödinger limit has also been found for the Klein-Gordon equation in the nonrelativistic limit [26]. In [7], it has been shown that (for the linear Dirac case) the splitting of the 'spinorfield' ψ as above may not be optimal in the nonrelativistic limit regime. Instead, another type splitting of the 'spinorfield' (cf. (2.10)) is suggested.

This work is devoted to the study of the nonrelativistic limit of the NLD (1.1) in the general form for sufficiently smooth initial data, employing the splitting suggested in [7]. We will identify the limit of the NLD (1.1) as $\varepsilon \to 0^+$ and show the convergence rates. As expected for the linear case, the Schrödinger limit should be a first order approximation of the NLD (1.1) in the nonrelativistic limit regime. Moreover, we shall present and analyze a second order approximation of the NLD (1.1) as $\varepsilon \to 0^+$ (referred as semi-nonrelativistic limit in the later discussion), which can be viewed as an intermediate step between the NLD (1.1) and the coupled nonlinear Schrödinger system when passing to the limit $\varepsilon \to 0^+$. As a common approach, Hilbert expansion can be used to construct an $O(\varepsilon)$ correction to the nonlinear Schrödinger system, which can yield an approximation to the NLD with $O(\varepsilon^2)$ error. However, as will be shown in the paper, our construction is more naturally and very convenient for numerical purposes (see the linear case [5] and the nonlinear case [9]).

This paper is organized as follows. In section 2, we will present the formal (semi-)nonrelativistic limit of the NLD (1.1) and the main results. The proof is shown in section 3. Finally, some conclusions and remarks are made in section 4. Throughout the paper, C

represents a constant independent of ε and may change from line to line. We use $p\lesssim q$ to denote that there exists a constant C independent of ε such that $|p|\le Cq$.

2 Formal limit and the main results

For simplicity of notation, following [7], we denote $D=(D_1,D_2,D_3)$ as the spatial derivative corresponding to the Fourier multiplier $\xi=(\xi_1,\xi_2,\xi_3)$ and $D_j=-i\partial_j (j=1,2,3)$ is the partial derivative associated to ξ_j. Let

$$\mathcal{T}^\varepsilon(D)=\sum_{j=1}^{3}\varepsilon\alpha_j D_j+\beta=-i\sum_{j=1}^{3}\varepsilon\alpha_j\partial_j+\beta, \tag{2.1}$$

where the domain of the 'free Dirac operator' $\mathcal{T}^\varepsilon$ is $(H^1(\mathbb{R}^3))^4$ and $\mathcal{W}:=\mathcal{W}(t,\mathbf{x})$

$$\mathcal{W}(t,\mathbf{x})=V(t,\mathbf{x})I_4-\sum_{j=1}^{3}A_j(t,\mathbf{x})\alpha_j,\quad \mathbf{x}\in\mathbb{R}^3. \tag{2.2}$$

The NLD (1.1) can be written as

$$i\partial_t\psi(t,\mathbf{x})=\frac{1}{\varepsilon^2}\mathcal{T}^\varepsilon\psi(t,\mathbf{x})+\mathcal{W}\psi(t,\mathbf{x})+F(\psi)\psi(t,\mathbf{x}). \tag{2.3}$$

We note that the operator $\mathcal{T}^\varepsilon$ is diagonalizable in the Fourier space. For

$$\mathcal{T}^\varepsilon(\xi)=\sum_{j=1}^{3}\varepsilon\alpha_j\xi_j+\beta,$$

there are two eigenvalues $\pm\lambda^\varepsilon(\xi)$ with $\lambda^\varepsilon(\xi)=\sqrt{1+\varepsilon^2|\xi|^2}$ and each eigenvalue has geometric multiplicity 2. The projectors associated to the eigenvalues $\pm\lambda^\varepsilon(\xi)$ are given as $\Pi_\pm^\varepsilon=\frac{1}{2}(Id\pm\frac{\mathcal{T}^\varepsilon}{\lambda^\varepsilon})$ (Id is the identity operator). Thus, $\mathcal{T}^\varepsilon$ can decomposed as

$$\mathcal{T}^\varepsilon=\sqrt{Id-\varepsilon^2\Delta}\,\Pi_+^\varepsilon-\sqrt{Id-\varepsilon^2\Delta}\,\Pi_-^\varepsilon, \tag{2.4}$$

where $\Delta=\nabla^2$ is the Laplace operator, Π_+^ε and Π_-^ε can be written as

$$\Pi_\pm^\varepsilon=\frac{1}{2}\Big[I_4\pm\big(Id-\varepsilon^2\Delta\big)^{-1/2}\mathcal{T}^\varepsilon\Big]. \tag{2.5}$$

Here $\lambda^\varepsilon(D)=\sqrt{Id-\varepsilon^2\Delta}$ is understood in the Fourier space by the symbol $\sqrt{1+\varepsilon^2|\xi|^2}$ ($\xi\in\mathbb{R}^3$) with domain $H^1(\mathbb{R}^3)$. It is easy to verify that $\Pi_+^\varepsilon+\Pi_-^\varepsilon=I_4$ and $\Pi_+^\varepsilon\Pi_-^\varepsilon=\Pi_-^\varepsilon\Pi_+^\varepsilon=\mathbf{0}$, $(\Pi_\pm^\varepsilon)^2=\Pi_\pm^\varepsilon$. Formally, it is easy to check that $\Pi_\pm^\varepsilon$ converges as $\varepsilon\to 0$ with

$$\Pi_+^0=\begin{pmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{pmatrix},\qquad \Pi_-^0=\begin{pmatrix}0&&&\\&0&&\\&&1&\\&&&1\end{pmatrix}. \tag{2.6}$$

2.1 Formal limit

Let $\psi^\varepsilon(t,\mathbf{x})$ be the solution of NLD (1.1), and define the projections of $\psi^\varepsilon(t,\mathbf{x})$ as

$$\psi_\pm^\varepsilon(t,\mathbf{x})=\Pi_\pm^\varepsilon(D)\psi^\varepsilon(t,\mathbf{x}). \tag{2.7}$$

Applying projectors $\Pi_\pm^\varepsilon(D)$ to the NLD (1.1), we obtain

$$i\partial_t\psi_\pm^\varepsilon=\pm\frac{1}{\varepsilon^2}\lambda^\varepsilon(D)\psi_\pm^\varepsilon+\Pi_\pm^\varepsilon\mathcal{W}(\psi_+^\varepsilon+\psi_-^\varepsilon)+\Pi_\pm^\varepsilon F(\psi_+^\varepsilon+\psi_-^\varepsilon)(\psi_+^\varepsilon+\psi_-^\varepsilon). \tag{2.8}$$

Making use of the identity $\frac{1}{\varepsilon^2}\lambda^\varepsilon=\frac{1}{\varepsilon^2}+\frac{|\xi|^2}{1+\lambda^\varepsilon}$, we have

$$\frac{1}{\varepsilon^2}\lambda^\varepsilon-\frac{1}{\varepsilon^2}=\frac{|\xi|^2}{1+\lambda^\varepsilon}\to\frac{|\xi|^2}{2},\quad \text{as } \varepsilon\to 0, \tag{2.9}$$

which shows $(\lambda^\varepsilon-1)/\varepsilon^2\to-\frac{1}{2}\Delta$. Introducing $\phi_\pm^\varepsilon$ as

$$\psi_\pm^\varepsilon=e^{\mp it/\varepsilon^2}\phi_\pm^\varepsilon=\Pi_\pm^\varepsilon(D)\psi^\varepsilon(t,\mathbf{x}), \tag{2.10}$$

