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The Fundamental and Rigidity Theorems for Pseudohermitian Submanifolds in the Heisenberg Groups

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Dedicated to Professors Sun-Yung Alice Chang and Paul C. Yang on their 70th birthdays

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Abstract. In this paper, we study some basic geometric properties of pseudohermitian submanifolds of the Heisenberg groups. In particular, we obtain the uniqueness and existence theorems, and some rigidity theorems.

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Key words: Motion equations, structure equations, Darboux frame, Darboux derivative.

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1 Introduction

In this paper, for $m \leq n$, we specify the ranges of indices as follows

$$
\begin{aligned}
&1 \leq \alpha,\beta,\gamma,\sigma,\rho,\cdots \leq n, \quad 1 \leq j,k,l,\cdots \leq m,\\
&m+1 \leq a,b,c,\cdots \leq n, \quad 1 \leq A,B,C,\cdots \leq 2n.
\end{aligned}
$$

1.1 The Heisenberg groups

The origin of pseudohermitian geometry came from the construction of a pseudohermitian connection, independently by N. Tanaka [15] and S. Webster [16]. In this paper, the Heisenberg group is a pseudohermitian manifold and it plays the role of the model in pseudohermitian geometry. That is, any pseudohermitian manifold with vanishing curvature and torsion locally is part of the Heisenberg group. Let H_n be the Heisenberg group, with coordinates (x_β,y_β,t). The group multiplication is defined by

$$(x,y,t)\circ(x',y',t') = (x+x',y+y',t+t'+yx'-xy').$$

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The associated standard CR structure J and contact form Θ are defined respectively by

$$J\mathring{e}_\beta = \mathring{e}_{n+\beta}, \qquad J\mathring{e}_{n+\beta} = -\mathring{e}_\beta,$$
$$\Theta = dt + \sum_{\beta=1}^{n} x_\beta dy_\beta - y_\beta dx_\beta,$$

where

$$\mathring{e}_\beta = \frac{\partial}{\partial x_\beta} + y_\beta \frac{\partial}{\partial t}, \; \mathring{e}_{n+\beta} = \frac{\partial}{\partial y_\beta} + x_\beta \frac{\partial}{\partial t}.$$

The contact bundle is $\xi = \ker\Theta$. We refer the reader to [2, 3, 5] for the details about the Heisenberg groups, and to [6, 11, 12, 15, 16] for pseudohermitian geometry.

The symmetry group $PSH(n)$ of H_n is the group consisting of all pseudohermitian transformations. Left translations L_p are symmetries. Another kind of examples are a rotation Φ_R around the t-axis which is defined by

$$\Phi_R \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix},$$

where $R = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in SO(2n)$. In [5], we showed that each symmetry $\Phi \in PSH(n)$ has the unique decomposition $\Phi = L_p \circ \Phi_R$, for some $p \in H_n$ and $R \in SO(2n)$. Since the action of $PSH(n)$ on H_n is transitive, the associated geometry is a kind of Klein geometry. The corresponding Cartan geometry is just pseudohermitian geometry.

1.2 Pseudohermitian submanifolds

We now give the definition of pseudohermitian submanifold.

Definition 1.1. *A $(2m+1)$-dimensional pseudohermitian manifold $(M, \hat{J}, \hat{\theta})$ is called a pseudohermitian*† *submanifold of H_n, $1 \le m \le n$, if*

- $\hat{\xi} = TM \cap \xi$;
- $\hat{J} = J|_{\hat{\xi}}$;
- $\hat{\theta} = \Theta|_M$,

where $\hat{\xi} = \ker\hat{\theta}$ is the contact structure on M. The number m is called the CR dimension of M.

Example 1.1. Suppose $M \hookrightarrow H_n$ is an embedded submanifold with CR dimension $n-1$. Then it is not hard to see that

†In [6], S. Dragomir and G. Tomassini call it *isopseudo-hermitian*, instead of *pseudohermitian*.

- In general, $\dim(T_pM\cap\xi_p)\geq 2n-2$, for all $p\in M$.
- $\dim(T_pM\cap\xi_p)=2n-2$, for a generic point $p\in M$.

All the generic points constitute the regular part of M, and those points p such that $\dim(T_pM\cap\xi_p)=2n-1$ are called the singular points. On the regular part M_{re}, assume that $T_pM\cap\xi$ is invariant under J, then it inherits a pseudohermitian structure $(\hat{J},\hat{\theta})$ from H_n such that $(M_{re},\hat{J},\hat{\theta})$ is a pseudohermitian submanifold of H_n.

In Section 3, we define some local invariants for pseudohermitian submanifolds, including the second fundamental form, the normal connection and the fundamental vector field ν (see the definition after Proposition 3.1). In addition, from Proposition 3.1, we see that the fundamental vector field ν actually describes the difference between the two Reeb vector fields T and $\hat{T}$, which are, respectively, associated with H_n and the pseudohermitian submanifold M. Hence if $\nu=0$, then $\hat{T}=T$. That means that $T=\frac{\partial}{\partial t}$ is always tangent to M at each point. Therefore, for such a submanifold, we call it *vertical*‡. On the other hand, if $\nu\neq 0$ at each point, we call it *completely non-vertical*.

Example 1.2. The subspace $H_m=\{(z,t)\in H_n \mid z_a=0\}\subset H_n$ is a pseudohermitian submanifold of H_n. It is easy to see that H_m is *vertical*.

Example 1.3. Let $S^{2n-1}\subset H_n$ be the sphere defined by

$$S^{2n-1}=\left\{(z,0)\in C^n\subset H_n \mid \sum_{\beta=1}^{n} z_\beta z_{\bar{\beta}}=1\right\}.$$

There are two pseudohermitian structures induced on S^{2n-1}, one is from H_n and the other is from C^n. In Subsection 4.2, we show that these two induced pseudohermitian structures coincide. In addition, S^{2n-1} is *completely non-vertical*.

1.3 Main theorems

There are many literatures which were given for the problem about CR embeddability of CR manifolds into spheres [4,9,10,17]. In this paper, we obtain the fundamental theorems and rigidity theorems for pseudohermitian submanifolds in the Heisenberg groups. We have

Theorem 1.1 (Theorem A). *The induced pseudohermitian structure, the second fundamental form, the normal connection, as well as the fundamental vector field constitute a complete set of invariants for pseudohermitian submanifolds of the Heisenberg groups.*

‡In [6], S. Dragomir and G. Tomassini call it *pseuo-Hermitian*, instead of *vertical pseudohermitian*.

Theorem A is shown in Section 5. It specifies that there are only four invariants for pseudohermitian submanifolds. That is, if two pseudohermitian submanifolds have the same such four invariants, then they are locally congruent with each other in the sense that they differ from each other by nothing more than an action of a symmetry.

Any pseudohermitian submanifold $M\subset H_n$ automatically satisfies a natural geometric condition which we call *the integrability condition* (defined in Subsection 5.2). Conversely, we will show that it is also a condition for an arbitrary pseudohermitian manifold to be (locally) embedded as a pseudohermitian submanifold of H_n.

Theorem 1.2 (Theorem B)**.** *Let (M^{2m+1},J_M,θ_M) be a simply connected pseudohermitian manifold satisfying the integrability condition for some $n\geq m$. Then M can be embedded as a pseudohermitian submanifold of the Heisenberg group H_n.*

Theorem B is shown in Section 6. In [9], S.-Y. Kim and J.-W. Oh also studied the problem of characterizing pseudohermitian manifolds which are pseudohermitian embeddable into the Heisenberg groups[§]. In the case that M is nondegenerate, S.-Y. Kim and J.-W. Oh used Cartan's prolongation method to show that the induced pseudohermitian structure constitutes a complete set of invariants. In addition, they gave a necessary and sufficient condition, in terms of Webster curvature and torsion tensor, for pseudohermitian manifolds to be embeddable into the Heisenberg groups nondegenerately. This condition is just equivalent to the integrability condition which we define in Subsection 5.2. However, S.-Y. Kim and J.-W. Oh did not deal with the degenerate cases.

In the case of CR codimension one, the nondegeneracy just means that the second fundamental form does not vanish at each point. In such a case, we basically recover the results of S.-Y. Kim and J.-W. Oh. Moreover, we give the rigidity theorems for pseudohermitian degenerate submanifolds, which are shown in Section 7. We have

Theorem 1.3 (Theorem C)**.** *Let $(M,\hat{J},\hat{\theta})$ be a* vertical*, simply connected pseudohermitian submanifold of H_n with CR dimension $m=n-1$. Then we have*
(i) if the second fundamental form $II\neq 0$ at each point, then the induced pseudohermitian structure $(\hat{J},\hat{\theta})$ constitutes a complete set of invariants.
(ii) if $II=0$, then M is an open part of $H_{n-1}=\{z_n=0\}$, after a Heisenberg rigid motion.

Theorem 1.4 (Theorem D)**.** *Let $(M,\hat{J},\hat{\theta})$ be a* completely non-vertical*, simply connected pseudohermitian submanifold of H_n with CR dimension $m=n-1$. Then we have*
(i) the induced pseudohermitian structure $(\hat{J},\hat{\theta})$ constitutes a complete set of invariants.
(ii) if the second fundamental form $II=0$ (or, equivalently, the pseudohermitian torsion $A_{jk}=0$, $1\leq j,k\leq m$), then M is an open part of the standard sphere $S^{2m+1}\subset H_n$, after a Heisenberg rigid motion.

Finally, in subsection 4.1, we study the general properties of vertical pseudohermitian submanifolds and obtain

[§]In their paper, they used the pseudohermitian flat sphere as the ambient space, instead of the Heisenberg group. But after a Cayley transformation, this two spaces are isomorphic as pseudohermitian manifolds

Theorem 1.5 (Theorem E)**.** *Let* $(M,\hat{J},\hat{\theta})$ *be a* vertical *pseudohermitian submanifold of* H_n. *Then we have that the Webster-torsion vanishes and the Webster-Ricci tensor is non-positive, as well as the pseudohermitian connection and the tangential connection (see the definition in subsection 3.3) coincide.*

For the fundamental theorems, we used Cartan's method of moving frame as well as calculus on Lie groups. And we prove (ii) of Theorem D by means of the motions equation of the Darboux frame. Therefore, in Section 2, we give a brief review of Cartan's method of moving frame, which includes the motion equations and the structure equations. We would like to end the introduction by pointing out that, in [4], Curry and Gover recently addressed the so called CR Bonnet theorem. They formulated and proved the theorem inspired by the conformal Bonnet theorem formulated and proved in terms of standard conformal tractors.

2 Cartan's method of moving frame

In this section, we give a brief review of Cartan's method of moving frame and Calculus on Lie groups. For the details, we refer the reader to [5]. Let (X,G) be a Klein geometry. The philosophy of Elie Cartan is that in many cases, the symmetry group G may be identified with a set of frame on X. Then to investigate the geometry of a submanifold M of X, one associates the submanifold with a natural set of frames. In this situation, the infinitesimal motion of this natural frame should contain all the geometric information of the submanifold M. Now we will go along the idea of Elie Cartan to get a complete set of invariants for M.

