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On a Coupled Cahn–Hilliard System for Copolymer/Homopolymer Mixtures

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Abstract. Our aim in this paper is to study a coupled Cahn–Hilliard system for copolymer/homopolymer mixtures. We prove the existence, uniqueness and regularity of solutions. We then prove the existence of finite dimensional global attractors.

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Key words: Coupled Cahn–Hilliard system, well-posedness, finite dimensional global attractors

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1 Introduction

Our aim in this paper is to study a coupled Cahn–Hillard model for copolymer/homopolymer mixtures.

The original Cahn–Hilliard equation,

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta f(u)=0,$$

was initially proposed to model phase separation processes in binary alloys (see [6,7]). Since then, this equation, or some of its variants, were successfully applied to many other applications than just phase separation in alloys. We can mention, for instance, dealloying (this can be observed in corrosion processes; see [14]), population dynamics (see [11]), tumor growth (see [1, 15, 16, 21, 26]), bacterial films (see [22]), thin films (see [31]), chemistry (see [35]), image processing (see [5, 8, 13]) and even astronomy,

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with the rings of Saturn (see [33]), and ecology (for instance, the clustering of mussels can be perfectly well described by the Cahn–Hilliard equation; see [24]). We refer the interested reader to [25, 28] for reviews on the Cahn–Hilliard equation and some of its variants, as well as their mathematical analysis. The numerical analysis of the Cahn–Hilliard equation was addressed, e.g., in [37, 38, 40, 41] (see also [25] for more references).

Block copolymer materials are important in engineering as they have the ability to create a wide variety of micro-structures resulting from a compromise between phase segregation and polymer architecture which prevents complete phase separation (see, e.g., [4,12,20]). In particular, diblock copolymers have been studied from a mathematical and numerical point of view (see, e.g., [9,17] and references therein).

In this paper, we consider a coupled Cahn–Hilliard system considered in [2,3] (see also [10]) to study the phase separation of mixtures consisting of a homopolymer and a copolymer. More precisely, the two phase variables were introduced to describe the macro-phase separation between the homopolymer and copolymer, as well as the micro-phase separation between the two components of the diblock copolymers. The model consists of the coupling of the Ohta–Kawasaki equation,

$$\frac{\partial u}{\partial t}+\Delta^2 u+\sigma\overline{u}-\Delta f(u)=0, \qquad \sigma>0,$$

where $\overline{u}$ is the difference between u and its spatial average (see [29]; this equation actually is a variant of the Cahn–Hilliard–Oono equation,

$$\frac{\partial u}{\partial t}+\Delta^2 u+\sigma u-\Delta f(u)=0, \qquad \sigma>0,$$

proposed in [30] to model long-ranged effects), describing microscopic phase segregation, and the Cahn–Hillard equation. A related sharp interface model was proposed in [34], where global energy minimizers were studied. More precisely, there, existence and characterization of minimizers, together with upper and lower bounds on their energy, were obtained in one space dimension. In higher space dimensions, one only has upper bounds. Another related three-components model was considered in [18,19], where the existence and stability of equilibria, which are minimizers of the energy, were studied (see also [36] for a similar model with nonlocal interactions).

Efficient numerical simulations for the coupled Cahn–Hilliard model were performed in [23], based on an uncoupled and second-order unconditionally energy stable scheme. More precisely, there, two time marching schemes were proposed and their unique solvability and unconditional energy stability were established.

In this paper, we address the mathematical analysis of the problem. More precisely, we first prove the existence, uniqueness and regularity of solutions. We then address the asymptotic behavior of the associated dynamical system, in terms of finite dimensional global attractors.

2 Setting of the problem

We consider the following initial and boundary value problem in a bounded and regular domain $\Omega\subset\mathbb{R}^n$, $n=1$, 2 or 3, with boundary Γ:

$$\frac{\partial u}{\partial t}+\Delta^2 u-\Delta\frac{\partial F}{\partial u}=0, \tag{2.1}$$

$$\frac{\partial v}{\partial t}+\Delta^2 v-\Delta\frac{\partial F}{\partial v}+\sigma\overline{v}=0, \qquad \sigma>0, \tag{2.2}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial \Delta u}{\partial \nu}=\frac{\partial v}{\partial \nu}=\frac{\partial \Delta v}{\partial \nu}=0 \quad \text{on } \Gamma, \tag{2.3}$$

$$u|_{t=0}=u_0, \qquad v|_{t=0}=v_0. \tag{2.4}$$

Here,

$$F(u,v)=H(u)+H(v)+\alpha uv+\beta uv^2, \qquad \alpha,\ \beta\in\mathbb{R},$$
$$\frac{\partial F}{\partial u}(u,v)=f(u,v)=h(u)+\alpha v+\beta v^2, \quad \frac{\partial F}{\partial v}(u,v)=g(u,v)=h(v)+\alpha u+2\beta uv,$$
$$H(s)=\frac{1}{4}(s^2-1)^2, \quad h(s)(=H'(s))=s^3-s.$$