we could derive from (2.8) that

$$\begin{aligned} i\partial_t\phi_\pm^\varepsilon=&\pm\mathcal{D}^\varepsilon\phi_\pm^\varepsilon+\Pi_\pm^\varepsilon\mathcal{W}(\phi_\pm^\varepsilon+e^{\pm 2it/\varepsilon^2}\phi_\mp^\varepsilon)\\ &+\Pi_\pm^\varepsilon F(e^{-it/\varepsilon^2}\phi_+^\varepsilon+e^{it/\varepsilon^2}\phi_-^\varepsilon)(\phi_\pm^\varepsilon+e^{\pm 2it/\varepsilon^2}\phi_\mp^\varepsilon), \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} F(e^{-it/\varepsilon^2}\phi_+^\varepsilon+e^{it/\varepsilon^2}\phi_-^\varepsilon)=&\lambda[(\phi_+^\varepsilon)^*\beta\phi_+^\varepsilon+(\phi_-^\varepsilon)^*\beta\phi_-^\varepsilon]\beta+2\lambda\mathrm{Re}\left(e^{2it/\varepsilon^2}(\phi_+^\varepsilon)^*\beta\phi_-^\varepsilon\right)\beta\\ &+\gamma(|\phi_+^\varepsilon|^2+|\phi_-^\varepsilon|^2)I_4+2\gamma\mathrm{Re}\left(e^{2it/\varepsilon^2}(\phi_+^\varepsilon)^*\phi_-^\varepsilon\right)I_4, \end{aligned} \tag{2.12}$$

with initial data $\phi_\pm^\varepsilon(t=0)=\Pi_\pm^\varepsilon(\psi^\varepsilon(t=0))$. Here, for the simplicity of notation, we denote the operator $\mathcal{D}^\varepsilon$ as

$$\mathcal{D}^\varepsilon=(\lambda^\varepsilon(D)-Id)/\varepsilon^2. \tag{2.13}$$

It can be checked that $\mathcal{D}^\varepsilon$ is a uniformly bounded operator (w.r.t. ε) from $(H^m(\mathbb{R}^3))^4$ to $(H^{m-2}(\mathbb{R}^3))^4$ ($m\geq 2$). Omitting the highly oscillatory terms in (2.11)-(2.12), we obtain the following system

$$i\partial_t\varphi_\pm^\varepsilon=\pm\mathcal{D}^\varepsilon\varphi_\pm^\varepsilon+\Pi_\pm^\varepsilon(\mathcal{W}\varphi_\pm^\varepsilon)+\Pi_\pm^\varepsilon(G(\varphi_+^\varepsilon,\varphi_-^\varepsilon)\varphi_\pm^\varepsilon), \tag{2.14}$$

where

$$G(\varphi_+^\varepsilon,\varphi_-^\varepsilon)=\lambda[(\varphi_+^\varepsilon)^*\beta\varphi_+^\varepsilon+(\varphi_-^\varepsilon)^*\beta\varphi_-^\varepsilon]\beta+\gamma(|\varphi_+^\varepsilon|^2+|\varphi_-^\varepsilon|^2)I_4, \tag{2.15}$$

and initial data $\varphi_\pm^\varepsilon(t=0)=\Pi_\pm^\varepsilon(\psi^\varepsilon(t=0))$. As we shall prove, $\varphi_\pm^\varepsilon-\phi_\pm^\varepsilon$ is of order $O(\varepsilon^2)$ (which has been validated numerically in [9]). Further, letting $\varepsilon\to0^+$ and noticing (2.6) as well as (2.9), since $\Pi_+^0\alpha_j\Pi_+^0=\mathbf{0}$ and $\Pi_-^0\alpha_j\Pi_-^0=\mathbf{0}$ $(j=1,2,3)$, we have the limit as

$$i\partial_t\varphi_\pm^0=\mp\frac{1}{2}\Delta\varphi_\pm^0+\Pi_\pm^0\left(V(t,\mathbf{x})\varphi_\pm^0\right)+\Pi_\pm^0\left(G(\varphi_+^0,\varphi_-^0)\varphi_\pm^0\right), \tag{2.16}$$

and $\varphi_\pm^0(t=0)=\Pi_\pm^0(\psi_I^0)$, where ψ_I^0 is the limit of ψ_I^ε and it is straightforward to check that $\varphi_+^0=(\varphi_1,\varphi_2,0,0)^T$ and $\varphi_-^0=(0,0,\varphi_3,\varphi_4)^T$ in view of (2.6). On the other hand, letting $\varphi=(\varphi_1,\varphi_2,\varphi_3,\varphi_4)^T$, as $\varepsilon\to0^+$, we shall have (at least formally)

$$\begin{aligned}\psi^\varepsilon(t,\mathbf{x})&=e^{-it/\varepsilon^2}\phi_+^\varepsilon+e^{it/\varepsilon^2}\phi_-^\varepsilon\to e^{-it/\varepsilon^2}\varphi_+^\varepsilon+e^{it/\varepsilon^2}\varphi_-^\varepsilon\\&\to e^{-it/\varepsilon^2}\varphi_+^0+e^{it/\varepsilon^2}\varphi_-^0=e^{-it\beta/\varepsilon^2}\varphi.\end{aligned}$$

Now, it is convenient to rewrite the limiting system (2.16) as

$$i\partial_t\varphi=-\frac{1}{2}\beta\Delta\varphi+V(t,\mathbf{x})\varphi+F(\varphi)\varphi, \tag{2.17}$$

with $\varphi(t=0)=\psi_I^0$, i.e., $e^{it\beta/\varepsilon^2}\psi^\varepsilon(t,\mathbf{x})$ converges to the solution of the above nonlinear Schrödinger system. It can be seen that by using the decomposition (2.10) instead of the splitting suggested in [30], we could obtain a semi-nonrelativistic limit with $O(\varepsilon^2)$ error, i.e., (2.14)-(2.15), which also recovers the nonrelativistic limit presented in the literature [24,27,30] with $O(\varepsilon)$ error.

In the subsequent discussion, we will validate the approximation (2.14)-(2.15) and (2.17), when $\varepsilon\to0^+$. In particular, we will show that the solution of (2.14) does not exhibit ε-dependent rapid oscillations when $\varepsilon\to0^+$. As a result, (2.14) is very well-suited for numerical purpose.

2.2 Main results

We make the following assumptions on the electronic potential $V(t,\mathbf{x})$ and magnetic potential $\mathbf{A}=(A_1(t,\mathbf{x}),A_2(t,\mathbf{x}),A_3(t,\mathbf{x}))^T$,

$$(A).\qquad V(t,\mathbf{x}),A_j(t,\mathbf{x})\in C\left([0,T_0];H^m(\mathbb{R}^3)\right),\quad j=1,2,3,\quad m\geq2, \tag{2.18}$$

where $0<T_0<\infty$ is an arbitrary fixed time. For the initial data ψ_I^ε in (1.1), we assume $\psi_I^\varepsilon\in(H^m(\mathbb{R}^3))^4$ and there exists $\psi_I^0\in(H^m(\mathbb{R}^3))^4$, such that

$$(B).\qquad \|\psi_I^\varepsilon-\psi_I^0\|_{H^m}\leq C\varepsilon^2,\quad \text{for some constant } C>0 \text{ and } \varepsilon\in(0,1],\quad m\geq2. \tag{2.19}$$

We remark here that assumption (B) is required to ensure (2.14) is an $O(\varepsilon^2)$ approximation of NLD (1.1).

Combining the linear [7,25] and the nonlinear theories [4,30], we could establish the following theorems.

Theorem 2.1. *Under the assumptions (A) and (B), there exists* $0<T_1\leq T_0$*, such that for any* $\varepsilon\in(0,1]$*, NLD* (1.1) *admits a unique solution* $\psi^\varepsilon\in C^1([0,T_1];(H^{m-1}(\mathbb{R}^3))^4)\cap C([0,T_1];(H^m(\mathbb{R}^3))^4)$ *(*$m\geq 2$*) with uniform estimates*

$$\sup_{\varepsilon\in(0,1]}\|\psi^\varepsilon\|_{C([0,T_1];(H^m(\mathbb{R}^3))^4)}\leq C, \tag{2.20}$$

where C is independent of ε*.*

We have the well-posedness of (2.14) as the semi-nonrelativistic limit of the NLD (1.1). The major advantage is that the solution to the system (2.14) does not have ε-dependent highly oscillatory behavior when $\varepsilon\to 0^+$.