2.1 The frames on H_n

An *frame* for H_n is a set of vectors of the form

$$(p;e_\beta,e_{n+\beta},T),$$

where $p\in H_n$, $e_\beta\in\xi(p)$ and $e_{n+\beta}=Je_\beta$, for $1\le\beta\le n$. In addition $\{e_\beta,e_{n+\beta},T\}$ is an orthonormal frame with respect to the adapted metric g_θ, which is defined by viewing the basis $\mathring{e}_\beta,\mathring{e}_{n+\beta},T$ as an orthonormal basis.

2.2 Identifying $PSH(n)$ with a set of frames

We identify a symmetry Φ with a frame $(p;e_\beta,e_{n+\beta},T)$, provided that Φ is the unique transformation on H_n mapping the frame $(0;\mathring{e}_\beta,\mathring{e}_{n+\beta},T)$ to the given frame $(p;e_\beta,e_{n+\beta},T)$. That is

$$\Phi_*(0;\mathring{e}_\beta,\mathring{e}_{n+\beta},T)=(\Phi(0);\Phi_*\mathring{e}_\beta,\Phi_*\mathring{e}_{n+\beta},\Phi_*T)=(p;e_\beta,e_{n+\beta},T).$$

2.3 The matrix group representation of $PSH(n)$

If we identify points of H_n and $1\times H_n$ by

$$p\leftrightarrow\left(\begin{array}{c}1\\p\end{array}\right),$$

hence a vector $X\in TH_n$ can be identified by

$$X\leftrightarrow\left(\begin{array}{c}0\\X\end{array}\right).$$

We thus identify Φ with a matrix $A\in GL(2n+2,R)$ by

$$\Phi\leftrightarrow(p;e_\beta,e_{n+\beta},T)\leftrightarrow A=\left(\begin{array}{cccc}1&0&0&0\\p&e_\beta&e_{n+\beta}&T\end{array}\right). \tag{2.1}$$

We have

$$A\left(\begin{array}{c}1\\q\end{array}\right)=\left(\begin{array}{c}1\\\tilde{q}\end{array}\right),\quad \tilde{q}=\Phi(q).$$

This shows that (2.1) gives a matrix group representation of $PSH(n)$.

2.4 The motion equations

Let ω be the (left) Maurer Cartan form of $PSH(n)$. This is a $psh(n)$-valued one form defined by

$$\omega(v)=L_{g^{-1}*}v, \tag{2.2}$$

for each $v\in T_gG$, where $G=PSH(n),g\in G$. That is, the Maurer Cartan form moves each vector v to the identity element by the left translations. It is a natural way for us to identify each vector v with a vector tangent to the identity. Since $PSH(n)$ has a matrix group representation, The Maurer Cartan form has the simple elegant expression

$$\omega=A^{-1}dA, \tag{2.3}$$

where $A\in PSH(n)$ is the moving point. This formula (2.3) is equivalent to

$$dA=A\omega, \tag{2.4}$$

which is called the motion equations of the Heisenberg group. Taking the exterior derivative of the motion equations, we get the structure equations

$$d\omega+\omega\wedge\omega=0. \tag{2.5}$$

2.5 The Darboux frames for pseudohermitian submanifolds

Let $U\subset M$ be an open subset. For each point $p\in U$, we always choose the frame $\{Z_\beta,T\}$ such that $Z_j\in\hat{\xi}_C$ and $Z_a\in\hat{\xi}_C^\perp$, here $\hat{\xi}_C=\hat{\xi}\otimes C$ and $\hat{\xi}_C^\perp=\hat{\xi}^\perp\otimes C$. Such a moving frame $p\to(p;Z_\beta,T)$ is called the *Darboux frame* (of complex version) over U.

Let $\{\theta^\beta,\theta\}$ be the dual of $\{Z_\beta,T\}$. Writing $Z_\beta=\frac{1}{2}(e_\beta-ie_{n+\beta})$ and $\theta^\beta=\omega^\beta+i\omega^{n+\beta}$. Then $\{e_A,T\}$ and $\{\omega^A,\theta\}$ are dual to each other. This frame field $p\to(p;e_A,T)$ is the *real version of the Darboux frame*. It is easy to see that $e_k,e_{n+k}\in\hat{\xi}$, $e_a,e_{n+a}\in\xi^\perp$, and $e_{n+\beta}=Je_\beta$.

Denoting $\hat{Z}_j=Z_j$ and writing $\hat{Z}_j=\frac{1}{2}(\hat{e}_j-i\hat{e}_{m+j})$. Then we have $\hat{e}_j=e_j$, $\hat{e}_{m+j}=e_{n+j}$. Let $\{\hat{\omega}^j,\hat{\omega}^{m+j},\hat{\theta}\}$ be the dual of $\{\hat{e}_j,\hat{e}_{m+j},\hat{T}\}$. We also denote $\hat{\theta}^j=\hat{\omega}^j+i\hat{\omega}^{m+j}$.

2.6 The Darboux derivative

Let $f:U\to PSH(n)$ be a Darboux frame $f(p)=(p;e_A,T)$. The Darboux derivative ω_f of f is defined by

$$\omega_f=\omega\circ f_*=f^*\omega. \tag{2.6}$$

Therefore, it is just the usual differential f_*, provided that we have identified each vector with a vector tangent to the identity element by left translations. From (2.3),

$$\omega_f=f^*\omega=f^*(A^{-1}dA)=(A\circ f)^{-1}d(A\circ f)=f^{-1}df, \tag{2.7}$$

or, equivalently

$$df=f\omega_f. \tag{2.8}$$

This is the motion equations for the Darboux frame f. Again, taking the exterior derivative, we obtain the structure equations (the integrability condition)

$$d\omega_f+\omega_f\wedge\omega_f=0. \tag{2.9}$$

Writing

$$f(p)=(p;e_\beta(p),e_{n+\beta}(p),T)$$

and

$$\omega_f=\begin{pmatrix} 0 & 0 & 0 & 0\\ \omega^\beta & {\omega_\alpha}^\beta & {\omega_{n+\alpha}}^\beta & 0\\ \omega^{n+\beta} & {\omega_\alpha}^{n+\beta} & {\omega_{n+\alpha}}^{n+\beta} & 0\\ \omega^{2n+1} & \omega^{n+\alpha} & -\omega^\alpha & 0\end{pmatrix}.$$

Since ω_f is a $psh(n)$-valued one form, the entry forms satisfy

$$\begin{aligned}&{\omega_a}^b=-{\omega_b}^a,\quad \text{for } 1\le a,b\le 2n,\\ &{\omega_{n+\alpha}}^{n+\beta}={\omega_\alpha}^\beta,\ \ {\omega_\alpha}^{n+\beta}=-{\omega_{n+\alpha}}^\beta,\quad \text{for } 1\le\alpha,\beta\le n.\end{aligned}$$

Then the motion equations and structure equations, respectively, read

$$\begin{aligned}
&dp = e_\beta \otimes \omega^\beta + e_{n+\beta} \otimes \omega^{n+\beta} + T \otimes \omega^{2n+1},\\
&de_\gamma = e_\beta \otimes {\omega_\gamma}^\beta + e_{n+\beta} \otimes {\omega_\gamma}^{n+\beta} + T \otimes \omega^{n+\gamma},\\
&de_{n+\gamma} = e_\beta \otimes {\omega_{n+\gamma}}^\beta + e_{n+\beta} \otimes {\omega_{n+\gamma}}^{n+\beta} - T \otimes \omega^\gamma, \qquad dT = 0;
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&d\omega^\beta = -{\omega_\alpha}^\beta \wedge \omega^\alpha - {\omega_{n+\alpha}}^\beta \wedge \omega^{n+\alpha}, \qquad d\omega^{n+\beta} = -{\omega_\alpha}^{n+\beta} \wedge \omega^\alpha - {\omega_{n+\alpha}}^{n+\beta} \wedge \omega^{n+\alpha},\\
&d\omega^{2n+1} = 2\sum_{\alpha=1}^{n} \omega^\alpha \wedge \omega^{n+\alpha}, \qquad d{\omega_\alpha}^\beta = -{\omega_\gamma}^\beta \wedge {\omega_\alpha}^\gamma - {\omega_{n+\gamma}}^\beta \wedge {\omega_\alpha}^{n+\gamma},\\
&d{\omega_{n+\alpha}}^\beta = -{\omega_\gamma}^\beta \wedge {\omega_{n+\alpha}}^\gamma - {\omega_{n+\gamma}}^\beta \wedge {\omega_{n+\alpha}}^{n+\gamma}.
\end{aligned} \tag{2.11}$$

2.7 The complex version

Writing

$$F(p) = (p; Z_\beta(p), T), \quad \text{where } Z_\beta = \frac{1}{2}(e_\beta - i e_{n+\beta}),$$

and

$$\omega_F = \begin{pmatrix} 0 & 0 & 0 \\ \vartheta^t & {\theta_\gamma}^\beta & 0 \\ \theta & i\bar{\vartheta} & 0 \end{pmatrix}, \tag{2.12}$$

where $\vartheta = (\theta^1, \cdots, \theta^n)$, $\theta^\beta = \omega^\beta + i\omega^{n+\beta}$, ${\theta_\gamma}^\beta = {\omega_\gamma}^\beta + i{\omega_\gamma}^{n+\beta}$. And hence we have ${\theta_\gamma}^\beta + {\theta_{\bar\beta}}^{\bar\gamma} = 0$. We have the complex version of motion equations

$$dp = Z_\beta \otimes \theta^\beta + Z_{\bar\beta} \otimes \theta^{\bar\beta} + T \otimes \theta, \qquad dZ_\gamma = Z_\beta \otimes {\theta_\gamma}^\beta + \frac{1}{2} T \otimes i\theta^{\bar\gamma}, \qquad dT = 0. \tag{2.13}$$

And the structure equations are equivalent to

$$d\omega_F + \omega_F \wedge \omega_F = 0, \tag{2.14}$$

or

$$d\theta^\beta = \theta^\gamma \wedge {\theta_\gamma}^\beta, \qquad d\theta = i\theta^\gamma \wedge \theta^{\bar\gamma}, \qquad d{\theta_\sigma}^\beta = {\theta_\sigma}^\gamma \wedge {\theta_\gamma}^\beta. \tag{2.15}$$

2.8 Calculus on Lie groups

Let M be a *simply connected* smooth manifold, $f: M \to PSH(n)$ be a smooth map. Recall that The (left) *Darboux derivative* ω_f of f is the $psh(n)$-valued 1-form defined by $\omega_f = \omega \circ f_* = f^*\omega$. The Darboux derivative plays an important role in the theory of calculus on Lie groups. The fundamental theorems are Theorem 2.1 and Theorem 2.2.

Theorem 2.1 (**The uniqueness theorem).** *Let* $f_1,f_2:M\to PSH(n)$ *be smooth maps. Then* $\omega_{f_1}=\omega_{f_2}$ *if and only if there exists* $g\in PSH(n)$ *such that* $f_2(x)=g\cdot f_1(x)$ *for all* $x\in M$.