Note in particular that

$$h'\geq -1. \tag{2.5}$$

Furthermore, it follows from Young's inequality that

$$c_1 s^4-c_2\leq H(s)\leq c_3 s^4+c_4, \qquad c_1,\ c_3>0,\ c_2,\ c_4\geq 0,\ s\in\mathbb{R}, \tag{2.6}$$

$$h(s)(s-m)\geq c_5 s^4-c_{6,m}, \qquad c_5>0, c_{6,m}\geq 0,\ s,\ m\in\mathbb{R}, \tag{2.7}$$

where $c_{6,m}$ depends continuously on m. We also set, for $w\in L^1(\Omega)$,

$$\langle w\rangle=\frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega w(x)\,dx,$$

and for $w\in H^{-1}(\Omega)$,

$$\langle w\rangle=\frac{1}{\mathrm{Vol}(\Omega)}\langle w,1\rangle_{H^{-1}(\Omega),H^1(\Omega)}.$$

We finally set, whenever it makes sense,

$$\overline{w}=w-\langle w\rangle.$$

Remark 2.1. We can more generally consider here any regular function H such that $h' \geq -c_0$, $c_0 \geq 0$, and satisfying (2.6)-(2.7). we can also consider any polynomial growth of the form as^{2p+2}, $p \in \mathbb{N}$, $a > 0$. We will however give a regularity result which requires $p = 1$ when $n = 3$.

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$. We also set $\|\cdot\|_{-1} = \|(-\Delta)^{-\frac{1}{2}}\cdot\|$, where $(-\Delta)^{-1}$ denotes the inverse of the minus Laplace operator associated with Neumann boundary conditions and acting on functions with null spatial average. More generally, we denote by $\|\cdot\|_X$ the norm on the Banach space X.

We note that

$$w \mapsto (\|\overline{w}\|_{-1}^2 + \langle w \rangle^2)^{\frac{1}{2}}, \qquad w \mapsto (\|\overline{w}\|^2 + \langle w \rangle^2)^{\frac{1}{2}},$$

$$w \mapsto (\|\nabla w\|^2 + \langle w \rangle^2)^{\frac{1}{2}} \text{ and } w \mapsto (\|\Delta w\|^2 + \langle w \rangle^2)^{\frac{1}{2}}$$

are norms on $H^{-1}(\Omega)$, $L^2(\Omega)$, $H^1(\Omega)$ and $H^2(\Omega)$, respectively, which are equivalent to the usual norms on these spaces. Furthermore, $\|\cdot\|_{-1}$ is a norm on $\{w \in H^{-1}(\Omega),\ \langle w \rangle = 0\}$ which is equivalent to the usual H^{-1}-norm.

Throughout this paper, the same letters c and c' denote (generally positive) constants which may vary from line to line, or even in a same line. The same holds for $c_{\delta_1,\delta_2,\cdots}$ which denotes a constant depending on the parameters $\delta_1, \delta_2, \cdots$.

3 Well-posedness

We have the following result.

Theorem 3.1. *We assume that* $(u_0,v_0) \in H^1(\Omega)^2$. *Then,* (2.1)-(2.4) *possesses a unique weak solution* (u,v) *such that,* $\forall T > 0$,

$$(u,v) \in L^\infty(0,T;H^1(\Omega)^2) \cap L^2(0,T;H^2(\Omega)^2)$$

and

$$(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}) \in L^2(0,T;H^{-1}(\Omega)^2).$$

Proof. **Existence:**

The proof of existence can be carried out via a standard Galerkin scheme. Here below, we only give formal estimates which can be justified by the aforementioned scheme.

First, note that, integrating (2.1) and (2.2) over Ω and by parts, we obtain, owing to the boundary conditions (2.3),

$$\langle u(t) \rangle = \langle u_0 \rangle, \quad \langle v(t) \rangle = \langle v_0 \rangle, \quad t \geq 0. \tag{3.1}$$

We then rewrite the problem in the following equivalent form:

$$(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Delta u+\overline{\frac{\partial F}{\partial u}}=0, \tag{3.2}$$

$$(-\Delta)^{-1}\frac{\partial v}{\partial t}-\Delta v+\overline{\frac{\partial F}{\partial v}}+\sigma(-\Delta)^{-1}\overline{v}=0, \tag{3.3}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0 \quad \text{on } \Gamma, \tag{3.4}$$

$$u|_{t=0}=u_0, \qquad v|_{t=0}=v_0. \tag{3.5}$$

Note indeed that $\langle\frac{\partial u}{\partial t}\rangle=\langle\frac{\partial v}{\partial t}\rangle=0$.