Theorem 2.2. *Under the assumptions (A) and (B), there exists* $0<T_2\leq T_0$*, such that for any* $\varepsilon\in(0,1]$*,* (2.14) *with initial data* $\varphi_\pm^\varepsilon(t=0)=\Pi_\pm^\varepsilon(\psi_I^\varepsilon)$ *admits a unique solution* $\varphi_\pm^\varepsilon\in C^1([0,T_2];(H^{m-1}(\mathbb{R}^3))^4)\cap C([0,T_2];(H^m(\mathbb{R}^3))^4)$ *(*$m\geq 2$*) with uniform estimates*

$$\sup_{\varepsilon\in(0,1]}\|\partial_t^k\varphi^\varepsilon\|_{C([0,T_2];(H^{m-2k}(\mathbb{R}^3))^4)}\leq C,\quad k=0,1. \tag{2.21}$$

where C is independent of ε*. Moreover,* $\varphi_\pm^\varepsilon$ *remain in the eigenspaces associated with* $\Pi_\pm^\varepsilon$*, respectively, and if* $V(t,\mathbf{x}),A_j(t,\mathbf{x})\in C^{J-1}([0,T_2];H^{m-J}(\mathbb{R}^3))$ *(*$j=1,2,3,\ J\leq\lfloor m/2\rfloor$*), we have* $\varphi_\pm^\varepsilon\in C^J([0,T_2];(H^{m-J}(\mathbb{R}^3))^4)$ *and*

$$\sup_{\varepsilon\in(0,1]}\|\partial_t^k\varphi^\varepsilon\|_{C([0,T_2];(H^{m-2k}(\mathbb{R}^3))^4)}\leq C,\quad 1\leq k\leq\lfloor m/2\rfloor. \tag{2.22}$$

In addition, the following estimates hold true when $m\geq 2$*,*

$$\left\|\psi^\varepsilon-e^{-it/\varepsilon^2}\varphi_+^\varepsilon-e^{it/\varepsilon^2}\varphi_-^\varepsilon\right\|_{C([0,T];(H^{m-2}(\mathbb{R}^3))^4)}\lesssim\varepsilon^2, \tag{2.23}$$

where ψ^ε *is the solution of the NLD* (1.1) *and* $T=\min\{T_1,T_2\}$*.*

Finally, we have the well-posedness of (2.17) as the nonrelativistic limit of the NLD (1.1).

Theorem 2.3. *Under Assumptions (A) and (B), there exists* $0<T_3\leq T_0$ *such that* (2.17) *with initial data* ψ_I^0 *admits a unique solution* $\varphi\in C^1([0,T_3];(H^{m-2}(\mathbb{R}^3))^4)\cap C([0,T_3];(H^m(\mathbb{R}^3))^4)$ *(*$m\geq 2$*). Moreover, if* $m\geq 3$*, we have*

$$\left\|\psi^\varepsilon-e^{-it\beta/\varepsilon^2}\varphi\right\|_{C([0,T_*];(H^{m-3}(\mathbb{R}^3))^4)}\lesssim\varepsilon, \tag{2.24}$$

where ψ^ε *is the solution of NLD* (1.1) *and* $T_*=\min\{T_1,T_2,T_3\}$*.*

Remark 2.1. Theorems 2.1-2.3 can be easily generalized to lower dimensions. For the semi-nonrelativistic limit (2.14), one can further expand λ^ε and $\Pi_\pm^\varepsilon$ w.r.t. ε and omit $O(\varepsilon^2)$ terms to derive other second order approximations of the NLD (1.1).

3 The (semi-)nonrelativistic limit

We recall the following lemma regarding the projectors $\Pi_\pm^\varepsilon$ from [7].

Lemma 3.1. *([7]) (i) The projectors $\Pi_\pm^\varepsilon(D)$ are uniformly bounded (w.r.t. ε) from $(H^m(\mathbb{R}^3))^4$ to $(H^m(\mathbb{R}^3))^4$.*

(ii) $\Pi_\pm^\varepsilon(D)$ can be expanded as

$$\Pi_\pm^\varepsilon(D)=\Pi_\pm^0\pm\varepsilon\mathcal{R}_1=\Pi_\pm^0\mp i\frac{\varepsilon}{2}\sum_{j=1}^{3}\alpha_j\partial_j\pm\varepsilon^2\mathcal{R}_2, \tag{3.1}$$

where $\mathcal{R}_1:(H^m(\mathbb{R}^3))^4\to(H^{m-1}(\mathbb{R}^3))^4$ and $\mathcal{R}_2:(H^m(\mathbb{R}^3))^4\to(H^{m-2}(\mathbb{R}^3))^4$ are uniformly (w.r.t. ε) bounded operators.

3.1 Cauchy problem

In this subsection, we focus on the Cauchy problems of NLD (1.1), semi-relativistic limit system (2.14) and the nonrelativistic limit (2.17), which are involved in Theorems 2.1, 2.2 and 2.3, respectively. Since the proofs are quite similar, we only prove Theorem 2.1 and sketch some of the estimates in Theorem 2.2.

Proof of Theorem 2.1. By the Duhamel's principle and using the equivalent form (2.3) of NLD (1.1), we seek a local solution $\psi^\varepsilon(t):=\psi^\varepsilon(t,\mathbf{x})$ satisfying

$$\psi^\varepsilon(t,\mathbf{x})=e^{-it\mathcal{T}^\varepsilon/\varepsilon^2}\psi_I^\varepsilon-i\int_0^t e^{-i(t-s)\mathcal{T}^\varepsilon/\varepsilon^2}\left(\mathcal{W}\psi^\varepsilon(s)+F(\psi^\varepsilon(s))\psi^\varepsilon(s)\right)ds. \tag{3.2}$$

It is obvious that $e^{-it\mathcal{T}^\varepsilon/\varepsilon^2}$ preserves the H^m norm and $(H^m(\mathbb{R}^3))^4\subset(L^\infty(\mathbb{R}^3))^4$ which implies $F(\psi)\psi$ is locally Lipschitz in $(H^m(\mathbb{R}^3))^4$, i.e.,

$$\|F(\psi)\psi-F(\widetilde{\psi})\widetilde{\psi}\|_{H^m}\le C\left(\|\psi\|_{H^m}^2+\|\widetilde{\psi}\|_{H^m}^2\right)\|\psi-\widetilde{\psi}\|_{H^m}. \tag{3.3}$$

Under assumption (A), for all $t\in[0,T_0]$, $\mathcal{W}\psi$ is Lipschitz in $(H^m(\mathbb{R}^3))^4$, i.e.,

$$\sup_{t\in[0,T_0]}\|\mathcal{W}\psi-\mathcal{W}\widetilde{\psi}\|_{H^m}\le C_{\mathcal{W}}\|\psi-\widetilde{\psi}\|_{H^m}, \tag{3.4}$$

where $C_{\mathcal{W}}$ depends on $\|V(t,\mathbf{x})\|_{L^\infty([0,T_0];(H^m(\mathbb{R}^3))^4)}$ and $\|A_j(t,\mathbf{x})\|_{L^\infty([0,T_0];(H^m(\mathbb{R}^3))^4)}$ (j=1,2,3). By a standard fixed-point argument (e.g. [10]), it is easy to show that there exists a maximal time $T^\varepsilon\in(0,\infty]$, such that the integral equation (3.2) admits a unique solution $\psi^\varepsilon(t,\mathbf{x})$ in the function space $C([0,T^\varepsilon];(H^m(\mathbb{R}^3))^4)$ ($m\ge2$). Using equation (1.1), we can verify that $\partial_t\psi^\varepsilon(t,\mathbf{x})\in C([0,T^\varepsilon];(H^{m-1}(\mathbb{R}^3))^4)$. If $T^\varepsilon<+\infty$, we have $\|\psi^\varepsilon(t,\cdot)\|_{H^m}\to+\infty$ when $t\nearrow T^\varepsilon$.