Theorem 2.1 says that two maps from M into $PSH(n)$ are congruent to each other if and only if they have the same infinitesimal motions. Recall that ω_f satisfies the integrability conditions

$$d\omega_f+\omega_f\wedge\omega_f=0.$$

Conversely, one has

Theorem 2.2 (**The existence theorem).** *Let* η *be a* $psh(n)$*-valued one form on* M *satisfying* $d\eta+\eta\wedge\eta=0$. *Then there is a smooth map* $f:U\to PSH(n)$ *such that* $\eta|_U=\omega_f$.

Theorem 2.2 totally depends on Frobenius Theorem. We will apply theorem 2.1 to the Darboux frames of pseudohermitian submanifolds. Then, to prove Theorem A, we are reduced to compute the Darboux derivatives of the Darboux frames. And using Theorem 2.2, we obtain Theorem B. For the details about calculus on Lie groups, we refer the reader to [1, 7, 8, 13, 14].

3 Local invariants of Pseudohermitian submanifolds

In this section, we define some geometric invariants for pseudohermitian submanifolds.

3.1 The fundamental vector field ν

Proposition 3.1. *Let* $(M,\hat{J},\hat{\theta})$ *be a pseudohermitian submanifold of* H_n. *Then there exists a unique horizontal vector field* $\nu\in\hat{\xi}^{\perp}$ *such that* $T+\nu\in TM$. *Actually, denoting* $\hat{T}=T+\nu$, *it is not hard to see that* $\hat{T}$ *is the Reeb vector field associated to* $\hat{\theta}$.

Proof. Let $\hat{T}=aT+\sum_{A=1}a^Ae_A$, for some coefficients a,a^A. Since $1=\hat{\theta}(\hat{T})=\theta(\hat{T})=a$ and $\hat{T}\perp\hat{\xi}$, we have $\hat{T}=T+a^ae_a+a^{n+a}e_{n+a}$, and hence we can choose $\nu=a^ae_a+a^{n+a}e_{n+a}$. Next, suppose $\tilde{\nu}\in\hat{\xi}^{\perp}$ is another vector such that $T+\tilde{\nu}\in TM$. Then we have $\nu-\tilde{\nu}\in TM\cap\xi$, hence $\nu=\tilde{\nu}$. □

We call ν in Proposition 3.1 the fundamental vector field.

- If $\nu\equiv 0$, then $\hat{T}=T$, and hence we call M^{2m+1} a *vertical* submanifold.
- If $\nu\neq 0$ at each point of M, then M is *completely non-vertical*.

Proposition 3.2. *We have*

$$\omega^j|_M=\hat{\omega}^j,\ \ \omega^{n+j}|_M=\hat{\omega}^{m+j},\ \ \omega^a|_M=\frac{1}{2}\langle\nu,e_a\rangle\hat{\theta},\ \ \omega^{n+a}|_M=\frac{1}{2}\langle\nu,e_{n+a}\rangle\hat{\theta}, \tag{3.1}$$

where $\langle \, , \, \rangle$ *is the Levi-metric, hence*

$$\theta^j|_M = \hat{\theta}^j \quad \theta^a|_M = \langle \nu, Z_a \rangle \hat{\theta}. \tag{3.2}$$

In particular, if $\nu = 0$*, then we have* $\theta^a|_M = 0$*.*

Proof. We compute

$$\omega^j(\hat{T}) = \omega^j(T+\nu) = 0, \qquad \omega^j(\hat{e}_k) = \omega^j(e_k) = \delta_{jk}, \qquad \omega^j(\hat{e}_{m+k}) = \omega^j(e_{n+k}) = 0, \tag{3.3}$$

and

$$\begin{aligned} &\omega^{n+j}(\hat{T}) = \omega^{n+j}(T+\nu) = 0, \qquad \omega^{n+j}(\hat{e}_k) = \omega^{n+j}(e_k) = 0, \\ &\omega^{n+j}(\hat{e}_{m+k}) = \omega^{n+j}(e_{n+k}) = \delta_{jk}. \end{aligned} \tag{3.4}$$

Therefore $\{\omega^j|_M, \omega^{n+j}|_M, \theta|_M\}$ is the dual frame of $\{\hat{e}_j, \hat{e}_{m+j}, \hat{T}\}$. Similar computation shows that

$$\omega^a|_M = \frac{1}{2}\langle \nu, e_a \rangle \hat{\theta}, \qquad \omega^{n+a}|_M = \frac{1}{2}\langle \nu, e_{n+a} \rangle \hat{\theta}. \tag{3.5}$$

This completes the proof. □

3.2 The normal connection

The normal connection $\nabla^\perp$ which is defined, on the normal complex bundle $\hat{\xi}^\perp \otimes C$ spanned by Z_a, by

$$\nabla^\perp Z_a = \theta_a{}^b \otimes Z_b, \tag{3.6}$$

which is the orthogonal projection of the pseudohermitian connection ∇Z_a onto the normal bundle.

3.3 The tangential connection

The tangential connection ∇^t which is defined, on the complex bundle $\hat{\xi}_C$ spanned by Z_j, by

$$\nabla^\perp Z_j = \theta_j{}^k \otimes Z_k, \tag{3.7}$$

which is the orthogonal projection of the pseudohermitian connection ∇Z_j onto the contact bundle.

- Let $\hat{\theta}_j{}^k$ be the pseudohermitian connection forms with respect to the frame field Z_j. Then, from (5.11), we have

$$\theta_j{}^k|_M = \hat{\theta}_j{}^k + i\delta_{jk}|\nu|^2\hat{\theta}. \tag{3.8}$$

  Therefore, in general, $\nabla^t \neq \nabla^{p.h.}$, the associated pseudohermitian connection of M.

- If M is vertical, then $\nabla^t = \nabla^{p.h.}$.

3.4 The second fundamental form

Define the bilinear form II^a on $\hat{\xi}_{1,0}$ by

$$II^a(X,Y) = -\langle X, \nabla_{\overline{Y}} Z_a \rangle. \tag{3.9}$$

- We have $II^a = \theta^j \otimes \theta_j{}^a$. If $\widetilde{Z}_a = C_a{}^b Z_b$ is another normal frame field, then $\widetilde{II}^a = C_{\bar{a}}{}^{\bar{b}} II^b$.
- $II^a \otimes Z_a$ is independent of the choice of the normal frame field Z_a.

The second fundamental form II for M is defined, to be a map

$$II : \hat{\xi}_{1,0} \times \hat{\xi}_{1,0} \to \hat{\xi}_{1,0}^{\perp}, \tag{3.10}$$

by

$$II = II^a \otimes Z_a = \theta^j \otimes \theta_j{}^a \otimes Z_a. \tag{3.11}$$

4 General properties

4.1 Pseudohermitian submanifolds with $\nu \equiv 0$

The canonical example is the Heisenberg subgroup H_m which is defined by $H_m = \{(z,t) \in H_n \mid z_a = 0\}$. Now we discuss the general properties of such kind of submanifolds. From Proposition 3.2, we have

$$\theta^j|_M = \hat{\theta}^j, \text{ and } \theta^a|_M = 0. \tag{4.1}$$

Therefore, we have the structure equations

$$\begin{aligned} &d\theta^j = \theta^k \wedge \theta_k{}^j, \qquad 0 = \theta^k \wedge \theta_k{}^a \quad (\because \theta^a = 0), \qquad d\theta = i\theta^k \wedge \theta^{\bar{k}}, \\ &d\theta_j{}^l = \theta_j{}^k \wedge \theta_k{}^l + \theta_j{}^c \wedge \theta_c{}^l, \qquad d\theta_j{}^a = \theta_j{}^k \wedge \theta_k{}^a + \theta_j{}^c \wedge \theta_c{}^a, \\ &d\theta_a{}^j = \theta_a{}^k \wedge \theta_k{}^j + \theta_a{}^c \wedge \theta_c{}^j, \qquad d\theta_a{}^b = \theta_a{}^k \wedge \theta_k{}^b + \theta_a{}^c \wedge \theta_c{}^b. \end{aligned} \tag{4.2}$$

- From the first equation $d\theta^j = \theta^k \wedge \theta_k{}^j$ of (4.2), together with $\theta_k{}^j + \theta_{\bar{j}}{}^{\bar{k}}$, we have

$$\hat{\tau}^j \equiv 0, \; \hat{\theta}_k{}^j = \theta_k{}^j, \tag{4.3}$$

  where $\hat{\tau}^j$, $\hat{\theta}_k{}^j$ are the pseudohermitian torsion forms and connection forms with respect to the admissible coframe $\{\hat{\theta}^j\}$.

- From the second equation $0 = \theta^k \wedge \theta_k{}^a$ of (4.2), together with Cartan lemma, we have

$$\theta_j{}^a = h^a_{jk} \theta^k, \tag{4.4}$$

  for some functions h^a_{jk} satisfying $h^a_{jk} = h^a_{kj}$. Therefore

$$II = \theta^j \otimes \theta_j{}^a \otimes Z_a = h^a_{jk} \theta^j \otimes \theta^k \otimes Z_a, \tag{4.5}$$

- The fourth equation of (4.2)

$$d\theta_j{}^l = \theta_j{}^k \wedge \theta_k{}^l + \theta_j{}^c \wedge \theta_c{}^l \tag{4.6}$$

is called the *Gauss-like equation*. Since $\theta_j{}^k = \hat{\theta}_j{}^k$ and $\theta_j{}^c = h^c_{jk}\theta^k$, it is easy to see that the Gauss-like equation is equivalent to

$$R_{j\bar{l}\zeta\bar{\eta}} = -\sum_c h^c_{j\zeta} h^{\bar{c}}_{\bar{l}\bar{\eta}}, \tag{4.7}$$

which implies $R_{\zeta\bar{\eta}} = -\sum_{c=m+1}^n h^c_{k\zeta} h^{\bar{c}}_{\bar{k}\bar{\eta}}$, and hence the Webster-Ricci tensor is *non-positive*.

- The fifth equation of (4.2)

$$d\theta_j{}^a = \theta_j{}^k \wedge \theta_k{}^a + \theta_j{}^c \wedge \theta_c{}^a \tag{4.8}$$

is equivalent to the sixth equation of (4.2)

$$d\theta_a{}^j = \theta_a{}^k \wedge \theta_k{}^j + \theta_a{}^c \wedge \theta_c{}^j. \tag{4.9}$$

Either one is called the *Codazzi-like equation*.

- The last equation of (4.2)

$$d\theta_a{}^b = \theta_a{}^k \wedge \theta_k{}^b + \theta_a{}^c \wedge \theta_c{}^b \tag{4.10}$$

is called the *Ricci-like equation*.