We multiply (3.2) by $\frac{\partial u}{\partial t}$ and (3.3) by $\frac{\partial v}{\partial t}$, integrate over Ω and by parts and sum the two resulting equalities to find

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}\Big)+\Big\|\frac{\partial u}{\partial t}\Big\|^2_{-1}+\Big\|\frac{\partial v}{\partial t}\Big\|^2_{-1}+((\frac{\partial F}{\partial u},\frac{\partial u}{\partial t}))+((\frac{\partial F}{\partial v},\frac{\partial v}{\partial t}))=0,$$

which yields, noting that

$$((\frac{\partial F}{\partial u},\frac{\partial u}{\partial t}))+((\frac{\partial F}{\partial v},\frac{\partial v}{\partial t}))=\frac{d}{dt}\int_\Omega F(u,v)\,dx,$$

the energy equality

$$\frac{d}{dt}\Big(\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}+2\int_\Omega F(u,v)\,dx\Big)+2\Big(\|\frac{\partial u}{\partial t}\|^2_{-1}+\|\frac{\partial v}{\partial t}\|^2_{-1}\Big)=0. \tag{3.6}$$

We set, in what follows, recalling (3.1),

$$\langle u_0\rangle=\kappa_1, \quad \langle v_0\rangle=\kappa_2, \qquad \kappa_1,\,\kappa_2\in\mathbb{R}.$$

In particular, the constants below may depend on κ_1 and κ_2. We however do not write such a dependence explicitly.

That said, we next multiply (3.2) by $\overline{u}$ and (3.3) by $\overline{v}$ and have, summing the two resulting equalities,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|\overline{u}\|^2_{-1}+\|\overline{v}\|^2_{-1})+\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}&\\ +((f(u,v),\overline{u}))+((g(u,v),\overline{v}))=0.&\end{aligned} \tag{3.7}$$

Note that

$$
\begin{aligned}
&((f(u,v),\overline{u}))+((g(u,v),\overline{v}))\\
=&((h(u),\overline{u}))+((h(v),\overline{v}))+2\alpha((u,v))+3\beta((u,v^2))\\
&-\alpha\kappa_1((v,1))-\beta\kappa_1((v,v))-\alpha\kappa_2((u,1))-2\beta\kappa_2((u,v))\\
\geq&\frac{c_5}{2}(\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)})-c,
\end{aligned}
\tag{3.8}
$$

owing to (2.7) and Young's inequality. It thus follows from (3.7)-(3.8) that

$$
\frac{d}{dt}(\|\overline{u}\|^2_{-1}+\|\overline{v}\|^2_{-1})+\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}+c_5(\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)})\leq c. \tag{3.9}
$$

Summing (3.6) and (3.9), we also deduce from (2.6) the differential inequality

$$
\frac{dE}{dt}+c\left(E+\left\|\frac{\partial u}{\partial t}\right\|^2_{-1}+\left\|\frac{\partial v}{\partial t}\right\|^2_{-1}\right)\leq c', \qquad c>0, \tag{3.10}
$$

where

$$
E=\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}+2\int_\Omega F(u,v)\,dx+\|\overline{u}\|^2_{-1}+\|\overline{v}\|^2_{-1}
$$

satisfies

$$
\begin{aligned}
&c\left(\|u\|^2_{H^1(\Omega)}+\|v\|^2_{H^1(\Omega)}+\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)}\right)-c'\leq E\\
&\qquad\leq c''(\|u\|^2_{H^1(\Omega)}+\|v\|^2_{H^1(\Omega)}+\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)})+c''',
\end{aligned}
$$

for c, $c''>0$, c', $c'''\geq 0$. Indeed, note that it follows from Young's inequality and (2.6) that

$$
\begin{aligned}
\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)}&\geq\frac{1}{2}(\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)})+\|u\|^2+\|v\|^2-c\\
&\geq c\int_\Omega F(u,v)\,dx+\|u\|^2+\|v\|^2-c', \qquad c>0,
\end{aligned}
$$

so that

$$
\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|\overline{v}\|^2_{-1}+c_5(\|u\|^4_{L^4(\Omega)}+\|v\|^4_{L^4(\Omega)})\geq cE-c',\ c>0.
$$

We finally multiply (3.2) by $-\Delta u$ and (3.3) by $-\Delta v$ and obtain, summing the two resulting equalities,

$$
\begin{aligned}
&\frac{d}{dt}(\|\overline{u}\|^2+\|\overline{v}\|^2)+\|\Delta u\|^2+\|\Delta v\|^2+\sigma\|\overline{v}\|^2\\
&\quad+((\frac{\partial f}{\partial u}\nabla u,\nabla u))+(((\frac{\partial f}{\partial v}+\frac{\partial g}{\partial u})\nabla u,\nabla v))+((\frac{\partial g}{\partial v}\nabla v,\nabla v))=0,
\end{aligned}
\tag{3.11}
$$

where

$$\frac{\partial f}{\partial u}=h'(u),\quad \frac{\partial f}{\partial v}=\alpha+2\beta v,\quad \frac{\partial g}{\partial u}=\alpha+2\beta v,\quad \frac{\partial g}{\partial v}=h'(v)+2\beta u.$$

We note that it follows from (2.5) that

$$\begin{aligned}&((\frac{\partial f}{\partial u}\nabla u,\nabla u))+(((\frac{\partial f}{\partial v}+\frac{\partial g}{\partial u})\nabla u,\nabla v))+((\frac{\partial g}{\partial v}\nabla v,\nabla v))\\ &\quad\geq -c(\|\nabla u\|^2+\|\nabla v\|^2)+4\beta((v\nabla u,\nabla v))+2\beta((u\nabla v,\nabla v)).\end{aligned} \tag{3.12}$$