Next, we would like to show T^ε has a uniform lower bound $T_1>0$, i.e., $T^\varepsilon\ge T_1$ ($\varepsilon\in(0,1]$). By the conservation of mass, we know $\|\psi^\varepsilon(t,\cdot)\|_{L^2}=\|\psi_I^\varepsilon\|_{L^2}\lesssim 1$ by assumption

(B). Multiplying both sides of (1.1) by $(-\Delta)^m\psi^*$ (we can take an approximation argument to make $(-\Delta)^m\psi^*$ make sense in L^2 or simply treat it as a $(H^{-m}(\mathbb{R}^3))^3$) and then integrating over $\mathbb{R}^3$ and taking the imaginary parts, using integral by parts and the Hermitian property of the "free Dirac operator" $\mathcal{T}^\varepsilon$, we have

$$\partial_t\|(-\Delta)^{m/2}\psi^\varepsilon(t,\cdot)\|_{L^2}^2 \le C_1\|\psi^\varepsilon(t,\cdot)\|_{H^m}^2 + C_2\|\psi^\varepsilon(t,\cdot)\|_{H^m}^4, \tag{3.5}$$

where C_1 depends on m, $\|V(t,\mathbf{x})\|_{L^\infty([0,T_0];(H^m(\mathbb{R}^3))^4)}$ and $\|A_j(t,\mathbf{x})\|_{L^\infty([0,T_0];(H^m(\mathbb{R}^3))^4)}$ ($j=1,2,3$), C_2 depends on m. Thus,

$$\partial_t\|\psi^\varepsilon(t,\cdot)\|_{H^m}^2 \le C_3(1+\|\psi^\varepsilon(t,\cdot)\|_{H^m}^2)\|\psi^\varepsilon(t,\cdot)\|_{H^m}^2. \tag{3.6}$$

From ODE theory, it follows that for $T'>0$ such that $\frac{1+\|\psi_I^\varepsilon\|_{H^m}^2}{\|\psi_I^\varepsilon\|_{H^m}^2}e^{-C_3T'}>1$,

$$\sup_{t\in[0,T']}\|\psi^\varepsilon(t,\cdot)\|_{H^m} \le \left(\frac{\|\psi_I^\varepsilon\|_{H^m}^2}{(1+\|\psi_I^\varepsilon\|_{H^m}^2)e^{-C_3T'}-\|\psi_I^\varepsilon\|_{H^m}^2}\right)^{1/2}. \tag{3.7}$$

Therefore, under assumption (B), we have a lower bound $T_1>0$ such that $T^\varepsilon \ge T_1$ ($\varepsilon\in(0,1]$) and the estimate (2.20) holds. □

Proof of the Cauchy problem in Theorem 2.2. Repeating the arguments in the proof of Theorem 2.1, using Lemma 3.1 and the fact that $\lambda^\varepsilon(D)-Id$ is Hermitian, we can find that there exists $T_2>0$ such that for all $\varepsilon\in(0,1]$, (2.14) admits a unique solution $\varphi_\pm^\varepsilon \in C^1([0,T_2];(H^{m-1}(\mathbb{R}^3))^4)\cap C([0,T_2];(H^m(\mathbb{R}^3))^4)$ with uniform estimates

$$\sup_{\varepsilon\in(0,1]}\|\varphi_\pm^\varepsilon\|_{C([0,T_2];(H^m(\mathbb{R}^3))^4)} \le C. \tag{3.8}$$

Using equation (2.14) and noticing the boundedness of operator $\mathcal{D}^\varepsilon$, we obtain

$$\sup_{\varepsilon\in(0,1]}\|\partial_t\varphi_+^\varepsilon\|_{C([0,T_2];(H^{m-2}(\mathbb{R}^3))^4)} \le C. \tag{3.9}$$

Similarly, we have the estimate for $\partial_t\varphi_-^\varepsilon$. In addition, if $V(t,\mathbf{x}),A_j(t,\mathbf{x}) \in C^{J-1}([0,T_2];H^{m-J}(\mathbb{R}^3))$ ($j=1,2,3$, $J\le\lfloor m/2\rfloor$) have higher regularities, by differentiating (2.14) in time t, we would have (2.22). □

3.2 Semi-nonrelativistic limit (2.14)

In this subsection, we want to prove Theorem 2.2. Since the Cauchy problem and the uniform estimates of $\varphi_\pm^\varepsilon$ have been considered, we only need to derive the error estimate (2.23).

For ψ^ε, i.e., the solution of NLD (1.1), let $\phi_\pm^\varepsilon$ be given in (2.10), i.e.,

$$\phi_\pm^\varepsilon = e^{\pm it/\varepsilon^2}\Pi_\pm^\varepsilon(D)\psi^\varepsilon(t,\mathbf{x}). \tag{3.10}$$

Then we find $\phi_\pm^\varepsilon$ satisfies the coupled system (2.11). Noticing Lemma 3.1, Theorem 2.1 and the boundedness of $\mathcal{D}^\varepsilon$, we have

$$\|\phi_\pm^\varepsilon\|_{C([0,T_1];(H^m(\mathbb{R}^3))^4)}+\|\partial_t\phi_\pm^\varepsilon\|_{C([0,T_1];(H^{m-2}(\mathbb{R}^3))^4)}\le C. \tag{3.11}$$

Now, let us check the difference between $\phi_\pm^\varepsilon$ and $\varphi_\pm^\varepsilon$ (the solution of (2.14)). Denote

$$\chi_\pm^\varepsilon=\chi_\pm^\varepsilon(t):=\chi_\pm^\varepsilon(t,\mathbf{x})=\phi_\pm^\varepsilon-\varphi_\pm^\varepsilon, \tag{3.12}$$

and from the choices of initial data, we have $\chi_\pm^\varepsilon(t=0)=0$. For $0<t\le T=\min\{T_1,T_2\}$, we can derive the equations for $\chi_\pm^\varepsilon$ by subtracting (2.14) from (2.11),

$$\begin{aligned}i\partial_t\chi_\pm^\varepsilon=&\pm\mathcal{D}^\varepsilon\chi_\pm^\varepsilon+\Pi_\pm^\varepsilon(\mathcal{W}\chi_\pm^\varepsilon)+\Pi_\pm^\varepsilon(G(\phi_+^\varepsilon,\phi_-^\varepsilon)\phi_\pm^\varepsilon-G(\varphi_+^\varepsilon,\varphi_-^\varepsilon)\varphi_\pm^\varepsilon)\\&+e^{\pm2it/\varepsilon^2}R_{\pm,1}+e^{\mp2it/\varepsilon^2}R_{\pm,2}+e^{\pm4it/\varepsilon^2}R_{\pm,3}+R_{\pm,4}\end{aligned} \tag{3.13}$$

where

$$\begin{aligned}R_{\pm,1}=&\Pi_\pm^\varepsilon(\mathcal{W}\phi_\mp^\varepsilon)+\Pi_\pm^\varepsilon(G(\phi_+^\varepsilon,\phi_-^\varepsilon)\phi_\mp^\varepsilon)+\lambda\Pi_\pm^\varepsilon(((\phi_\pm^\varepsilon)^*\beta\phi_\mp^\varepsilon)\beta\phi_\pm^\varepsilon)\\&+\gamma\Pi_\pm^\varepsilon(((\phi_\pm^\varepsilon)^*\phi_\mp^\varepsilon)\phi_\pm^\varepsilon),\end{aligned} \tag{3.14}$$

$$R_{\pm,2}=\lambda\Pi_\pm^\varepsilon(((\phi_\mp^\varepsilon)^*\beta\phi_\pm^\varepsilon)\beta\phi_\pm^\varepsilon)+\gamma\Pi_\pm^\varepsilon(((\phi_\mp^\varepsilon)^*\phi_\pm^\varepsilon)\phi_\pm^\varepsilon), \tag{3.15}$$

$$R_{\pm,3}=\lambda\Pi_\pm^\varepsilon(((\phi_\pm^\varepsilon)^*\beta\phi_\mp^\varepsilon)\beta\phi_\mp^\varepsilon)+\gamma\Pi_\pm^\varepsilon(((\phi_\pm^\varepsilon)^*\phi_\mp^\varepsilon)\phi_\mp^\varepsilon), \tag{3.16}$$

$$R_{\pm,4}=\lambda\Pi_\pm^\varepsilon(((\phi_\mp^\varepsilon)^*\beta\phi_\pm^\varepsilon)\beta\phi_\mp^\varepsilon)+\gamma\Pi_\pm^\varepsilon(((\phi_\mp^\varepsilon)^*\phi_\pm^\varepsilon)\phi_\mp^\varepsilon). \tag{3.17}$$