4.2 Pseudohermitian submanifolds with ν nowhere zero

The canonical example is the standard sphere $S^{2n-1} \subset C^n \subset H_n = C^n \times R$, $n \geq 2$. It is defined by

$$S^{2n-1}(r) = \left\{ (z_1, \cdots, z_n, 0) \in H_n \mid \sum_{\beta=1}^n z_\beta z_{\bar{\beta}} = r^2 \right\}.$$

Let L_p be a left translation, we compute the image of $(z,0) \in S^{2n-1}(r)$,

$$L_p(z,0) = p + x_\beta \mathring{e}_\beta(p) + y_\beta \mathring{e}_{n+\beta}(p),$$

where $z_\beta = x_\beta + i y_\beta$, and hence the image of $S^{2n-1}(r)$ under L_p is

$$L_p\left(S^{2n-1}(r)\right) = \{q \in H_n \mid q - p \in \xi(p), \text{ and } |q-p| = r\}, \tag{4.11}$$

where the norm $|\cdot|$ is measured by the levi metric. Next, there are two pseudohermitian structures induced on S^{2n-1}, one is from the Heisenberg group H_n, denoted by $(\hat{J}, \hat{\theta})$, and the other is from C^n. It is easy to see that these two induced pseudohermitian structures coincide on $S^{2n-1}(r)$ as the following specifies. Let $u=\left(\sum_{\beta=1}^{n} z_\beta z_{\bar\beta}\right)-r^2$ be the defining function. We have

$$\hat\theta=\Theta|_{S^{2n-1}}=x_\beta dy_\beta - y_\beta dx_\beta=\frac{i(\bar\partial u-\partial u)}{2}, \tag{4.12}$$

hence

$$\hat\xi=\ker\hat\theta=TS^{2n-1}\cap J_{C^n}(TS^{2n-1})\subset TC^n, \tag{4.13}$$

where J_{C^n} is the standard complex structure on C^n.

Lemma 4.1. *Let $p=(z,t)\in S^{2n-1}$. If a vector $X=a_\beta\mathring{e}_\beta+a_{n+\beta}\mathring{e}_{n+\beta}\in\hat\xi(p)$, then $a_\beta y_\beta-a_{n+\beta}x_\beta=0$, where $z_\beta=x_\beta+iy_\beta$. In addition, we have*

$$X=a_\beta\mathring{e}_\beta+a_{n+\beta}\mathring{e}_{n+\beta}=a_\beta\frac{\partial}{\partial x_\beta}+a_{n+\beta}\frac{\partial}{\partial y_\beta}, \tag{4.14}$$

for all $X\in\hat\xi$.

Proof. We compute

$$X=a_\beta\mathring{e}_\beta+a_{n+\beta}\mathring{e}_{n+\beta}=a_\beta\frac{\partial}{\partial x_\beta}+a_{n+\beta}\frac{\partial}{\partial y_\beta}+(a_\beta y_\beta-a_{n+\beta}x_\beta)\frac{\partial}{\partial t}. \tag{4.15}$$

Since $\hat\xi\subset TC^n$, we get $a_\beta y_\beta-a_{n+\beta}x_\beta=0$. □

For all $X\in\hat\xi$,

$$\begin{aligned}\hat J(X)&=J(a_\beta\mathring{e}_\beta+a_{n+\beta}\mathring{e}_{n+\beta})=a_\beta\mathring{e}_{n+\beta}-a_{n+\beta}\mathring{e}_\beta\\&=a_\beta\frac{\partial}{\partial y_\beta}-a_{n+\beta}\frac{\partial}{\partial x_\beta}=J_{C^n}\left(a_\beta\frac{\partial}{\partial x_\beta}+a_{n+\beta}\frac{\partial}{\partial y_\beta}\right)\\&=J_{C^n}(X),\end{aligned} \tag{4.16}$$

which shows that $\hat J$ is also induced from J_{C^n}. On the other hand, from (4.12), we have

$$\hat T=\frac{i\left(z_\beta\frac{\partial}{\partial z_\beta}-z_{\bar\beta}\frac{\partial}{\partial z_{\bar\beta}}\right)}{r^2}=\frac{\partial}{\partial t}+\nu, \tag{4.17}$$

which implies that

$$\nu=\frac{i\left(z_\beta\frac{\partial}{\partial z_\beta}-z_{\bar\beta}\frac{\partial}{\partial z_{\bar\beta}}\right)}{r^2}-\frac{\partial}{\partial t}=\frac{i\left(x_\beta\frac{\partial}{\partial y_\beta}-y_{\bar\beta}\frac{\partial}{\partial x_{\bar\beta}}\right)}{r^2}-\frac{\partial}{\partial t}=\frac{x_\beta\mathring{e}_{n+\beta}-y_\beta\mathring{e}_\beta}{r^2}. \tag{4.18}$$

This shows that the standard sphere $S^{2n-1}(r)$ is completely non-vertical.

5 The uniqueness theorem

In this section, we are going to prove Theorem A. Let M and N be two pseudohermitian submanifolds with the same CR dimension m. Suppose Φ is a Heisenberg rigid motion such that $\Phi(M)=N$ and denote $\varphi=\Phi|_M$.

Let $\{Z_\beta\}$ be a frame field over M, and suppose $\widetilde{Z}_\beta=\Phi_*Z_\beta$, the set $\{\widetilde{Z}_\beta\}$ is a frame field over N. Suppose $\{\theta^\beta,\Theta\}$ and $\{\widetilde{\theta}^\beta,\Theta\}$ are the dual frame fields of $\{Z_\beta,T\}$ and $\{\widetilde{Z}_\beta,T\}$, respectively. Then we have

$$\theta^\beta=\Phi^*\widetilde{\theta}^\beta,\qquad \Theta=\Phi^*\Theta. \tag{5.1}$$

In particular, we have

$$\theta^j=\varphi^*\widetilde{\theta}^j,\qquad \hat{\theta}=\varphi^*\widetilde{\theta}, \tag{5.2}$$

where $\hat{\theta}$ and $\widetilde{\theta}$ are the induced contact form on M and N, respectively. (5.2) implies that φ preserves the induced pseudohermitian structures.

From the structure equation on H_n, we compute

$$\begin{array}{l} d\theta^\beta=\theta^\gamma\wedge{\theta_\gamma}^\beta=\left(\Phi^*\widetilde{\theta}^\gamma\right)\otimes{\theta_\gamma}^\beta\\ \| \\ d(\Phi^*\widetilde{\theta}^\beta)=\Phi^*(d\widetilde{\theta}^\beta), \end{array} \tag{5.3}$$

which is equivalent to

$$d\widetilde{\theta}^\beta=\widetilde{\theta}^\gamma\otimes(\Phi^{-1})^*{\theta_\gamma}^\beta. \tag{5.4}$$

Together with

$$(\Phi^{-1})^*{\theta_\gamma}^\beta+(\Phi^{-1})^*{\theta_{\bar{\beta}}}^{\bar{\gamma}}=(\Phi^{-1})^*({\theta_\gamma}^\beta+{\theta_{\bar{\beta}}}^{\bar{\gamma}})=0, \tag{5.5}$$

and by the uniqueness, we get

$${\theta_\gamma}^\beta=\Phi^*{\widetilde{\theta}_\gamma}{}^\beta. \tag{5.6}$$

In particular, we have

$${\theta_\gamma}^\beta=\varphi^*{\widetilde{\theta}_\gamma}{}^\beta, \tag{5.7}$$

and hence

$$\langle II,V\rangle=\varphi^*\langle\widetilde{II},\Phi_*V\rangle, \tag{5.8}$$

for all $V\in\hat{\xi}_C^\perp$.

The defferential Φ_* defines a vector bundle isomorphism

$$\begin{array}{ccc} \hat{\xi}_{1,0}^\perp & \longrightarrow & \widetilde{\xi}_{1,0}^\perp\\ \downarrow & & \downarrow\\ M & \longrightarrow & N, \end{array} \tag{5.9}$$

which preserving the hermitian structures induced from the levi-metric and cover φ, such that Φ_* preserves the normal connections, i.e.,

$$\Phi_*(\nabla^\perp_X Z_a)=\widetilde{\nabla}^\perp_{\varphi_*X}(\Phi_*Z_a), \tag{5.10}$$

for all $X\in TM$, where $\nabla^{\perp}$ and $\widetilde{\nabla}^{\perp}$ are the induced normal connections on M and N, respectively. Finally, it is easy to see that $\Phi_* \nu = \widetilde{\nu}$.

Definition 5.1. *Suppose that M and N are two pseudohermitian submanifolds of H_n with the same CR dimension m. We say that M and N have* the same (induced) pseudohermitian structures, the second fundamental forms, the normal connections and the fundamental vector fields *if there exists a vector bundle isomorphism $F: \hat{\xi}^{\perp}_{1,0} \to \widetilde{\xi}^{\perp}_{1,0}$, which preserves the induced hermitian structures and covers a map $\varphi: M \to N$, such that*

- *F preserves the induced pseudohermitian structures: $\varphi_* \circ \hat{J} = \widetilde{J} \circ \varphi_*$; and $\varphi^* \widetilde{\theta} = \hat{\theta}$;*
- *F preserves the second fundamental forms: $\langle II, V\rangle_{\hat{\xi}^{\perp}_{1,0}} = \varphi^* \langle \widetilde{II}, FV\rangle_{\widetilde{\xi}^{\perp}_{1,0}}$, for all $V \in \hat{\xi}^{\perp}_{1,0}$.*
- *F preserves the normal connections: $F(\nabla^{\perp}_X V) = \widetilde{\nabla}^{\perp}_{\varphi_* X}(FV)$, for all $X \in TM$, $V \in \hat{\xi}^{\perp}_{1,0}$.*
- *F preserves the fundamental vector field: $F\nu = \widetilde{\nu}$.*

Therefore we conclude that if M is congruent to N, then they have the same such four invariants. Conversely, we have

Theorem 5.1. *Let $(M,\hat{J},\hat{\theta})$ and $(N,\widetilde{J},\widetilde{\theta})$ be two simply connected pseudohermitian submanifolds of H_n with CR dimension m. Suppose that they have the same (induced) pseudohermitian structures, the second fundamental forms, the normal connections and the fundamental vector fields. Then they differ by a Heisenberg rigid motion.*

Corollary 5.1. *If M and N are* vertical, *then the (induced) pseudohermitian structures, the second fundamental forms and the normal connections constitute a complete set of invariants.*

5.1 The proof of Theorem 5.1

Let $(M,\hat{J},\hat{\theta})$ be a pseudohermitian submanifold of H_n. Recall that we always choose the frame field $\{Z_\beta, T\}$ over M such that $Z_j \in \hat{\xi}_{1,0}$ and $Z_a \in \hat{\xi}^{\perp}_{1,0}$. This is a Darboux frame. Let $\{\theta^\beta, \theta\}$ be the dual of $\{Z_\beta, T\}$. We would like to show that the restrictions of θ^β and ${\theta_\beta}^\gamma$ to M are expressed as follows:

$$\begin{aligned} &\theta^j|_M = \hat{\theta}^j, \qquad \theta^a|_M = \langle \nu, Z_a\rangle \hat{\theta}, \qquad \theta|_M = \hat{\theta}, \\ &{\theta_j}^k|_M = {\hat{\theta}_j}{}^k + i\delta_{jk}|\nu|^2\hat{\theta}, \qquad {\theta_j}^a|_M = h^a_{jk}\hat{\theta}^k + i\delta_{jk}\langle \nu, Z_a\rangle \hat{\theta}^{\bar{k}} + \langle \nabla^{\perp}_{\hat{Z}_j}\nu, Z_a\rangle \hat{\theta}; \end{aligned} \tag{5.11}$$

and here $h^a_{jk} = II^a(\hat{Z}_j, \hat{Z}_k)$, and ${\theta_a}^b|_M$ is the normal connection forms w.r.t. $\{Z_a\}$. This shows that the Darboux derivative of the Draboux frame is completely determined by the induced pseudohermitian structure, the second fundamental form, the normal connection and the fundamental vector field.