Furthermore, owing to Hölder's inequality and a proper Sobolev embedding,

$$\begin{aligned}&|4\beta((v\nabla u,\nabla v))+2\beta((u\nabla v,\nabla v))|\\ \leq& c(\|v\|_{L^4(\Omega)}\|\nabla u\|_{L^4(\Omega)}\|\nabla v\|+\|u\|_{L^4(\Omega)}\|\nabla v\|_{L^4(\Omega)}\|\nabla v\|)\\ \leq& c(\|v\|_{H^1(\Omega)}\|u\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)}+\|u\|_{H^1(\Omega)}\|v\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)})\\ \leq&\frac{1}{2}(\|\Delta u\|^2+\|\Delta v\|^2)+c(\|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)}+\|v\|^4_{H^1(\Omega)}+1).\end{aligned} \tag{3.13}$$

It thus follows from (3.11)-(3.13) that

$$\begin{aligned}&\frac{d}{dt}(\|\overline{u}\|^2+\|\overline{v}\|^2)+c(\|u\|^2_{H^2(\Omega)}+\|v\|^2_{H^2(\Omega)})\\ \leq& c'(\|u\|^2_{H^1(\Omega)}\|v\|^2_{H^1(\Omega)}+\|v\|^4_{H^1(\Omega)}+1),\quad c>0.\end{aligned} \tag{3.14}$$

It follows from (3.10) that, for $T>0$ given, $(u,v)\in L^\infty(0,T;H^1(\Omega)^2)$ and $(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t})\in L^2(0,T;H^{-1}(\Omega)^2)$. Then, (3.14) yields that $(u,v)\in L^2(0,T;H^2(\Omega)^2)$.

Uniqueness:

Let (u_1,v_1) and (u_2,v_2) be two solutions with initial data $(u_{1,0},v_{1,0})$ and $(u_{2,0},v_{2,0})$, respectively, such that

$$\langle u_{1,0}\rangle=\langle u_{2,0}\rangle,\ \langle v_{1,0}\rangle=\langle v_{2,0}\rangle.$$

We have, setting $(u,v)=(u_1,v_1)-(u_2,v_2)$ and $(u_0,v_0)=(u_{1,0},v_{1,0})-(u_{2,0},v_{2,0})$,

$$(-\Delta)^{-1}\frac{\partial u}{\partial t}-\Delta u+\overline{f(u_1,v_1)-f(u_2,v_2)}=0, \tag{3.15}$$

$$(-\Delta)^{-1}\frac{\partial v}{\partial t}-\Delta v+\overline{g(u_1,v_1)-g(u_2,v_2)}+\sigma(-\Delta)^{-1}\overline{v}=0, \tag{3.16}$$

$$\frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0\quad \text{on } \Gamma, \tag{3.17}$$

$$u|_{t=0}=u_0,\qquad v|_{t=0}=v_0. \tag{3.18}$$

Multiplying (3.15) by u and (3.16) by v, we obtain, summing the two resulting equalities and noting that $\langle u\rangle=\langle v\rangle=0$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|u\|_{-1}^2+\|v\|_{-1}^2)+\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|v\|_{-1}^2\\&\quad+((f(u_1,v_1)-f(u_2,v_2),u))+((g(u_1,v_1)-g(u_2,v_2),v))=0.\end{aligned}\tag{3.19}$$

Note that, employing (2.5) and Young's inequality,

$$\begin{aligned}&((f(u_1,v_1)-f(u_2,v_2),u))+((g(u_1,v_1)-g(u_2,v_2),v))=((h(u_1)-h(u_2),u))\\&\qquad+((h(v_1)-h(v_2),v))+2\alpha((u,v))+\beta(((v_1+v_2)v,u))+2\beta((u_1v+uv_2,v))\\&\geq -c(\|u\|^2+\|v\|^2)+\beta(((v_1+v_2)v,u))+2\beta((u_1v+uv_2,v)).\end{aligned}\tag{3.20}$$

Furthermore, employing Ladyzhenskaya's (we take $n=3$; the other cases can be dealt with in a similar (even easier) way), Hölder's and Young's inequalities,

$$\begin{aligned}|\beta(((v_1+v_2)v,u))|&\leq c(\|v_1\|+\|v_2\|)\|u\|_{L^4(\Omega)}\|v\|_{L^4(\Omega)}\\&\leq c(\|v_1\|+\|v_2\|)\|u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}}\|v\|^{\frac{1}{4}}\|\nabla v\|^{\frac{3}{4}}\\&\leq c_T(\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{3}{2}}+\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{3}{2}})\\&\leq \epsilon(\|\nabla u\|^2+\|\nabla v\|^2)+c_{\epsilon,T}(\|u\|^2+\|v\|^2),\quad \forall\epsilon>0.\end{aligned}\tag{3.21}$$

Proceeding similarly for the other term, it follows from (3.19)-(3.21) that

$$\frac{d}{dt}(\|u\|_{-1}^2+\|v\|_{-1}^2)+\frac{3}{2}(\|\nabla u\|^2+\|\nabla v\|^2)\leq c_T(\|u\|^2+\|v\|^2).$$