Applying Duhamel's principle to (3.13), we have

$$\begin{aligned}\chi_\pm^\varepsilon(t)=&e^{\mp it\mathcal{D}^\varepsilon}\chi_\pm^\varepsilon(0)-i\int_0^t e^{\mp i(t-s)\mathcal{D}^\varepsilon}(\Pi_\pm^\varepsilon(\mathcal{W}\chi_\pm^\varepsilon(s)))\,ds\\&-i\int_0^t e^{\mp i(t-s)\mathcal{D}^\varepsilon}\Pi_\pm^\varepsilon(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))\phi_\pm^\varepsilon(s)-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_\pm^\varepsilon(s))\,ds\\&-i\int_0^t e^{\mp i(t-s)\mathcal{D}^\varepsilon}\Big(e^{\pm\frac{2is}{\varepsilon^2}}R_{\pm,1}(s)+e^{\mp\frac{2is}{\varepsilon^2}}R_{\pm,2}(s)+e^{\pm\frac{4is}{\varepsilon^2}}R_{\pm,3}(s)+R_{\pm,4}(s)\Big)\,ds.\end{aligned} \tag{3.18}$$

We shall estimate $\|\chi_\pm^\varepsilon(t)\|_{H^{m-2}}$ using (3.18). Taking H^{m-2} norm on both sides of (3.18), noticing $e^{-it\mathcal{D}^\varepsilon}$ preserves the H^s ($s\ge0$) norm, applying Lemma 3.1 and assumption (A), we find

$$\begin{aligned}\|\chi_+^\varepsilon(t)\|_{H^{m-2}}\le&\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}\Pi_+^\varepsilon(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))\phi_+^\varepsilon(s)-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_+^\varepsilon(s))\,ds\right\|_{H^{m-2}}\\&+\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{2is}{\varepsilon^2}}R_{+,1}(s)\,ds\right\|_{H^{m-2}}+\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{-2is}{\varepsilon^2}}R_{+,2}(s)\,ds\right\|_{H^{m-2}}\\&+\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{4is}{\varepsilon^2}}R_{+,3}(s)\,ds\right\|_{H^{m-2}}+\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}R_{+,4}(s)\,ds\right\|_{H^{m-2}}\\&+C\int_0^t\|\chi_+^\varepsilon(s)\|_{H^{m-2}}\,ds.\end{aligned} \tag{3.19}$$

Before estimating each terms in (3.19) further, we investigate the properties of the nonlinear terms involved.

Lemma 3.2. *Under assumptions (A) and (B), for $m\geq 2$, $s\in[0,T]$ and $\varepsilon\in(0,1]$, we have*

$$\begin{aligned}&\|\Pi_\pm^\varepsilon(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))\phi_\pm^\varepsilon(s)-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_\pm^\varepsilon(s))\|_{H^{m-2}}\\ \leq& C(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}}),\end{aligned}\tag{3.20}$$

and

$$\|R_{\pm,1}(s)\|_{H^m}+\|R_{\pm,2}(s)\|_{H^m}+\|R_{\pm,3}(s)\|_{H^m}\leq C,\tag{3.21}$$
$$\|\partial_s R_{\pm,1}(s)\|_{H^{m-2}}+\|\partial_s R_{\pm,2}(s)\|_{H^{m-2}}+\|\partial_s R_{\pm,3}(s)\|_{H^{m-2}}\leq C,\tag{3.22}$$
$$\|R_{\pm,4}(s)\|_{H^{m-2}}\leq C\varepsilon^2.\tag{3.23}$$

Proof. Under the assumptions of Lemma 3.2, noticing (3.8), (3.9) and (3.11), we have for $s\in[0,T]$,

$$\|\phi_\pm^\varepsilon(s)\|_{H^m}\leq C,\quad \|\varphi_\pm^\varepsilon(s)\|_{H^m}\leq C,\quad \|\partial_s\phi_\pm^\varepsilon(s)\|_{H^{m-2}}\leq C,\quad \|\partial_s\varphi_\pm^\varepsilon(s)\|_{H^{m-2}}\leq C.\tag{3.24}$$

By the identity

$$\begin{aligned}&G(\phi_+^\varepsilon,\phi_-^\varepsilon)-G(\varphi_+^\varepsilon,\varphi_-^\varepsilon)\\ =&\lambda((\chi_+^\varepsilon)^*\beta\phi_+^\varepsilon+(\chi_-^\varepsilon)^*\beta\phi_-^\varepsilon)\beta+\lambda((\varphi_+^\varepsilon)^*\beta\chi_+^\varepsilon+(\varphi_-^\varepsilon)^*\beta\chi_-^\varepsilon)\beta\\ &+\gamma((\chi_+^\varepsilon)^*\phi_+^\varepsilon+(\chi_-^\varepsilon)^*\phi_-^\varepsilon)I_4+\gamma((\varphi_+^\varepsilon)^*\chi_+^\varepsilon+(\varphi_-^\varepsilon)^*\chi_-^\varepsilon)I_4,\end{aligned}$$

for $m\geq 2$, it is straightforward to check that

$$\begin{aligned}&\|\Pi_+^\varepsilon(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))\phi_+^\varepsilon(s)-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_+^\varepsilon(s))\|_{H^{m-2}}\\ \leq&\|(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s)))\phi_+^\varepsilon(s)\|_{H^{m-2}}\\ &+\|(G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s)))\chi_+^\varepsilon(s)\|_{H^{m-2}}\\ \leq& C(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}}).\end{aligned}$$

Similarly, we can get the estimate for $\Pi_-^\varepsilon(G(\phi_+^\varepsilon(s),\phi_-^\varepsilon(s))\phi_-^\varepsilon(s)-G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_-^\varepsilon(s))$. Hence, the estimates in (3.20) hold true.

Recalling the definitions of $R_{\pm,k}$ ($k=1,2,3$), (3.21) and (3.22) is a direct consequence of (3.24).

It remains to show (3.23). Again, it suffices to consider $\|R_{+,4}\|_{H^{m-2}}$. As the first step, we claim

$$\|(\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s)\|_{H^{m-1}}+\|(\phi_+^\varepsilon(s))^*\phi_-^\varepsilon(s)\|_{H^{m-1}}\leq C\varepsilon.\tag{3.25}$$

Using Lemma 3.1 and $(\Pi_+^0)^*\beta\Pi_-^0=\mathbf{0}$, we can write

$$\begin{aligned}(\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s)&=e^{-\frac{2is}{\varepsilon^2}}(\Pi_+^\varepsilon(D)\psi^\varepsilon)^*\beta(\Pi_-^\varepsilon(D)\psi^\varepsilon)=e^{-\frac{2is}{\varepsilon^2}}\left((\Pi_+^0+\varepsilon\mathcal{R}_1)\psi^\varepsilon\right)^*\beta(\Pi_-^\varepsilon\psi^\varepsilon)\\ &=e^{-\frac{2is}{\varepsilon^2}}\varepsilon\left((\mathcal{R}_1\psi^\varepsilon)^*\beta(\Pi_-^\varepsilon\psi^\varepsilon)\right)+e^{-\frac{2is}{\varepsilon^2}}\left(\Pi_+^0\psi^\varepsilon\right)^*\beta\left((\Pi_-^0-\varepsilon\mathcal{R}_1)\psi^\varepsilon\right)\\ &=e^{-\frac{2is}{\varepsilon^2}}\varepsilon\left((\mathcal{R}_1\psi^\varepsilon)^*\beta(\Pi_-^\varepsilon\psi^\varepsilon)\right)-e^{-\frac{2is}{\varepsilon^2}}\varepsilon\left((\Pi_+^0\psi^\varepsilon)^*\beta(\mathcal{R}_1\psi^\varepsilon)\right),\end{aligned}$$

and it follows that

$$\|(\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s)\|_{H^{m-1}}\le C\varepsilon\|\mathcal{R}_1\psi^\varepsilon(s)\|_{H^{m-1}}\|\psi^\varepsilon(s)\|_{H^m}\le C\varepsilon\|\psi^\varepsilon(s)\|_{H^m}\le C\varepsilon.$$

Similarly, we can show $\|(\phi_+^\varepsilon(s))^*\phi_-^\varepsilon(s)\|_{H^{m-1}}\le C\varepsilon$ and we only need modify the above arguments by replacing β with I_4, in light of the relation $(\Pi_+^0)^*I_4\Pi_-^0=\mathbf{0}$. Therefore, we have proved (3.25).