Now we prove (5.11).

$$d\theta^a = \theta^j \wedge {\theta_j}^a + \theta^b \wedge {\theta_b}^a = \hat{\theta}^j \wedge {\theta_j}^a + \langle \nu, Z_b\rangle \hat{\theta} \wedge {\theta_b}^a. \tag{5.12}$$

On the other hand,

$$\begin{aligned} d\theta^a &= d\big(\langle \nu, Z_a\rangle \hat{\theta}\big) = d\langle \nu, Z_a\rangle \wedge \hat{\theta} + \langle \nu, Z_a\rangle d\hat{\theta} \\ &= \big(\langle \nabla \nu, Z_a\rangle + \langle \nu, \nabla_{\cdot} Z_a\rangle\big)\wedge\hat{\theta} + \langle \nu, Z_a\rangle d\hat{\theta} \\ &= \Big(\langle \nabla^{\perp}\nu, Z_a\rangle + \langle \nu, \nabla^{\perp}_{\cdot} Z_a\rangle\Big)\wedge\hat{\theta} + \langle \nu, Z_a\rangle d\hat{\theta}, \end{aligned} \tag{5.13}$$

and

$$\langle \nu, \nabla^{\perp}_{\cdot} Z_a\rangle = \langle \nu, {\theta_a}^{b}(\cdot)\otimes Z_b\rangle = \langle \nu, Z_b\rangle {\theta_{\bar{a}}}^{\bar{b}}. \tag{5.14}$$

From (5.12)–(5.14), we obtain

$$\hat{\theta}^j \wedge {\theta_j}^a = \langle \nabla^{\perp}\nu, Z_a\rangle \wedge \hat{\theta} + \langle \nu, Z_a\rangle d\hat{\theta}. \tag{5.15}$$

That is,

$$\begin{aligned} &{\theta_j}^a(\hat{Z}_k)\hat{\theta}^j\wedge\hat{\theta}^k + {\theta_j}^a(\hat{Z}_{\bar{k}})\hat{\theta}^j\wedge\hat{\theta}^{\bar{k}} + {\theta_j}^a(\hat{T})\hat{\theta}^j\wedge\hat{\theta} \\ =&\langle \nabla^{\perp}_{\hat{Z}_k}\nu, Z_a\rangle\hat{\theta}^k\wedge\hat{\theta} + \langle \nabla^{\perp}_{\hat{Z}_{\bar{k}}}\nu, Z_a\rangle\hat{\theta}^{\bar{k}}\wedge\hat{\theta} + i\langle \nu, Z_a\rangle\hat{\theta}^j\wedge\hat{\theta}^{\bar{j}}, \end{aligned} \tag{5.16}$$

which implies

$${\theta_j}^a(\hat{T}) = \langle \nabla^{\perp}_{\hat{Z}_j}\nu, Z_a\rangle, \qquad 0 = \langle \nabla^{\perp}_{\hat{Z}_{\bar{j}}}\nu, Z_a\rangle, \tag{5.17a}$$

$${\theta_j}^a(\hat{Z}_{\bar{k}}) = i\delta_{jk}\langle \nu, Z_a\rangle, \qquad {\theta_j}^a(\hat{Z}_k) = {\theta_k}^a(\hat{Z}_j) = h^a_{jk}, \tag{5.17b}$$

and thus

$${\theta_j}^a = h^a_{jk}\hat{\theta}^k + i\delta_{jk}\langle \nu, Z_a\rangle\hat{\theta}^{\bar{k}} + \langle \nabla^{\perp}_{\hat{Z}_j}\nu, Z_a\rangle\hat{\theta}. \tag{5.18}$$

Now we compute

$$d\theta^k = \theta^j\wedge{\theta_j}^k + \theta^a\wedge{\theta_a}^k = \hat{\theta}^j\wedge{\theta_j}^k + \hat{\theta}\wedge\Big(\langle \nu, Z_a\rangle{\theta_a}^k\Big). \tag{5.19}$$

On the other hand,

$$d\theta^k = d\hat{\theta}^k = \hat{\theta}^j\wedge{\hat{\theta}_j}{}^k + \hat{\theta}\wedge\tau^k. \tag{5.20}$$

By Cartan lemma, there exists functions $B^k_{jl}, B^k_{j(m+1)}, B^k_{(m+1)l}$ and $B^k_{(m+1)(m+1)}$ such that

$${\hat{\theta}_j}{}^k = {\theta_j}^k + B^k_{jl}\hat{\theta}^l + B^k_{j(m+1)}\hat{\theta}, \qquad \hat{\tau}^k = \Big(\langle \nu, Z_a\rangle{\theta_a}^k\Big) + B^k_{(m+1)l}\hat{\theta}^l + B^k_{(m+1)(m+1)}\hat{\theta}, \tag{5.21}$$

where $B^k_{jl} = B^k_{lj}$ and $B^k_{j(m+1)} = B^k_{(m+1)j}$, for $1 \le j,k,l \le m$. Since $\hat{\tau}^k = {A^k}_{\hat{l}}\hat{\theta}^{\bar{l}}$, comparing with (5.21), we get

$${A^k}_{\bar{l}} = \sum_{a=m+1}^{n}\langle \nu, Z_a\rangle{\theta_a}^k(\hat{Z}_{\bar{l}}) = -\sum_{a=m+1}^{n}\langle \nu, Z_a\rangle h^{\bar{a}}_{\bar{k}\bar{l}}, \tag{5.22a}$$

$$B^k_{(m+1)l} = -\sum_{a=m+1}^{n}\langle \nu, Z_a\rangle{\theta_a}^k(\hat{Z}_l) = -i\delta_{kl}|\nu|^2, \tag{5.22b}$$

$$B^k_{(m+1)(m+1)} = -\sum_{a=m+1}^{n}\langle \nu, Z_a\rangle{\theta_a}^k(\hat{T}) = \sum_{a=m+1}^{n}\langle \nu, Z_a\rangle\langle \nabla^{\perp}_{\hat{Z}_{\bar{k}}}\nu, Z_{\bar{a}}\rangle. \tag{5.22c}$$

Finally, since ${\theta_j}^k + {\theta_{\bar{k}}}^{\bar{j}} = 0$ and ${\hat{\theta}_j}^k + {\hat{\theta}_{\bar{k}}}^{\bar{j}} = 0$, we have, from (5.21), $B^k_{jl} = 0$, and hence

$$ {\hat{\theta}_j}^k = {\theta_j}^k - i\delta_{jk}|\nu|^2\hat{\theta}. \tag{5.23} $$

This completes the proof of (5.11).

5.2 The integrability condition

Let $(M,\hat{J},\hat{\theta})$ be a pseudohermitian submanifold of H_n. We choose a Darboux frame $\{Z_\beta,T\}$ over M. Let $\{\theta^\beta,\theta\}$ be the dual of $\{Z_\beta,T\}$.

Definition 5.2. *The* restriction *to M of the structure equations of H_n,*

$$ d\theta^\beta = \theta^\gamma \wedge {\theta_\gamma}^\beta, \qquad d\theta = i\theta^\gamma \wedge \theta^{\bar{\gamma}}, \qquad d{\theta_\sigma}^\beta = {\theta_\sigma}^\gamma \wedge {\theta_\gamma}^\beta, \tag{5.24} $$

is defined to be the integrability condition *of M. Note that the restrictions of θ^β and ${\theta_\beta}^\gamma$ to M have the expressions of the forms as (5.11) specifies..*

6 The existence theorem

In this section, we would like to show Theorem B. Let (M,J_M,θ_M) be a pseudohermitian manifold with CR dimension m. Since the existence theory is local, we assume that M is simply connected. Putting $\xi_M = \ker\theta_M$ and $\eta = \theta_M$.

- Let $\xi^\perp_M$ be a complex vector bundle over M, of complex dimension $n-m$, with a Hermitian metric $h^\perp_M$ and a connection ∇^M compatible with $h^\perp_M$.
- Suppose $\{W_1,\cdots,W_m \in T_{1,0}M\}$ is an orthonormal CR holomorphic frame field of M. Its dual is denoted by $\{\eta^1,\cdots,\eta^m\}$. Let ${\hat{\eta}_j}^k$ be the pseudohermitian conection forms w.r.t. W_j. We have ${\hat{\eta}_j}^k + {\hat{\eta}_{\bar{k}}}^{\bar{j}} = 0$.
- Suppose $\{W_{m+1},\cdots,W_n\}$ is an orthonormal frame field of $\xi^\perp_M$ w.r.t. $h^\perp_M$ and ${\eta_a}^b$ are the connection forms w.r.t. $\{W_a\}$, i.e.,

  $$ \nabla^M W_a = {\eta_a}^b \otimes W_b. \tag{6.1} $$

  We have ${\eta_a}^b + {\eta_{\bar{b}}}^{\bar{a}} = 0$.
- Let $II_M : T_{1,0}M \times T_{1,0}M \to \xi^\perp_M$ be a $\xi^\perp_M$-valued complex bilinear form.
- Let μ be a real section of the bundle $\xi^\perp_M$ over M. And define

  $$ \eta^a = \langle \mu, W_a \rangle \eta, \quad {\eta_j}^k = {\hat{\eta}_j}^k + i\delta_{jk}|\mu|^2\eta, \quad {\eta_j}^a = g^a_{jk}\eta^k + i\delta_{jk}\langle \mu, W_a\rangle \eta^{\bar{k}} + \langle \nabla^M_{W_j}\mu, W_a\rangle \eta, \tag{6.2} $$

  where $\langle\ ,\ \rangle = h^\perp_M$ and $g^a_{jk} = \langle II_M(W_j,W_k), W_a\rangle$.

- Finally, define ${\eta_a}^j$ by ${\eta_j}^a+{\eta_{\bar a}}^{\bar j}=0$.