Employing finally the interpolation inequality

$$\|w\|^2\leq c\|w\|_{-1}\|\nabla w\|,\quad w\in H^1(\Omega),\quad \langle w\rangle=0,$$

and Young's inequality, we deduce that

$$\frac{d}{dt}(\|u\|_{-1}^2+\|v\|_{-1}^2)+\|\nabla u\|^2+\|\nabla v\|^2\leq c_T(\|u\|_{-1}^2+\|v\|_{-1}^2),\quad t\in[0,T].\tag{3.22}$$

In particular, it follows from Gronwall's lemma that

$$\begin{aligned}&\|u_1(t)-u_2(t)\|_{-1}+\|v_1(t)-v_2(t)\|_{-1}\\&\leq c_T(\|u_{1,0}-u_{2,0}\|_{-1}+\|v_{1,0}-v_{2,0}\|_{-1}),\qquad t\in[0,T],\end{aligned}\tag{3.23}$$

which yields the continuous dependence with respect to the initial data in the H^{-1}-topology, as well as the uniqueness. □

We then have the following.

Theorem 3.2. *We further assume that* $(u_0,v_0)\in H^2(\Omega)^2$*, with* $\frac{\partial u_0}{\partial\nu}=\frac{\partial v_0}{\partial\nu}=0$ *on* Γ*. Then, the solution* (u,v) *given in Theorem* 3.1 *satisfies,* $\forall T>0$*,*

$$(u,v)\in\mathcal{C}([0,T];H^2(\Omega)^2)\cap L^2(0,T;H^4(\Omega)^2)$$

and

$$(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t})\in L^2(0,T;L^2(\Omega)^2).$$

Proof. We multiply (3.2) by $-\Delta^3 u$ and (3.3) by $-\Delta^3 v$ and obtain, summing the two resulting equalities,

$$\frac{1}{2}\frac{d}{dt}(\|\Delta u\|^2+\|\Delta v\|^2)+\|\Delta^2 u\|^2+\|\Delta^2 v\|^2+\sigma\|\Delta v\|^2$$
$$-((\Delta f(u,v),\Delta^2 u))-((\Delta g(u,v),\Delta^2 v))=0.$$

Note that

$$((\Delta f(u,v),\Delta^2 u))$$
$$=((\Delta h(u),\Delta^2 u))+\alpha((\Delta v,\Delta^2 u))+2\beta((v\Delta v,\Delta^2 u))+2\beta((|\nabla v|^2,\Delta^2 u)).$$

It is proved in [25] (we consider the most difficult case $n=3$), based on Agmon's and several interpolation inequalities, that

$$|((\Delta h(u),\Delta^2 u))|\le\epsilon\|\Delta^2 u\|^2+c_\epsilon(1+\|u\|_{H^1(\Omega)}^{14}).$$

Indeed, note that

$$|((\Delta h(u),\Delta^2 u))|\le\|\Delta h(u)\|\|\Delta^2 u\|$$

and

$$\Delta h(u)=h'(u)\Delta u+h''(u)\nabla u\cdot\nabla u.$$

Moreover, with our choice of the function h, we have $h'(s)=3s^2-1$ and $h''(s)=6s$. Therefore,

$$\|\Delta h(u)\|\le c(\|u^2\Delta u\|+\|\Delta u\|+\|u\nabla u\cdot\nabla u\|).$$

Employing the interpolation inequality

$$\|u\|_{H^2(\Omega)}\le c\|u\|_{H^1(\Omega)}^{\frac{2}{3}}\|u\|_{H^4(\Omega)}^{\frac{1}{3}}$$

and Agmon's inequality

$$\|u\|_{L^\infty(\Omega)}\le c\|u\|_{H^1(\Omega)}^{\frac{1}{2}}\|u\|_{H^2(\Omega)}^{\frac{1}{2}},$$

it follows that

$$\|u\|_{L^\infty(\Omega)}\le c\|u\|_{H^1(\Omega)}^{\frac{5}{6}}\|u\|_{H^4(\Omega)}^{\frac{1}{6}}.$$

We thus find

$$\|u^2\Delta u\| \le \|u\|^2_{L^\infty(\Omega)}\|\Delta u\| \le c\|u\|^2_{L^\infty(\Omega)}\|u\|_{H^2(\Omega)},$$

so that

$$\|u^2\Delta u\| \le c\|u\|^{\frac{7}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)}. \tag{3.24}$$

Next, noting that $H^4(\Omega) \subset H^1(\Omega)$ with continuous embedding and

$$\|\Delta u\| \le c\|u\|_{H^2(\Omega)} \le c\|u\|^{\frac{2}{3}}_{H^1(\Omega)}\|u\|^{\frac{1}{3}}_{H^4(\Omega)},$$

it is easy to see that

$$\|\Delta u\| \le c\|u\|^{\frac{1}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)}. \tag{3.25}$$

We now have

$$\|u\nabla u\cdot\nabla u\| \le \|u\|_{L^\infty(\Omega)}\|\nabla u\|^2_{L^4(\Omega)} \le c\|u\|^{\frac{5}{6}}_{H^1(\Omega)}\|u\|^{\frac{1}{6}}_{H^4(\Omega)}\|\nabla u\|^2_{L^4(\Omega)}.$$