Next, we claim

$$\begin{aligned}&\|\Pi_+^\varepsilon(((\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s))\beta\phi_-^\varepsilon(s))\|_{H^{m-2}}+\|\Pi_+^\varepsilon(((\phi_+^\varepsilon(s))^*\phi_-^\varepsilon(s))\phi_-^\varepsilon(s))\|_{H^{m-2}}\\ \le&C\varepsilon^2.\end{aligned}\tag{3.26}$$

Analogous to the proof of (3.25), denote $u(s)=((\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s))\in C([0,T];H^m(\mathbb{R}^3))$, then we can write

$$\begin{aligned}&\Pi_+^\varepsilon(((\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s))\beta\phi_-^\varepsilon(s))=e^{-\frac{is}{\varepsilon^2}}\Pi_+^\varepsilon(u(s)\beta(\Pi_-^\varepsilon\psi^\varepsilon))\\ =&e^{-\frac{is}{\varepsilon^2}}\varepsilon\mathcal{R}_1(u(s)\beta(\Pi_-^\varepsilon\psi^\varepsilon))+e^{-\frac{is}{\varepsilon^2}}u(s)\Pi_+^0\beta\left((\Pi_-^0-\varepsilon\mathcal{R}_1)\psi^\varepsilon\right)\\ =&e^{-\frac{is}{\varepsilon^2}}\varepsilon\mathcal{R}_1(u(s)\beta(\Pi_-^\varepsilon\psi^\varepsilon))-e^{-\frac{is}{\varepsilon^2}}\varepsilon u(s)\Pi_+^0\beta(\mathcal{R}_1\psi^\varepsilon),\end{aligned}$$

where we have used that fact $\Pi_+^0\beta\Pi_-^0=\mathbf{0}$. From (3.25), we know $\|u(s)\|_{H^{m-1}}\le C\varepsilon$ and

$$\begin{aligned}&\|\Pi_+^\varepsilon(((\phi_+^\varepsilon(s))^*\beta\phi_-^\varepsilon(s))\beta\phi_-^\varepsilon(s))\|_{H^{m-2}}\\ \le&C\varepsilon(\|\mathcal{R}_1(u(s)\beta(\Pi_-^\varepsilon\psi^\varepsilon(s)))\|_{H^{m-2}}+\|u(s)\mathcal{R}_1\psi^\varepsilon(s)\|_{H^{m-2}})\\ \le&C\varepsilon(\|u(s)\|_{H^{m-1}}\|\psi^\varepsilon(s)\|_{H^m}+\|u(s)\|_{H^{m-2}}\|\psi^\varepsilon(s)\|_{H^m})\le C\varepsilon^2.\end{aligned}$$

Similarly, we can prove that $\|\Pi_+^\varepsilon(((\phi_+^\varepsilon(s))^*\phi_-^\varepsilon(s))\phi_-^\varepsilon(s))\|_{H^{m-2}}\le C\varepsilon^2$ and (3.26) holds true. From the definition of $R_{+,4}$ (3.17), we immediately get $\|R_{+,4}(s)\|\le C\varepsilon^2$. $R_{-,4}$ can be estimated in the same way and we omit the proof here for brevity. □

Having Lemma 3.2, we can obtain from (3.19) that

$$\begin{aligned}\|\chi_+^\varepsilon(t)\|_{H^{m-2}}\le&C\int_0^t(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}})ds+C\varepsilon^2\\ &+\left\|\int_0^te^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{2is}{\varepsilon^2}}R_{+,1}(s)ds\right\|_{H^{m-2}}+\left\|\int_0^te^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{-2is}{\varepsilon^2}}R_{+,2}(s)ds\right\|_{H^{m-2}}\\ &+\left\|\int_0^te^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{4is}{\varepsilon^2}}R_{+,3}(s)ds\right\|_{H^{m-2}}.\end{aligned}\tag{3.27}$$

Next, we deal with the integrals. Using integration by parts, we have

$$\begin{aligned}&\int_0^te^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{2is}{\varepsilon^2}}R_{+,1}(s)ds\\ =&-\frac{i}{2}\varepsilon^2e^{\frac{2is}{\varepsilon^2}}e^{-i(t-s)\mathcal{D}^\varepsilon}R_{+,1}(s)\Big|_{s=0}^t+\frac{i}{2}\varepsilon^2\int_0^te^{\frac{2is}{\varepsilon^2}}\partial_s\left(e^{-i(t-s)\mathcal{D}^\varepsilon}R_{+,1}(s)\right)ds\\ =&-\frac{i\varepsilon^2}{2}(e^{\frac{2it}{\varepsilon^2}}R_{+,1}(t)-e^{-it\mathcal{D}^\varepsilon}R_{+,1}(0))+\frac{i}{2}\varepsilon^2\int_0^te^{\frac{2is}{\varepsilon^2}}\left(e^{-i(t-s)\mathcal{D}^\varepsilon}(\partial_sR_{+,1}(s)+i\mathcal{D}^\varepsilon R_{+,1}(s))\right)ds,\end{aligned}$$

which yields for $t\in[0,T]$,

$$\begin{aligned}&\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{2is}{\varepsilon^2}}R_{+,1}(s)ds\right\|_{H^{m-2}}\\ \le&\frac{\varepsilon^2}{2}\left(\|R_{+,1}(t)\|_{H^{m-2}}+\|R_{+,1}(0)\|_{H^{m-2}}+\int_0^t(\|\partial_sR_{+,1}(s)\|_{H^{m-2}}+\|\mathcal{D}^\varepsilon R_{+,1}(s)\|_{H^{m-2}})ds\right)\\ \le&\varepsilon^2\left(C+\int_0^t\|R_{+,1}(s)\|_{H^m}ds\right)\le C\varepsilon^2.\end{aligned}$$

Here, we have used Lemma 3.2 and that $\mathcal{D}^\varepsilon:(H^m(\mathbb{R}^3))^4\to(H^{m-2}(\mathbb{R}^3))^4$ is uniformly bounded. In a similar way, we can prove

$$\left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{-2is}{\varepsilon^2}}R_{+,2}(s)ds\right\|_{H^{m-2}}\le C\varepsilon^2,\qquad \left\|\int_0^t e^{-i(t-s)\mathcal{D}^\varepsilon}e^{\frac{4is}{\varepsilon^2}}R_{+,3}(s)ds\right\|_{H^{m-2}}\le C\varepsilon^2.$$

Thus, (3.27) leads to

$$\|\chi_+^\varepsilon(t)\|_{H^{m-2}}\le C\varepsilon^2+C\int_0^t(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}})ds. \tag{3.28}$$

For χ_-^ε in (3.18), using the same arguments, we arrive at

$$\|\chi_-^\varepsilon(t)\|_{H^{m-2}}\le C\varepsilon^2+C\int_0^t(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}})ds. \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\begin{aligned}&\|\chi_+^\varepsilon(t)\|_{H^{m-2}}+\|\chi_-^\varepsilon(t)\|_{H^{m-2}}\\ \le&C\varepsilon^2+C\int_0^t\left(\|\chi_+^\varepsilon(s)\|_{H^{m-2}}+\|\chi_-^\varepsilon(s)\|_{H^{m-2}}\right)ds,\quad 0\le t\le T.\end{aligned} \tag{3.30}$$

Gronwall's inequality implies

$$\|\chi_+^\varepsilon(t)\|_{H^{m-2}}+\|\chi_-^\varepsilon(t)\|_{H^{m-2}}\le C\varepsilon^2,\quad \forall t\in[0,T]. \tag{3.31}$$

Since

$$\psi^\varepsilon-e^{-it/\varepsilon^2}\varphi_+^\varepsilon-e^{it/\varepsilon^2}\varphi_-^\varepsilon=e^{-it/\varepsilon^2}\chi_+^\varepsilon(t)+e^{it/\varepsilon^2}\chi_-^\varepsilon(t), \tag{3.32}$$

we conclude that (2.23) is true and Theorem 2.2 is proved.