Theorem 6.1. *Suppose that the pseudohermitian manifold (M^{2m+1},J_M,θ_M), together with II_M, the compatible connection ∇^M and the section μ satisfies the* integrability condition, *in the sense that η^β,η and ${\eta_\gamma}^\beta$ satisfy* (5.24).*Then*

- *There exists an embedding ϕ such that (M,J_M,θ_M) can be embedded into H_n with CR dimensionn m.*
- *In addition, there exists a vector bundle isomorphism $\Psi:\xi_M^\perp\to\hat{\xi}_{1,0}^\perp$, covering ϕ, such that $\Psi^*II=II_M$, $\Psi^*\nabla^\perp=\nabla^M$, and $\Psi^*\nu=\mu$, where $\hat{\xi}_{1,0}^\perp,II,\nabla^\perp$ and ν are, respectively, the induced normal bundle, second fundamental form, normal connection and fundamental vector field ν over $\phi(M)$.*

Proof. Let $\flat=(\eta^1,\cdots,\eta^n),\bar\flat=(\eta^{\bar 1},\cdots,\eta^{\bar n})$. Define the matrix Π by

$$\Pi=\begin{pmatrix} 0 & 0 & 0\\ \flat^t & {\eta_\gamma}^\beta & 0\\ \eta & i\bar\flat & 0\end{pmatrix}. \tag{6.3}$$

The integrability condition means that

$$d\Pi+\Pi\wedge\Pi=0. \tag{6.4}$$

Taking the real version ζ of Π,

$$\zeta=\begin{pmatrix} 0 & 0 & 0 & 0\\ \lambda^\beta & {\lambda_\alpha}^\beta & {\lambda_{n+\alpha}}^\beta & 0\\ \lambda^{n+\beta} & {\lambda_\alpha}^{n+\beta} & {\lambda_{n+\alpha}}^{n+\beta} & 0\\ \lambda & \lambda^{n+\alpha} & -\lambda^\alpha & 0\end{pmatrix},$$

where $\lambda=\eta,\eta^\beta=\lambda^\beta+i\lambda^{n+\beta}$ and ${\eta_\gamma}^\beta={\lambda_\gamma}^\beta+i{\lambda_\gamma}^{n+\beta}$. Then ${\eta_\gamma}^\beta+{\eta_{\bar\beta}}^{\bar\gamma}=0$ implies that ζ is a $psh(n)$-valued one form. And (6.4) is equivalent to $d\zeta+\zeta\wedge\zeta=0$. Therefore, by calculus on Lie groups, we have that ζ is the Darboux derivative of some map $f:M\to PSH(n)$, that is,

$$\zeta=f^*\omega. \tag{6.5}$$

Define a map $\phi:M\to H_n$ by $\phi=\pi\circ f$, where π is the bundle projection $\pi:PSH(n)\to H_n$, and define a bundle map $\Psi:\xi_M^\perp\to\hat{\xi}_{1,0}^\perp$ by $\Psi(p,W_a)=(\phi(p),Z_a)$. Then, using (6.5), it is easy to check that Ψ and ϕ satisfy all what we want. This completes the proof. □

7 Rigidity theorems for submanifolds with CR co-dimension one

In this section, we prove some rigidity theorems for pseudohermitian submanifolds, including both the nondegenerate and degenerate cases.

Theorem 7.1. *Let $(M,\hat{J},\hat{\theta})$ be a* vertical*, simply connected pseudohermitian submanifold of H_n with CR dimension $m=n-1$. Suppose that the second fundamental form $II=0$.Then M is an open subset U of $H_{n-1}=\{z_n=0\}$ after a Heisenberg rigid motion.*

Proof. In the case $m=n-1$, we write $\theta_j{}^n=h_{jk}\theta^k$, here h_{jk} are the coefficients of the second fundamental form II. If $II=0$, then $\theta_j{}^n=0$. On the other hand, $\nu=0$ implies $\theta^n=0$. Hence the structure equations of H_n, restricting to M, are reduced to

$$d\theta^j=\theta^k\wedge\theta_k{}^j,\qquad d\theta=i\theta^k\wedge\theta^{\bar{k}},\qquad d\theta_j{}^l=\theta_j{}^k\wedge\theta_k{}^l,\qquad d\theta_n{}^n=0. \tag{7.1}$$

The last equation of (7.1) says that $\theta_n{}^n$ is closed, and hence locally is exact. By the transformation law of the normal connection, we can choose a normal frame Z_n such that the corresponding connection form $\theta_n{}^n$ vanishes. On the other hand, the first three equations of (7.1) is just the structure equations of H_{n-1}. This means that M is an open part U of $H_{n-1}\subset H_n$, up to a pseudohermitian transformatio φ from M to U. Define F by $F(x,Z_n(x))=(\varphi(x),\mathring{Z}_n)$. Then F defines the normal bundle isomorphism covering φ which preserving the induced pseudohermitian structures, the second fundamental forms and the normal connections of M and U, respectively. Hence φ is just the restriction of a Heisenberg rigid motion. □

For a vertical pseudohermitian submanifold of H_n, we define a flat point of M to be a point such that $II=0$ at that point. Theorem 7.4 says that the induced pseudohermitian structure is the only invariant for vertical pseudohermitian submanifolds without flat points.

Theorem 7.2. *Let $(M,\hat{J},\hat{\theta})$ and $(N,\widetilde{J},\widetilde{\theta})$ be two* vertical*, simply connected pseudohermitian submanifolds of H_n without flat points. Suppose both of them are of CR dimension $m=n-1$. If there exists a pseudohermitian transformation $\phi:\ M\to N$, then $\phi=\Phi|_M$ for some Heisenberg rigid motion Φ.*

Proof. By Theorem 5.1, it suffices to show that both the second fundamental form and the normal connection are completely determined by the induced pseudohermitian structure. We write $\theta_j{}^n=h_{jk}\theta^k$. Then, from the Gauss-like equation, we have

$$d\theta_j{}^l-\theta_j{}^k\wedge\theta_k{}^l=-h_{jp}h_{\bar{l}\bar{q}}\theta^p\wedge\theta^{\bar{q}}. \tag{7.2}$$

On the other hand

$$d\theta_j{}^l-\theta_j{}^k\wedge\theta_k{}^l=d\hat{\theta}_j{}^l-\hat{\theta}_j{}^k\wedge\hat{\theta}_k{}^l=R_j{}^l{}_{p\bar{q}}\theta^p\wedge\theta^{\bar{q}}. \tag{7.3}$$

From (7.2) and (7.3), we see that the Gauss-like equation is equivalent to

$$R_{j\bar{l}p\bar{q}}=-h_{jp}h_{\bar{l}\bar{q}}, \tag{7.4}$$

which implies that

$$R_{j\bar{l}p\bar{q}}=0\Leftrightarrow II=0. \tag{7.5}$$

If $II\neq 0$, then there exists $h_{jk}\neq 0$ for some j,k. Since II is a symmetric bilinear form, after a frame transformation, we can assume, w.l.o.g., that $h_{11}\neq 0$. Then we have

$$R_{j\bar{1}p\bar{1}}=-h_{jp}h_{\bar{1}\bar{1}}. \tag{7.6}$$

In particular,

$$h_{11}=\sqrt{-R_{1\bar{1}1\bar{1}}}e^{i\varphi},\quad \text{for some } \varphi. \tag{7.7}$$

On the other hand, if we take another orthonormal frame field $\{\tilde{Z}_\beta\}$ such that

$$\tilde{Z}_j=Z_j,\qquad \tilde{Z}_n=e^{i\psi}Z_n, \tag{7.8}$$

for some ψ. Then we have the transformation law for connection forms

$$\tilde{\theta}_j{}^k=\theta_j{}^k,\qquad \tilde{\theta}_j{}^n=e^{-i\psi}\theta_j{}^n,\qquad \tilde{\theta}_n{}^j=e^{i\psi}\theta_n{}^j,\qquad \tilde{\theta}_n{}^n=\theta_n{}^n+id\psi. \tag{7.9}$$

Notice that $\theta_j{}^n=h_{jk}\theta^k$, $\tilde{\theta}_j{}^n=\tilde{h}_{jk}\tilde{\theta}^k$ and $\tilde{\theta}^k=\theta^k$, hence we immediately have

$$\tilde{h}_{jk}=e^{-i\psi}h_{jk},\quad \text{for all } j,k. \tag{7.10}$$

In particular, $\tilde{h}_{11}=e^{-i\psi}h_{11}=e^{i(\varphi-\psi)}\sqrt{-R_{1\bar{1}1\bar{1}}}$. Taking $\psi=\varphi$, we have

$$\tilde{h}_{11}=\sqrt{-R_{1\bar{1}1\bar{1}}}=\sqrt{-\tilde{R}_{1\bar{1}1\bar{1}}}. \tag{7.11}$$

Formula (7.11) means that we can always choose a frame field $\{Z_\beta\}$ such that $h_{11}=\sqrt{-R_{1\bar{1}1\bar{1}}}$, and hence

$$h_{jk}=-\frac{R_{j\bar{1}k\bar{1}}}{\sqrt{-R_{1\bar{1}1\bar{1}}}},\quad \text{for all } j,k. \tag{7.12}$$

This means that the second fundamental form II is completely determined by the induced pseudohermitian structure.

We proceed to show that the normal connection is also completely determined by the induced pseudohermitian structure. For each j,

$$d\theta_j{}^n=d(h_{jk}\theta^k)=(dh_{jk}-h_{jl}\theta_k{}^l)\wedge\theta^k. \tag{7.13}$$

On the other hand,

$$d\theta_j{}^n=\theta_j{}^k\wedge\theta_k{}^n+\theta_j{}^n\wedge\theta_n{}^n=(h_{lk}\theta_j{}^l-h_{jk}\theta_n{}^n)\wedge\theta^k. \tag{7.14}$$

From (7.13) and (7.14), we have, for each j,k,

$$dh_{jk}-h_{jl}\theta_k{}^l-h_{lk}\theta_j{}^l+h_{jk}\theta_n{}^n=\sum_{l=1}^{n-1}B_{jkl}\theta^l, \tag{7.15}$$

for some B_{jkl}, which satisfying $B_{jkl}=B_{jlk}$. In particular

$$h_{11}\theta_n{}^n=-(dh_{11}-h_{1l}\theta_1{}^l-h_{l1}\theta_1{}^l)+\sum_{l=1}^{n-1}B_{11l}\theta^l. \tag{7.16}$$

The conjugate of (7.16) is,

$$-h_{11}\theta_n{}^n=h_{\bar{1}\bar{1}}\theta_{\bar{n}}{}^{\bar{n}}=-(dh_{\bar{1}\bar{1}}-h_{\bar{1}\bar{l}}\theta_{\bar{1}}{}^{\bar{l}}-h_{\bar{l}\bar{1}}\theta_{\bar{1}}{}^{\bar{l}})+\sum_{l=1}^{n-1}B_{\bar{1}\bar{1}\bar{l}}\theta^{\bar{l}}. \tag{7.17}$$

Taking the sum of (7.16) and (7.17)

$$\sum_{l=1}^{n-1}B_{11l}\theta^l+\sum_{l=1}^{n-1}B_{\bar{1}\bar{1}\bar{l}}\theta^{\bar{l}}=(h_{11,l}\theta^l+h_{11,\bar{l}}\theta^{\bar{l}}+h_{11,0}\theta)+\text{ conjugate}, \tag{7.18}$$

which implies that $B_{11l}=h_{11,l}+h_{\bar{1}\bar{1},l}$. Substituting this into (7.16), we get

$$\theta_n{}^n=\frac{h_{\bar{1}\bar{1},l}\theta^l-h_{11,\bar{l}}\theta^{\bar{l}}-h_{11,0}\theta}{h_{11}}, \tag{7.19}$$

which means that $\theta_n{}^n$ is completely determined by the induced pseudohermitian structure. □

Remark 7.1. (i) From (7.18), we also get $h_{11,0}+h_{\bar{1}\bar{1},0}=0$. Therefore, in the case $n=2$, we have $Th_{11}=0$ or $TR=R_0=0$.
(ii) Also, for $n=2$, we have that $\theta_2{}^2=2\theta_1{}^1-d(\ln h_{11})+2Z_1(\ln h_{11})\theta^1$.