As already mentioned, we concentrate on the most difficult case $n=3$. Note that, in three space dimensions, $H^{\frac{3}{4}}(\Omega) \subset L^4(\Omega)$ with continuous embedding, so that

$$\|\nabla u\|_{L^4(\Omega)} \le c\|u\|_{H^{\frac{7}{4}}(\Omega)}.$$

Employing the interpolation inequality

$$\|u\|_{H^{\frac{7}{4}}(\Omega)} \le c\|u\|^{\frac{3}{4}}_{H^1(\Omega)}\|u\|^{\frac{1}{4}}_{H^4(\Omega)},$$

it follows that

$$\|u\nabla u\cdot\nabla u\| \le c\|u\|^{\frac{5}{6}}_{H^1(\Omega)}\|u\|^{\frac{1}{6}}_{H^4(\Omega)}\|u\|^{\frac{3}{2}}_{H^1(\Omega)}\|u\|^{\frac{1}{2}}_{H^4(\Omega)}$$

and

$$\|u\nabla u\cdot\nabla u\| \le c\|u\|^{\frac{7}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)}. \tag{3.26}$$

Collecting the above estimates, we obtain, employing Young's inequality,

$$\begin{aligned}\|\Delta h(u)\| &\le c(\|u\|^{\frac{7}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)} + \|u\|^{\frac{1}{3}}_{H^1(\Omega)}\|u\|^{\frac{2}{3}}_{H^4(\Omega)})\\ &\le c(1+\|u\|^{\frac{7}{3}}_{H^1(\Omega)})\|u\|^{\frac{2}{3}}_{H^4(\Omega)}.\end{aligned}$$

We thus deduce, employing again Young's inequality, that

$$\begin{aligned}|((\Delta h(u),\Delta^2 u))| &\le c(1+\|u\|^{\frac{7}{3}}_{H^1(\Omega)})\|u\|^{\frac{2}{3}}_{H^4(\Omega)}\|\Delta^2 u\|\\ &\le c(1+\|u\|^{\frac{7}{3}}_{H^1(\Omega)})(\|u\|_{H^1(\Omega)}+\|\Delta^2 u\|)^{\frac{5}{3}}.\end{aligned}$$

Employing once more Young's inequality, we find

$$|((\Delta h(u),\Delta^2 u))| \le \epsilon \|\Delta^2 u\|^2 + c_\epsilon (1+\|u\|^{14}_{H^1(\Omega)}). \tag{3.27}$$

The other terms can be handled in a similar way (and are actually easier to be dealt with, as they are of lower order with respect to u and v). The term $((\Delta g(u,v),\Delta^2 v))$ can also be handled in a similar way. We finally end up with the differential inequality

$$\frac{d}{dt}(\|\Delta u\|^2+\|\Delta v\|^2)+\|\Delta^2 u\|^2+\|\Delta^2 v\|^2 \le c(1+\|u\|^{14}_{H^1(\Omega)}+\|v\|^{14}_{H^1(\Omega)}). \tag{3.28}$$

It follows from (3.28) that $(u,v)\in L^\infty(0,T;H^2(\Omega)^2)\cap L^2(0,T;H^4(\Omega)^2)$, while the regularity $(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t})\in L^2(0,T;L^2(\Omega)^2)$ can be read from (2.1)-(2.2). Finally, employing Lions–Magenes's theorem (see, e.g., [25]), we deduce that $(u,v)\in \mathcal{C}([0,T];H^2(\Omega)^2)$. □

Remark 3.1. (i) In one and two space dimensions, we can deal with a polynomial of the form

$$h(s)=\sum_{i=1}^{2p+1} a_i s^i,\ p\in\mathbb{N},\ a_{2p+1}>0.$$

However, the restriction $p=1$ is needed in three space dimensions (see [25]).

(ii) Actually, we can also take p arbitrary in three space dimensions, employing more refined estimates and techniques. We refer the reader to [25] for more details.

4 Existence of finite dimensional global attractors

We set

$$\Phi=\{(w,z)\in H^1(\Omega)^2,\ \langle w\rangle=\kappa_1,\ \langle z\rangle=\kappa_2\},\ \kappa_1,\ \kappa_2\in\mathbb{R}.$$

It follows from Theorem 3.1 that we can define the family of continuous (for the H^{-1}-topology) solving operators $S(t):\Phi\to\Phi$, $(u_0,v_0)\mapsto(u(t),v(t))$, $t\ge 0$. Furthermore, this family of operators forms a semigroup, i.e., $S(0)=I$ and $S(t)\circ S(\tau)=S(t+\tau)$, t, $\tau\ge 0$.

Next it follows from (3.10) and Gronwall's lemma that $S(t)$ possesses a bounded absorbing set $\mathcal{B}_0$ in Φ, i.e., $\forall B\subset\Phi$ bounded, there exists $t_0=t_0(B)$ such that $t\ge t_0$ implies $S(t)B\subset\mathcal{B}_0$. Furthermore, it follows from (3.10) again, (3.28) and the uniform Gronwall's lemma (see, e.g., [32]) that $S(t)$ possesses a bounded absorbing set which is compact in Φ and bounded in $H^2(\Omega)^2$. It thus follows from standard results (see, e.g., [27,32]) that we have the following.