3.3 Non-relativistic limit (2.17)

Here, we would like to show the nonrelativistic limit of NLD (1.1) is (2.17). As mentioned, it suffices to prove that the coupled system (2.14)-(2.15) converges to (2.16) as $\varepsilon\to0^+$. Now, we shall assume $m\ge3$.

Noticing

$$\frac{\lambda^\varepsilon(\xi)-1}{\varepsilon^2}=\frac{|\xi|^2}{1+\sqrt{1+\varepsilon^2|\xi|^2}}=\frac{|\xi|^2}{2}-\frac{\varepsilon^2|\xi|^4}{2(1+\sqrt{1+\varepsilon^2|\xi|^2})^2}, \tag{3.33}$$

we have

$$\mathcal{D}^\varepsilon=\frac{\lambda^\varepsilon(D)-Id}{\varepsilon^2}=-\frac{1}{2}\Delta+\varepsilon\mathcal{L}^\varepsilon, \tag{3.34}$$

where $\mathcal{L}^\varepsilon:(H^m(\mathbb{R}^3))^4\to(H^{m-3}(\mathbb{R}^3))^4$ is uniformly bounded w.r.t. $\varepsilon\in(0,1]$. Denote

$$\delta^\varepsilon_\pm(t)=\varphi^\varepsilon_\pm(t)-\varphi^0_\pm(t), \tag{3.35}$$

then assumption (B) and Lemma 3.1 yield

$$\begin{aligned}\|\delta^\varepsilon_\pm(0)\|_{H^{m-1}}&\le\|\Pi^\varepsilon_\pm(\psi^\varepsilon_I)-\Pi^0_\pm(\psi^\varepsilon_I)\|_{H^{m-1}}+\|\Pi^0_\pm(\psi^\varepsilon_I-\psi^0_I)\|_{H^{m-1}}\\&\le C\left(\varepsilon\|\psi^\varepsilon_I\|_{H^m}+\varepsilon^2\right)\le C\varepsilon.\end{aligned} \tag{3.36}$$

Subtracting (2.16) from (2.14) and using Lemma 3.1, we have

$$\begin{aligned}i\partial_t\delta^\varepsilon_\pm=&\mp\frac{1}{2}\Delta\delta^\varepsilon_\pm+\Pi^0_\pm(V(t,\mathbf{x})\delta^\varepsilon_\pm)+\Pi^0_\pm\big(G(\varphi^\varepsilon_+,\varphi^\varepsilon_-)\varphi^\varepsilon_\pm-G(\varphi^0_+,\varphi^0_-)\varphi^0_\pm\big),\\&\pm\varepsilon\mathcal{L}^\varepsilon\varphi^\varepsilon_\pm+L_{\pm,1}+L_{\pm,2}+L_{\pm,3},\end{aligned} \tag{3.37}$$

where

$$L_{\pm,1}=\pm\varepsilon\mathcal{R}_1(V(t,\mathbf{x})\varphi^\varepsilon_\pm),\qquad L_{\pm,2}=\Pi^\varepsilon_\pm(\sum_{j=1}^3A_j(t,\mathbf{x})\alpha_j\varphi^\varepsilon_\pm), \tag{3.38a}$$

$$L_{\pm,3}=\pm\varepsilon\mathcal{R}_1(G(\varphi^\varepsilon_+,\varphi^\varepsilon_-)\varphi^\varepsilon_\pm), \tag{3.38b}$$

Therefore, we get from (3.37) that

$$\begin{aligned}\delta^\varepsilon_+(t)=&e^{it\Delta/2}\delta^\varepsilon_+(0)-i\int_0^t e^{i(t-s)\Delta/2}\big(\Pi^0_+(V(s,\mathbf{x})\delta^\varepsilon_+(s))\big)\,ds\\&-i\int_0^t e^{i(t-s)\Delta/2}\Pi^0_+\big(G(\varphi^\varepsilon_+(s),\varphi^\varepsilon_-(s))\varphi^\varepsilon_+(s)-G(\varphi^0_+(s),\varphi^0_-(s))\varphi^0_+(s)\big)\,ds\\&-i\int_0^t e^{i(t-s)\Delta/2}(\varepsilon\mathcal{L}^\varepsilon\varphi^\varepsilon_+(s)+L_{+,1}(s)+L_{+,2}(s)+L_{+,3}(s))\,ds,\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\|\delta^\varepsilon_+(t)\|_{H^{m-3}}\le&\|\delta^\varepsilon_+(0)\|_{H^{m-1}}+C\int_0^t\|\delta^\varepsilon_+(s)\|_{H^{m-3}}ds\\&+\int_0^t\big\|G(\varphi^\varepsilon_+(s),\varphi^\varepsilon_-(s))\varphi^\varepsilon_+(s)-G(\varphi^0_+(s),\varphi^0_-(s))\varphi^0_+(s)\big\|_{H^{m-3}}ds\\&+\int_0^t(\varepsilon\|\mathcal{L}^\varepsilon\varphi^\varepsilon_+(s)\|_{H^{m-3}}+\|L_{+,1}(s)+L_{+,2}(s)+L_{+,3}(s)\|_{H^{m-3}})\,ds.\end{aligned} \tag{3.40}$$

Similarly to Lemma 3.2, for $0\le s\le t\le T_*$, there holds

$$\begin{aligned}&\|G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_+^\varepsilon(s)-G(\varphi_+^0(s),\varphi_-^0(s))\varphi_+^0(s)\|_{H^{m-3}}\\ \le& C\Big(\|\delta_+^\varepsilon(s)\|_{H^{m-3}}+\|\delta_-^\varepsilon(s)\|_{H^{m-3}}\Big).\end{aligned} \tag{3.41}$$

It is easy to verify that

$$\begin{aligned}&\|L_{+,1}(s)\|_{H^{m-3}}+\|L_{+,3}(s)\|_{H^{m-3}}\\ \le&\varepsilon\Big(\|V(s,\cdot)\varphi_+^\varepsilon\|_{H^{m-2}}+\|G(\varphi_+^\varepsilon(s),\varphi_-^\varepsilon(s))\varphi_+^\varepsilon(s)\|_{H^{m-2}}\Big)\le C\varepsilon.\end{aligned}$$

For $L_{+,2}$, since $\Pi_+^\varepsilon\varphi_+^\varepsilon=\varphi_+^\varepsilon$ and $\Pi_+^0\alpha_j\Pi_+^0=\mathbf{0}$, we find

$$\begin{aligned}&\Pi_+^\varepsilon\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon\Big)\\ =&\Pi_+^0\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j(\Pi_+^\varepsilon\varphi_+^\varepsilon)\Big)+\varepsilon\mathcal{R}_1\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon\Big)\\ =&\Pi_+^0\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j(\Pi_+^0\varphi_+^\varepsilon)\Big)+\varepsilon\Pi_+^0\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j(\mathcal{R}_1\varphi_+^\varepsilon)\Big)+\varepsilon\mathcal{R}_1\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon\Big)\\ =&\varepsilon\Pi_+^0\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j(\mathcal{R}_1\varphi_+^\varepsilon)\Big)+\varepsilon\mathcal{R}_1\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon\Big),\end{aligned}$$

which implies

$$\begin{aligned}&\|\Pi_+^\varepsilon\Big(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon\Big)\|_{H^{m-3}}\\ \le&\varepsilon\left(\|\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j(\mathcal{R}_1\varphi_+^\varepsilon)\|_{H^{m-3}}+\|\mathcal{R}_1(\sum_{j=1}^3 A_j(s,\mathbf{x})\alpha_j\varphi_+^\varepsilon)\|_{H^{m-3}}\right)\\ \le& C\varepsilon\|\varphi_+^\varepsilon\|_{H^{m-2}}\Big(\sum_{j=1}^3\|A_j(s,\cdot)\|_{H^{m-1}}\Big)\le C\varepsilon.\end{aligned}$$