Theorem 7.3. *Let $(M,\hat{J},\hat{\theta})$ and $(N,\widetilde{J},\widetilde{\theta})$ be two simply connected pseudohermitian submanifolds of H_n with CR dimension $m=n-1$. Suppose, in addition, that their fundamental vector fields are* nowhere zero. *If they have the same (induced) pseudohermitian structures. Then they locally differ by a Heisenberg rigid motion. More explicitly, if there exists a pseudohermitian transformation $\phi:M\to N$, then $\phi=\Phi|_M$ for some Heisenberg rigid motion Φ.*

Proof. The key point is that if $\nu(p)\neq 0$ for each point $p\in M$, then we can always choose a Darboux frame $p\to(p;e_\beta,Je_\beta,T)$ such that

$$e_n=-\frac{\nu}{|\nu|},\ \ e_{2n}=Je_n. \tag{7.20}$$

Then we would like to compute the Darboux derivative of the Barboux frame. It is equivalent to computing the restrictions of $\theta^\beta,\theta_\gamma{}^\beta$ to M. To finish the proof, we need to show the Darboux derivative is completely determined by the induced pseudohermitian structure.

$$d\theta^\beta=\theta^\gamma\wedge\theta_\gamma{}^\beta=\hat{\theta}^k\wedge\theta_k{}^\beta+\theta^n\wedge\theta_n{}^\beta=\hat{\theta}^k\wedge\theta_k{}^\beta+\hat{\theta}\wedge(-|\nu|\theta_n{}^\beta). \tag{7.21}$$

On the other hand,

$$d\theta^j = d\hat{\theta}^j = \hat{\theta}^k \wedge \hat{\theta}_k{}^j + \hat{\theta} \wedge \hat{\tau}^j, \tag{7.22}$$

and

$$d\theta^n = -d(|\nu|\hat{\theta}) = \hat{\theta}^k \wedge (-i|\nu|\hat{\theta}^{\bar{k}}) + \hat{\theta} \wedge (d|\nu|). \tag{7.23}$$

From (7.21), (7.22) and (7.23), there exists complex-valued functions $a^j_{\beta\gamma}$ such that $a^j_{\beta\gamma} = a^j_{\gamma\beta}$ and

$$\theta_k{}^j = \hat{\theta}_k{}^j + a^j_{kl}\hat{\theta}^l + a^j_{kn}\hat{\theta}, \qquad -|\nu|\theta_n{}^j = \hat{\tau}^j + a^j_{nl}\hat{\theta}^l + a^j_{nn}\hat{\theta}, \tag{7.24}$$

Also, there exists complex-valued functions $b_{\beta\gamma}$ such that $b_{\beta\gamma} = b_{\gamma\beta}$ and

$$\theta_k{}^n = -i|\nu|\hat{\theta}^{\bar{k}} + b_{kl}\hat{\theta}^l + b_{kn}\hat{\theta}, \qquad -|\nu|\theta_n{}^n = d|\nu| + b_{nl}\hat{\theta}^l + b_{nn}\hat{\theta}. \tag{7.25}$$

From (7.24),

$$0 = \theta_k{}^j + \theta_{\bar{j}}{}^{\bar{k}} = (\hat{\theta}_k{}^j + \hat{\theta}_{\bar{j}}{}^{\bar{k}}) + a^j_{kl}\hat{\theta}^l + a^{\bar{k}}_{\bar{j}\bar{l}}\hat{\theta}^{\bar{l}} + (a^j_{kn} + a^{\bar{k}}_{\bar{j}\bar{n}})\hat{\theta}, \tag{7.26}$$

hence

$$a^j_{kl} = 0, \ \ a^j_{kn} + a^{\bar{k}}_{\bar{j}\bar{n}} = 0, \ \text{ for all } 1 \le j,k,l \le m. \tag{7.27}$$

Similarly, and notice that we write $\hat{\tau}^j = A^j{}_{\bar{k}}\hat{\theta}^{\bar{k}}$, we have

$$\begin{aligned} &b_{nl} = -2(\hat{Z}_l|\nu|), \quad b_{nn} + b_{\bar{n}\bar{n}} = -2(\hat{T}|\nu|), \quad a^j_{nl} = i\delta_{jl}|\nu|^2, \\ &b_{jl} = \frac{A^{\bar{j}}{}_l}{|\nu|}, \quad a^j_{nn} = |\nu| b_{\bar{j}\bar{n}} = -2|\nu|(\hat{Z}_{\bar{j}}|\nu|), \end{aligned} \tag{7.28}$$

for all $1 \le j,l \le m$. From (7.24), (7.27), (7.28), we have, for all $1 \le j,k \le m$,

$$\theta_k{}^j = \hat{\theta}_k{}^j + (i\delta_{jk}|\nu|^2)\hat{\theta}, \qquad -|\nu|\theta_n{}^j = \hat{\tau}^j + (i\delta_{jl}|\nu|^2)\hat{\theta}^l - 2|\nu|(\hat{Z}_{\bar{j}}|\nu|)\hat{\theta}. \tag{7.29}$$

From (7.25), (7.28),

$$\theta_k{}^n = -i|\nu|\hat{\theta}^{\bar{k}} + \frac{A^{\bar{k}}{}_l}{|\nu|}\hat{\theta}^l - 2(\hat{Z}_k|\nu|)\hat{\theta}, \qquad -|\nu|\theta_n{}^n = (d|\nu|) - 2(\hat{Z}_l|\nu|)\hat{\theta}^l + b_{nn}\hat{\theta}. \tag{7.30}$$

From the look of (7.29) and (7.30), there is only one term b_{nn} not determined yet. In order to complete the proof, we need to show that both b_{nn} and $|\nu|$ are completely determined by the induced pseudohermitian structure. For this, using (7.29) and (7.30), we compute

$$\begin{aligned} d\theta_k{}^j &= \theta_k{}^l \wedge \theta_l{}^j + \theta_k{}^n \wedge \theta_n{}^j \\ &= \Big(\hat{\theta}_k{}^l + (i\delta_{lk}|\nu|^2)\hat{\theta}\Big) \wedge \Big(\hat{\theta}_l{}^j + (i\delta_{jl}|\nu|^2)\hat{\theta}\Big) \\ &\quad + \frac{1}{|\nu|^2}\Big(i|\nu|^2\hat{\theta}^{\bar{k}} - A^{\bar{k}}{}_l\hat{\theta}^l + 2|\nu|(\hat{Z}_k|\nu|)\hat{\theta}\Big) \wedge \Big(A^j{}_{\bar{l}}\hat{\theta}^{\bar{l}} + i|\nu|^2\hat{\theta}^j - 2|\nu|(\hat{Z}_{\bar{j}}|\nu|)\hat{\theta}\Big), \end{aligned}$$

that is,

$$\begin{aligned}&d\theta_k{}^j-\hat{\theta}_k{}^l\wedge\hat{\theta}_l{}^j\\=&\frac{1}{|\nu|^2}\Big(-iA^{\bar{k}}{}_l|\nu|^2\hat{\theta}^l\wedge\hat{\theta}^j-iA^j{}_{\bar{l}}|\nu|^2\hat{\theta}^{\bar{l}}\wedge\hat{\theta}^{\bar{k}}+(-A^{\bar{k}}{}_lA^j{}_{\bar{q}}+\delta^l_j\delta^q_k|\nu|^4)\hat{\theta}^l\wedge\hat{\theta}^{\bar{q}}+(2A^{\bar{k}}{}_l|\nu|(\hat{Z}_{\bar{j}}|\nu|)\\&2i\delta^l_j|\nu|^3(\hat{Z}_k|\nu|))\hat{\theta}^l\wedge\hat{\theta}+(-2A^j{}_{\bar{q}}|\nu|(\hat{Z}_k|\nu|)-2i\delta^q_k|\nu|^3(\hat{Z}_{\bar{j}}|\nu|))\hat{\theta}^{\bar{q}}\wedge\hat{\theta}\Big).\end{aligned}\tag{7.31}$$

On the other hand, from (7.29) and using the structure equations of the pseudohermitian structure, we have

$$\begin{aligned}d\theta_k{}^j-\hat{\theta}_k{}^l\wedge\hat{\theta}_l{}^j=&d\hat{\theta}_k{}^j-\hat{\theta}_k{}^l\wedge\hat{\theta}_l{}^j+d(i\delta_{jk}|\nu|^2\hat{\theta})\\=&R_k{}^j{}_{p\bar{q}}\hat{\theta}^p\wedge\hat{\theta}^{\bar{q}}+W_k{}^j{}_p\hat{\theta}^p\wedge\hat{\theta}-W^j{}_{k\bar{p}}\hat{\theta}^{\bar{p}}\wedge\hat{\theta}+i\hat{\theta}_k\wedge\hat{\tau}^j-\hat{\tau}_k\wedge\hat{\theta}^j\\&+i\delta_{jk}\Big((\hat{Z}_l|\nu|^2)\hat{\theta}^l\wedge\hat{\theta}+(\hat{Z}_{\bar{l}}|\nu|^2)\hat{\theta}^{\bar{l}}\wedge\hat{\theta}\Big)-\delta_{jk}|\nu|^2\hat{\theta}^l\wedge\hat{\theta}^{\bar{l}}.\end{aligned}\tag{7.32}$$

Comparing the coefficients of the same terms in (7.31) and (7.32), and notice that $\hat{\tau}^j=A^j{}_{\bar{k}}\hat{\theta}^{\bar{k}}$, we get

$$R_k{}^j{}_{l\bar{q}}-\delta_{jk}\delta_{lq}|\nu|^2=-\frac{A^{\bar{k}}{}_lA^j{}_{\bar{q}}}{|\nu|^2}+\delta^l_j\delta^q_k|\nu|^2,\tag{7.33a}$$

$$W_k{}^j{}_l+i\delta_{jk}(\hat{Z}_l|\nu|^2)=2\frac{A^{\bar{k}}{}_l}{|\nu|}(\hat{Z}_{\bar{j}}|\nu|)-2i\delta^l_j|\nu|(\hat{Z}_k|\nu|),\tag{7.33b}$$

$$-W^j{}_{k\bar{l}}+i\delta_{jk}(\hat{Z}_{\bar{l}}|\nu|^2)x=-2\frac{A^j{}_{\bar{l}}}{|\nu|}(\hat{Z}_k|\nu|)-2i\delta^l_k|\nu|(\hat{Z}_{\bar{j}}|\nu|),\tag{7.33c}$$

for all $1\le j,k,l,q\le m$. From the first equation of (7.33),

$$\begin{aligned}R_{k\bar{j}}=R_{k\bar{j}l\bar{l}}=&-\sum_{l=1}^m\frac{A^{\bar{k}}{}_lA^j{}_{\bar{l}}}{|\nu|^2}+\sum_{l=1}^m(\delta_{jk}\delta_{ll}+\delta^l_j\delta^l_k)|\nu|^2\\=&\begin{cases}-\sum_{l=1}^m\frac{A^{\bar{k}}{}_lA^j{}_{\bar{l}}}{|\nu|^2}, & \text{for } k\neq j\\-\left(\sum_{l=1}^m\frac{A^{\bar{k}}{}_lA^j{}_{\bar{l}}}{|\nu|^2}\right)+(m+1)|\nu|^2, & \text{for } k=j.\end{cases}\end{aligned}\tag{7.34}$$