Theorem 4.1. *The semigroup $S(t)$ possesses the global attractor $\mathcal{A}$ such that*

(i) $\mathcal{A}$ is compact in Φ and bounded in $H^2(\Omega)^2$;

(ii) $\mathcal{A}$ is invariant, $S(t)\mathcal{A}=\mathcal{A}$, $t\ge 0$;

(iii) $\mathcal{A}$ attracts the bounded subsets of Φ in the H^{-1}-topology,

$$\forall B\subset\Phi \text{ bounded, } \operatorname{dist}(S(t)B,\mathcal{A})\to 0 \text{ as } t\to+\infty,$$

where dist *denotes the Hausdorff semidistance between sets defined as*

$$\operatorname{dist}(A,B)=\sup_{a\in A}\inf_{b\in B}\|a-b\|_{H^{-1}(\Omega)^2}.$$

Remark 4.1. The global attractor is the smallest (for the inclusion) bounded set which attracts the bounded sets of trajectories as time goes to infinity. It can thus be seen as a suitable object in view of the study of the asymptotic behavior of the system; it contains all possible dynamics (see, e.g., [27,32] for more details).

We now recall the following (see [39]).

Proposition 4.1. *Let X be a compact subset of the Banach space E. We assume that there exist a Banach space E_1 such that E_1 is compactly embedded into E and a mapping $L:X\to X$ such that $L(X)=X$ and L satisfies the following smoothing property on the difference of two solutions:*

$$\|Lx_1-Lx_2\|_{E_1}\le c\|x_1-x_2\|_E,\quad \forall x_1,\ x_2\in X,\ \ c>0.$$

Then, the fractal dimension of X is finite (in the topology of E).

Remark 4.2. The fractal dimension is defined as follows. Let $X\subset E$ be a (relatively) compact set. For $\epsilon>0$, let $N_\epsilon(X)$ be the minimal number of balls in E of radius ϵ which are necessary to cover X. Then, the fractal dimension of X is the quantity (which belongs to $[0,+\infty]$)

$$\dim_{\mathrm{F}}X=\limsup_{\epsilon\to 0^+}\frac{\log_2 N_\epsilon(X)}{\log_2\frac{1}{\epsilon}}\left(=\limsup_{\epsilon\to 0^+}\frac{\ln N_\epsilon(X)}{\ln\frac{1}{\epsilon}}\right).$$

We can then prove the following.

Theorem 4.2. *The global attractor $\mathcal{A}$ has finite fractal dimension in the H^{-1}-topology.*

Proof. We again consider two solutions (u_1,v_1) and (u_2,v_2), now with initial data $(u_{1,0},v_{1,0})$ and $(u_{2,0},v_{2,0})$ belonging to $\mathcal{A}$ and such that

$$\langle u_{1,0}\rangle=\langle u_{2,0}\rangle,\ \langle v_{1,0}\rangle=\langle v_{2,0}\rangle.$$

Note that it follows from the invariance property that the trajectories are globally (in time) bounded in $H^2(\Omega)^2$ and, thus, in $L^\infty(\Omega)^2$.

First, note that, integrating (3.22) over $[0,1]$, it follows from (3.23) (we take here $T=1$) that

$$\int_0^1(\|\nabla u\|^2+\|\nabla v\|^2)\,dx\le c(\|u_0\|_{-1}^2+\|v_0\|_{-1}^2). \tag{4.1}$$

Next, we multiply (3.15) by $t\frac{\partial u}{\partial t}$ and (3.16) by $t\frac{\partial v}{\partial t}$ and obtain, summing the two resulting equalities,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2+t\|\nabla v\|^2+t\sigma\|v\|_{-1}^2)+t\|\frac{\partial u}{\partial t}\|_{-1}^2+t\|\frac{\partial v}{\partial t}\|_{-1}^2\\
&\qquad +t((f(u_1,v_1)-f(u_2,v_2),\frac{\partial u}{\partial t}))+t((g(u_1,v_1)-g(u_2,v_2),\frac{\partial v}{\partial t}))\\
&=\frac{1}{2}(\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|v\|_{-1}^2).
\end{aligned} \tag{4.2}$$

Note that

$$|((f(u_1,v_1)-f(u_2,v_2),\frac{\partial u}{\partial t}))|\le\|\nabla(f(u_1,v_1)-f(u_2,v_2))\|\|\frac{\partial u}{\partial t}\|_{-1}$$