Finally, $\|\mathcal{L}^\varepsilon\varphi_+^\varepsilon\|_{H^{m-3}}\le C\|\varphi_+^\varepsilon\|_{H^m}\le C$. Noticing (3.36), we can obtain from (3.40) that

$$\|\delta_+^\varepsilon(t)\|_{H^{m-3}}\le C\varepsilon+C\int_0^t\Big(\|\delta_+^\varepsilon(s)\|_{H^{m-3}}+\|\delta_-^\varepsilon(s)\|_{H^{m-3}}\Big)ds,\quad t\le T_*. \tag{3.42}$$

Similarly, we can derive from that

$$\|\delta_-^\varepsilon(t)\|_{H^{m-3}}\le C\varepsilon+C\int_0^t\Big(\|\delta_+^\varepsilon(s)\|_{H^{m-3}}+\|\delta_-^\varepsilon(s)\|_{H^{m-3}}\Big)ds,\quad t\le T_*. \tag{3.43}$$

In view of the Gronwall inequality, (3.42) and (3.43) yield

$$\|\delta_+^\varepsilon(t)\|_{H^{m-3}}+\|\delta_-^\varepsilon(t)\|_{H^{m-3}}\le C\varepsilon. \tag{3.44}$$

Theorem 2.2 would suggest that

$$\begin{aligned}&\|\psi^\varepsilon-e^{-it\beta/\varepsilon^2}\varphi\|_{H^{m-3}}\\ \le&\|\psi^\varepsilon-e^{-it/\varepsilon^2}\varphi_+^\varepsilon-e^{it/\varepsilon^2}\varphi_-^\varepsilon\|_{H^{m-3}}+\|\delta_+^\varepsilon(t)\|_{H^{m-3}}+\|\delta_-^\varepsilon(t)\|_{H^{m-3}}\le C\varepsilon,\end{aligned}$$

and Theorem 2.3 is proved. □

4 Conclusion

The nonlinear Dirac equation with external electro-magnetic potentials was considered. There is a dimensionless parameter ε inversely proportional to the speed of light in the NLD. We investigated the NLD in the nonrelativistic limit regime, i.e., when $0<\varepsilon\ll 1$. By projecting the solution into the eigenspaces of the "free Dirac operator", we obtained the coupled nonlinear Schrödinger system as the nonrelativistic limit and a second order approximation as the semi-nonrelativistic limit, when $\varepsilon\to 0^+$. The analysis provided the detailed structure of the NLD in the nonrelativistic limit. In particular, the semi-nonrelativistic limit, which is an $O(\varepsilon^2)$ approximation to the NLD, is well-suited for numerical implementations, since the differential operators can be easily calculated in Fourier space.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grants No. 91630204 and 11771036), and the Fundamental Research Funds for the Central Universities (Grants No. CCNU19TD010).

References

[1] Ablowitz M J, Zhu Y. Nonlinear waves in shallow honeycomb lattices. SIAM J Appl Math, 2012, 72: 240-260.
[2] Alvarez A, Kuo P Y, Vazquez L. The numerical study of a nonlinear one-dimensional Dirac equation. Appl Math Comput, 1983, 13: 1-15.
[3] Anderson C D. The positive electron. Phys Rev, 1933, 43: 491-498.
[4] Bao W, Cai Y. Mathematical theory and numerical methods for Bose-Einstein condensation. Kinet Relat Mod, 2013, 6: 1-135.
[5] Bao W, Cai Y, Jia X, *et al.* A uniformly accurate multiscale time integrator pseudospectral method for the Dirac equation in the nonrelativistic limit regime. SIAM J Numer Anal, 2016, 54: 1785-1812.

[6] Bao W, Cai Y, Jia X, *et al.* Error estimates of numerical methods for the nonlinear Dirac equation in the nonrelativistic limit regime. Sci China Math, 2016, 59: 1461-1494.
[7] Bechouche P, Mauser N, Poupaud F. (Semi)-nonrelativistic limits of the Dirac eqaution with external time-dependent electromagnetic field. Commun Math Phys, 1998, 197: 405-425.
[8] Bournaveas N. Local existence of energy class solutions for the Dirac-Klein-Gordon equations. Commun Part Diff Eq, 1999, 24: 1167-1193.
[9] Cai Y, Wang Y. A uniformly accurate (UA) multiscale time integrator pseudospectral method for the nonlinear Dirac equation in the nonrelativistic limit regime. ESAIM: M2AN, 2018, 52: 543-566.
[10] Cazenave T. Semilinear Schrödinger Equations. Courant Lect Notes Math, 10, Amer Math Soc, Providence, RI, 2003.
[11] Cirincione R J, Chernoff P R. Dirac and Klein-Gordon equations: convergence of solutions in the nonrelativistic limit. Commun Math Phys, 1981, 79: 33-46.
[12] Chang S J, Ellis S D, Lee B W. Chiral confinement: An exact solution of the massive Thirring model. Phys Rev D, 1975, 11: 3572-3582.
[13] Dias J P, Figueira M. On the existence of weak solutions for a nonlinear time dependent Dirac equation. Proc Roy Soc Edinburgh, 1989, 113: 149-158.
[14] Dirac P A M. The quantum theory of the electron. Proc Roy Soc A, 1928, 117: 610-624.
[15] Dirac P A M. A theory of electrons and protons. Proc Roy Soc A, A, 1930, 126: 360-365.
[16] Escobedo M, Vega L. A semilinear Dirac equation in $H^s(\mathbb{R}^3)$ for $s>1$. SIAM J Math Anal, 1997, 28: 338-362.
[17] Esteban M, Séré E. An overview on linear and nonlinear Dirac equations. Discrete Contin Dyn Syst, 2002, 8: 381-397.
[18] Fefferman C L, Weistein M I. Honeycomb lattice potentials and Dirac points. J Amer Math Soc, 2012, 25: 1169-1220.
[19] Fefferman C L, Weistein M I. Wave packets in honeycomb structures and two-dimensional Dirac equations. Commun Math Phys, 2014, 326: 251-286.
[20] Foldy L L, Wouthuysen S A. On the Dirac theory of spin 1/2 particles and its nonrelativistic limit. Phys Rev, 1950, 78: 29-36.
[21] Gross L. The Cauchy problem for the coupled Maxwell and Dirac equations. Comm Pure Appl Math, 1966, 19: 1-15.
[22] Haddad L H, Carr L D. The nonlinear Dirac equation in Bose-Einstein condensates: Foundation and symmetries. Phys. D, 2009, 238: 1413-1421.
[23] Hunziker W. On the nonrelativistic limit of the Dirac theory. Commun Math Phys, 1975, 40: 215-222.
[24] Machihara S, Nakanishi K, Ozawa T. Small global solutions and the nonrelativistic limit for the nonlinear Dirac equation. Rev Mat Iberoamericana, 2003, 19: 179-194.
[25] Masmoudi N, Mauser N J. The selfconsistent Pauli equation. Monatsh Math, 2001, 132: 19-24.
[26] Masmoudi N, Nakanishi K. From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations. Math Ann, 2002, 324: 359-389.
[27] Matsuyama T. Rapidly decreasing solutions and nonrelativistic limit of semilinear Dirac equation. Rev Math Phys, 1995, 7: 243-267.
[28] Mauser N J. Rigorous derivation of the Pauli equation with time-dependent electromagnetic field. VLSI Des, 1999, 9: 415-426.
[29] Merkl M, Jacob A, Zimmer F E, *et al.* Chiral confinement in quasi relativistic Bose-Einstein condensates. Phys Rev Lett, 2010, 104: 073603.

[30] Najman B. The nonrelativistic limit of the nonlinear Dirac equation. Ann Inst Henri Poincaré, 1992, 9: 3-12.
[31] Schoene A Y. On the nonrelativistic limits of the Klein-Gordon and Dirac equations. J Math Anal Appl, 1979, 71: 36-74.
[32] Soler M. Classical, stable, nonlinear spinor field with positive rest energy. Phys Rev D, 1970, 1: 2766-2769.
[33] White G B. Splitting of the Dirac operator in the nonrelativistic limit. Ann Inst Henri Poincaré, 1990, 53: 109-121.
[34] Xu J, Shao S, Tang H. Numerical methods for nonlinear Dirac equation. J Comput Phys, 2013, 245: 131-149.