In particlar

$$R=R_{k\bar{k}}=-\frac{|A|^2}{|\nu|^2}+m(m+1)|\nu|^2.\tag{7.35}$$

Formula (7.35) is equivalent to

$$|\nu|^2=\frac{R+\sqrt{R^2+4m(m+1)|A|^2}}{2m(m+1)}.\tag{7.36}$$

Finally, we would like to compute b_{nn}. From (7.28), we see that $b_{nn}+b_{\bar{n}\bar{n}}=-2(\hat{T}|\nu|)$, i.e., $b_{nn}=(-\hat{T}|\nu|)+i(\mathrm{Im}b_{nn})$. So we only compute $\mathrm{Im}b_{nn}$. For this, using (7.29) and (7.30)

$$\begin{aligned} d\theta_j{}^n &= \theta_j{}^k\wedge\theta_k{}^n+\theta_j{}^n\wedge\theta_n{}^n=(\theta_j{}^k-\delta_j^k\theta_n{}^n)\wedge\theta_k{}^n \\ &= \frac{1}{|\nu|^2}\Big(|\nu|\theta_j{}^k+\delta_j^k(-|\nu|\theta_n{}^n)\Big)\wedge(|\nu|\theta_k{}^n) \\ &= \frac{1}{|\nu|^2}\Big(|\nu|\hat{\theta}_j{}^k+\delta_j^k\big[-(\hat{Z}_l|\nu|)\hat{\theta}^l+(\hat{Z}_{\bar{l}}|\nu|)\hat{\theta}^{\bar{l}}+i(|\nu|^3+\mathrm{Im}b_{nn})\hat{\theta}\big]\Big) \\ &\quad\wedge\Big(-i|\nu|^2\hat{\theta}^{\bar{k}}+A^{\bar{k}}{}_l\hat{\theta}^l-2|\nu|(\hat{Z}_k|\nu|)\hat{\theta}\Big) \\ &= -i|\nu|\hat{\theta}_j{}^k\wedge\hat{\theta}^l+\hat{\theta}_j{}^k\wedge\left(\frac{A^{\bar{k}}{}_l}{|\nu|}\hat{\theta}-2(\hat{Z}_k)\hat{\theta}\right)+\frac{1}{|\nu|^2}\Big(-(\hat{Z}_l|\nu|)\hat{\theta}^l+(\hat{Z}_{\bar{l}}|\nu|)\hat{\theta}^{\bar{l}} \\ &\quad+i(|\nu|^3+\mathrm{Im}b_{nn})\hat{\theta}\Big)\wedge\Big(-i|\nu|^2\hat{\theta}^{\bar{j}}+A^{\bar{j}}{}_l\hat{\theta}^l-2|\nu|(\hat{Z}_j|\nu|)\hat{\theta}\Big). \end{aligned} \tag{7.37}$$

On the other hand, using (7.30) and the structure equations of the pseudohermitian structure,

$$\begin{aligned} d\theta_j{}^n &= d\left(i|\nu|\hat{\theta}^{\bar{j}}+\frac{A^{\bar{j}}{}_l}{|\nu|}\hat{\theta}^l-2(\hat{Z}_j|\nu|)\hat{\theta}\right) \\ &= -i|\nu|\hat{\theta}_j{}^k\wedge\hat{\theta}^{\bar{k}}+i|\nu|\hat{\tau}^{\bar{j}}\wedge\hat{\theta}-i(d|\nu|)\wedge\hat{\theta}^{\bar{j}} \\ &\quad-2d(\hat{Z}_j|\nu|)\wedge\hat{\theta}-2(\hat{Z}_j|\nu|)d\hat{\theta}+d\left(\frac{A^{\bar{j}}{}_l}{|\nu|}\right)\wedge\hat{\theta}^l+\left(\frac{A^{\bar{j}}{}_l}{|\nu|}\right)d\hat{\theta}^l. \end{aligned} \tag{7.38}$$

For each j, $1\le j\le m$, comparing the coefficients of the terms in (7.37) and (7.38), we get

$$2\Big(\hat{Z}_{\bar{j}}(\hat{Z}_j|\nu|)-\hat{\theta}_j{}^k(\hat{Z}_{\bar{j}})(\hat{Z}_k|\nu|)\Big)-i(\hat{T}|\nu|)=2\frac{\big|\hat{Z}_j|\nu|\big|^2}{|\nu|}+|\nu|^3+\mathrm{Im}b_{nn}-\frac{\sum_{l=1}^m|A_{jl}|^2}{|\nu|}.$$

Writing

$$|\nu|_{j\bar{j}}=\hat{Z}_{\bar{j}}(\hat{Z}_j|\nu|)-\hat{\theta}_j{}^k(\hat{Z}_{\bar{j}})(\hat{Z}_k|\nu|),$$

and taking the sum for j over 1 to m, we have

$$-\hat{\Delta}_b|\nu|=-\overline{\Box}_b|\nu|-im(\hat{T}|\nu|)=2\frac{\big|\hat{\partial}_b|\nu|\big|^2}{|\nu|}+m(|\nu|^3+\mathrm{Im}b_{nn})-\frac{|A|^2}{|\nu|}, \tag{7.39}$$

where $\Box_b$ and $\hat{\partial}_b$ are the Kohn Laplacian and ∂_b-operator on M^{2m+1}. Hence b_{nn} is determined. Substituting (7.39) into (7.30), we get

$$|\nu|\theta_n{}^n=\hat{\partial}_b|\nu|-\bar{\hat{\partial}}_b|\nu|+i\left(\frac{\hat{\Delta}_b|\nu|}{m}+|\nu|^3+\frac{(2\big|\hat{\partial}_b|\nu|\big|^2-|A|^2)}{m|\nu|}\right)\hat{\theta}. \tag{7.40}$$

This completes the proof. □

Theorem 7.3 says that for pseudohermitian submanifolds of CR dimension $m=n-1$, in which there is no zero for ν, the induced pseudohermitian structure constitute a complete set of invariant. Moreover, we have that if the pseudohermitian torsion of M vanishes, then M locally is part of the standard sphere as Theorem 7.3 describes.

Theorem 7.4. *Let $(M,\hat{J},\hat{\theta})$ be a simply connected pseudohermitian submanifold with CR dimension $m=n-1$. Suppose that the fundamental vector fields is* nowhere zero. *If $A_{\beta\gamma}\equiv 0$, then the Webster curvature R is constant, hence it is part of the standard sphere after a Heisenberg rigid motion.*

Proof. Suppose that $A_{\beta\gamma}=0$. From (7.33) and (7.35), we get $R=m(m+1)|\nu|^2=constant$. Next we claim

$$\omega_a{}^{2n}=-|\nu|\omega^a, \quad \text{for } a=1,\cdots,2n-1. \tag{7.41}$$

From (7.30), we have for $1\le k\le n-1$,

$$\theta_k{}^n=-i|\nu|\hat{\theta}^{\bar{k}}=-i\nu|\theta^{\bar{k}}=-i|\nu|(\omega^k-i\omega^{n+k}). \tag{7.42}$$

And from (7.40),

$$\theta_n{}^n=i|\nu|^2\hat{\theta}=-i\nu|\theta^n=-i|\nu|(\omega^n+i\omega^{2n}). \tag{7.43}$$

On the other hand, we see that for $1\le k\le n$, we have

$$\theta_k{}^n=\omega_k{}^n+i\omega_k{}^{2n}=\omega_{n+k}{}^{2n}+i\omega_k{}^{2n}. \tag{7.44}$$

Comparing (7.42)–(7.44), we get the claim (7.41). In addition, we also have

$$\omega^{2n}=0,\ \ \omega^n=-|\nu|\theta. \tag{7.45}$$

Substituting (7.41) and (7.45) into the motion equation

$$\begin{aligned} de_{2n}&=e_\beta\otimes\omega_{2n}{}^\beta+e_{n+\beta}\otimes\omega_{2n}{}^{n+\beta}-T\otimes\omega^n\\ &=\sum_{\beta=1}^{n-1}e_\beta\otimes(|\nu|\omega^\beta)+e_n\otimes(|\nu|\omega^n)+\sum_{\beta=1}^{n-1}e_{n+\beta}\otimes(|\nu|\omega^{n+\beta})+T\otimes(|\nu|\theta). \end{aligned} \tag{7.46}$$

That is

$$d\left(\frac{e_{2n}}{|\nu|}\right)=e_A\otimes\omega^A+T\otimes\theta=dX, \quad \text{on M}. \tag{7.47}$$

We conclude that on M

$$X-X^0=\frac{e_{2n}}{|\nu|}, \quad \text{for some } \ X^0\in H_n. \tag{7.48}$$

Writing

$$\begin{aligned} &\frac{e_{2n}}{|\nu|}=a_A(X)\mathring{e}_A(X), \quad \text{for some coefficient functions } a_A,\\ &X=(X_1,\cdots,X_{2n},X_{2n+1}), \qquad X^0=(X_1^0,\cdots,X_{2n}^0,X_{2n+1}^0). \end{aligned} \tag{7.49}$$

From (7.48), we have

$$a_A(X) = X_A - X_A^0, \quad A = 1,\cdots,2n, \tag{7.50}$$

and hence

$$\begin{aligned}
X - X^0 = \frac{e_{2n}}{|\nu|} &= a_A(X)\mathring{e}_A(X) \\
&= (X_\beta - X_\beta^0)\left(\frac{\partial}{\partial x_\beta} + X_{n+\beta}\frac{\partial}{\partial t}\right) + (X_{n+\beta} - X_{n+\beta}^0)\left(\frac{\partial}{\partial y_\beta} - X_\beta\frac{\partial}{\partial t}\right) \\
&= (X_\beta - X_\beta^0)\left(\frac{\partial}{\partial x_\beta} + X_{n+\beta}^0\frac{\partial}{\partial t}\right) + (X_{n+\beta} - X_{n+\beta}^0)\left(\frac{\partial}{\partial y_\beta} - X_\beta^0\frac{\partial}{\partial t}\right) \\
&= a_A(X)\mathring{e}_A(X^0),
\end{aligned} \tag{7.51}$$

with $\sum_{A=1}^{2n} a_A^2 = \frac{1}{|\nu|^2}$. This completes the proof. □

Theorem 7.4 is the same as Theorem 7.5.

Theorem 7.5. *Let $(M,\hat{J},\hat{\theta})$ be a simply connected pseudohermitian submanifold with CR dimension $m = n-1$. Suppose that the fundamental vector fields is* nowhere zero. *If $II = 0$, then the Webster curvature R is constant, hence it is part of the standard sphere after a Heisenberg rigid motion.*

Proof. Note that in the proof of Theorem 7.1, we choose a Darboux frame such that

$$e_n = -\frac{\nu}{|\nu|}.$$

This implies that $\langle \nu, Z_n \rangle = -|\nu|$. Hence, from (5.11), we have

$$\theta_j{}^n = h_{jk}\hat{\theta}^k - i|\nu|\theta^{\bar{j}}, \quad \text{mod } \hat{\theta}.$$

Comparing with (7.30), we get

$$h_{jk} = \frac{A_{jk}}{|\nu|}, \quad 1 \le j,k \le m.$$

Therefore

$$A_{jk} = 0 \Leftrightarrow II = 0.$$

This complete the proof. □

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