and

$$\begin{aligned}
&\|\nabla(f(u_1,v_1)-f(u_2,v_2))\|=\|\nabla((u_1^2+u_2^2+u_1u_2-1)u+\alpha v+\beta(v_1+v_2)v)\|\\
=&\|(2u_1\nabla u_1+2u_2\nabla u_2+u_1\nabla u_2+u_2\nabla u_1)u+(u_1^2+u_2^2+u_1u_2-1)\nabla u\\
&\quad+\alpha\nabla v+\beta(\nabla v_1+\nabla v_2)v+\beta(v_1+v_2)\nabla v\|\\
\le& c(\|u_1\|_{L^\infty(\Omega)}\|\nabla u_1\|_{L^4(\Omega)}+\|u_2\|_{L^\infty(\Omega)}\|\nabla u_2\|_{L^4(\Omega)}+\|u_1\|_{L^\infty(\Omega)}\|\nabla u_2\|_{L^4(\Omega)}\\
&\quad+\|u_2\|_{L^\infty(\Omega)}\|\nabla u_1\|_{L^4(\Omega)})\|u\|_{L^4(\Omega)}\\
&\quad+(\|u_1\|_{L^\infty(\Omega)}^2+\|u_2\|_{L^\infty(\Omega)}^2+\|u_1\|_{L^\infty(\Omega)}\|u_2\|_{L^\infty(\Omega)}+1)\|\nabla u\|\\
&\quad+\|\nabla v\|+(\|\nabla v_1\|_{L^4(\Omega)}+\|\nabla v_2\|_{L^4(\Omega)})\|v\|_{L^4(\Omega)}+(\|v_1\|_{L^\infty(\Omega)}+\|v_2\|_{L^\infty(\Omega)})\|\nabla v\|\\
\le& c(\|u_1\|_{L^\infty(\Omega)}\|u_1\|_{H^2(\Omega)}+\|u_2\|_{L^\infty(\Omega)}\|u_2\|_{H^2(\Omega)}+\|u_1\|_{L^\infty(\Omega)}\|u_2\|_{H^2(\Omega)}\\
&\quad+\|u_2\|_{L^\infty(\Omega)}\|u_1\|_{H^2(\Omega)})\|\nabla u\|+(\|u_1\|_{L^\infty(\Omega)}^2+\|u_2\|_{L^\infty(\Omega)}^2+\|u_1\|_{L^\infty(\Omega)}\|u_2\|_{L^\infty(\Omega)}+1)\|\nabla u\|\\
&\quad+\|\nabla v\|+(\|v_1\|_{H^2(\Omega)}+\|v_2\|_{H^2(\Omega)})\|\nabla v\|+(\|v_1\|_{L^\infty(\Omega)}+\|v_2\|_{L^\infty(\Omega)})\|\nabla v\|\\
\le& c(\|\nabla u\|+\|\nabla v\|),
\end{aligned}$$

owing to the continuous embedding $H^1(\Omega)\subset L^4(\Omega)$ and recalling that $\mathcal{A}$ is bounded in $H^2(\Omega)$ and $L^\infty(\Omega)$. Proceeding in a similar way for the other term in the left-hand side of (4.2), it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2+t\|\nabla v\|^2+t\sigma\|v\|_{-1}^2)+t\|\frac{\partial u}{\partial t}\|_{-1}^2+t\|\frac{\partial v}{\partial t}\|_{-1}^2\\
&\le ct(\|\nabla u\|+\|\nabla v\|)(\|\frac{\partial u}{\partial t}\|_{-1}+\|\frac{\partial v}{\partial t}\|_{-1})+\frac{1}{2}(\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|v\|_{-1}^2),
\end{aligned}$$

which yields

$$\begin{aligned}
&\frac{d}{dt}(t\|\nabla u\|^2+t\|\nabla v\|^2+t\sigma\|v\|_{-1}^2)+t\|\frac{\partial u}{\partial t}\|_{-1}^2+t\|\frac{\partial v}{\partial t}\|_{-1}^2\\
&\le ct(\|\nabla u\|^2+\|\nabla v\|^2)+\|\nabla u\|^2+\|\nabla v\|^2+\sigma\|v\|_{-1}^2.
\end{aligned} \tag{4.3}$$

Employing Gronwall's lemma (over $[0,1]$), we find, in view of (4.1),

$$\|\nabla u(1)\|^2+\|\nabla v(1)\|^2\le c(\|u_0\|_{-1}^2+(\|v_0\|_{-1}^2). \tag{4.4}$$

We finally deduce from (4.4) that the assumptions of Proposition 4.1 are satisfied, taking $X=\mathcal{A}$, $L=S(1)$ (recall that $\mathcal{A}$ is invariant), $E=H^{-1}(\Omega)$ and $E_1=H^1(\Omega)$, which finishes the proof. □

Remark 4.3. The finite dimensionality means, roughly speaking, that, even though the phase space has infinite dimension, the reduced dynamics (on the global attractor) can be described, in some proper sense, by a finite number of parameters. Theorem 4.2 thus suggests that the large time behavior of the system is, in some proper sense, finite dimensional. This can, in particular, be useful in view of numerical simulations. Indeed, an upper bound on the fractal dimension of the global attractor, in terms of the physical parameters in the equations, would give an estimate on the number of unknowns which are necessary to capture all possible dynamics.

Remark 4.4. Note that the estimates derived above are uniform with respect to σ, for, say, $\sigma\in(0,1]$. It would thus be interesting to study the dynamics of the model as $\sigma\to 0^+$. In particular, as far as the Cahn–Hilliard–Oono equation is concerned, the dynamics is close, in some proper sense, to that of the Cahn–Hilliard equation when σ is small (see [25]). More precisely, one can construct robust families of exponential attractors (see also [27]). This will be addressed elsewhere.

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