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Data Recovery from Cauchy Measurements in Transient Heat Transfer

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Abstract. We study the ill-posedness degree of the reconstruction processes of missing boundary data or initial states in the transient heat conduction. Both problems are severely ill-posed. This is a powerful indicator about the way the instabilities will affect the computations in the numerical recovery methods. We provide rigorous proofs of this result where the conductivites are space dependent. The theoretical work is concerned with the unsteady heat equation in one dimension even though most of the results obtained here are readily extended to higher dimensions.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07.

Key words: Data completion process, ill-posedness degree, Cauchy matrix, convolution equations, parabolic regularity.

1 Introduction

Computational recovery processes of missing boundary data or initial states from Cauchy measurements in transient heat transfer seem recurrent in many areas in sciences and engineering (see [5, 6, 23]). They turn out to be among few pertinent ways to proceed, if not the only one, when engineers are interested for example in quantifying front surface heat inputs of a (thin) plate from back surface outputs. Mounting measurements set-up along the front surface may not be feasible due to harsh environmental conditions. Practitioners are therefore led to place sensors at the back surface; they are then left with the evaluation, by means of affordable (analytical and/or numerical) tools, of the heat

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transfer up to the front surface. The particularity of the related mathematical problem is the ill-posedness. Missing data to reconstruct suffer from high instabilities generated by unavoidable perturbations affecting the available data because of the finite accuracy of measuring instruments (see [2,3,14,17,18,22,31,32]). As a consequence, computing methods crudely employed for numerical handling of these inverse problems are most often doomed to fail unless they are used with relevant regularization techniques combined to suitable automatic selection rules of the regularization parameter(s). Readers are referred to [4,15,19,24,25,33] for a general exposition of these issues. We are exclusively focused on the issue of ill-posedness degree for the reconstruction problem of either the initial or the boundary data from Cauchy's measurements. The purpose is then to perform a rigorous analysis on the severe ill-posed of the inverse problems under scope. We develop in details methodologies for the transient heat equation set in a rod. This choice is made only for seek of simplicity. We do not see why the central ideas discussed here cannot be effective for higher dimensions.

The contents of the paper are as follows. Section 2 is dedicated to the study of the initial state recovery from Cauchy's conditions. After defining the linear operator to invert, we study some of its marked features. Expanding this operator along Fourier basis shows that the ill-posedness degree is connected with a Cauchy matrix. This matrix is spectrally equivalent to a Pick matrix. Using the theory elaborated in [7], we exhibit the asymptotics of its eigenvalues. They decay exponentially fast towards zero which is an indication of the severe ill-posedness of the reconstruction process of the initial state. In Section 3, we turn to the recovery of a missing boundary condition at one extremity of the rod, where the initial state is known. We follow a similar approach, we define the linear operator to analyze and underline its distinctive properties. The key point is to put it under an integral form using a suitable Green kernel. The smoothness and the flatness at the origin of that kernel is the clue to the analysis of the ill-posedness degree. It is derived in the appendix thanks to the Laplace transform and some comparison results. In Section 4, we provide two numerical illustrations by MATLAB of the technical results stated in the previous sections.

Notation 1.1. Let X be a Banach space endowed with its norm $\|\cdot\|_X$. We denote by $L^2(0,T;X)$ the space of measurable functions v from $(0,T)$ in X such that

$$\|v\|_{L^2(0,T;X)} = \Big(\int_{(0,T)} \|v(s)\|_X^2 ds\Big)^{1/2} < +\infty.$$

We also use the space $\mathscr{C}(0,T;X)$ of continuous functions v from $[0,T]$ in X. Denote by I a given interval in $\mathbb{R}$, the Sobolev space $H^1(I)$ is the space all the functions that belong to $L^2(I)$ together with their first derivatives.

2 Initial condition reconstruction

Let a rod be geometrically represented by the segment $I=(0,\pi)$ and $T>0$ be a fixed real-number. We set $Q=I\times]0,T[$. The generic point in I is denoted by x and the generic time is t. Assume now be given ψ in $L^2(I)$ and denote by y_ψ the unique solution of the heat problem

$$\begin{aligned} &\partial_t y_\psi-(\gamma y_\psi')'=0, && \text{in } Q,\\ &y_\psi(0,t)=0, \quad (\gamma y_\psi)'(\pi,t)=0, && \forall t\in(0,T),\\ &y_\psi(x,0)=\psi, && \forall x\in I. \end{aligned} \tag{2.1}$$

The symbol $'$ is used for the space derivative ∂_x. The conductivity parameter $\gamma(\cdot)\in L^\infty(I)$ is piecewise continuously differentiable and is supposed to be positive and bounded away from zero. This means that $\gamma_m=\min_{x\in I}\gamma(x)>0$. We also set $\gamma_\infty=\max_{x\in I}\gamma(x)=\|\gamma\|_{L^\infty(I)}$. The function y_ψ exists and lies in $L^2(0,T;V)\cap\mathbb{C}([0,T];L^2(I))$ (see [26, Chap. 4]), where the space V is set to be

$$V=\Big\{z\in H^1(I);\ z(0)=0\Big\}.$$

The inverse problem to deal with here is the reconstruction of the initial condition ψ from an additional boundary data at $x=\pi$, related to abundant observations at that extreme-point. Hence, for a given function $h=h(t)\in L^2(0,T)$, we intend to find $\psi\in L^2(I)$ satisfying the following Dirichlet condition

$$y_\psi(\pi,t)=h(t), \qquad \forall t\in(0,T). \tag{2.2}$$

The point we focus on is the ill-posedness degree of this observation problem, an indication of how unstable it is. We refer to [36] for the general introduction of this notion and to [34] (see also [24, 35]) for a definition specifically adapted to the problems under investigation.

2.1 Uniqueness and admissible data

That problem (2.2) is ill-posed is widely known. No existence is guaranteed and the hypothetical solution $\psi(\cdot)$ does not depend continuously on the data $h(\cdot)$. The identifiability is the only result one may obtain. Indeed, we have

Lemma 2.1. *The inverse problem* (2.2) *has at most one solution.*

The methodology followed in the subsequent consists in the investigation of the linear operator

$$B:\ \psi\mapsto y_\psi(\pi,\cdot),$$

that is at the heart of the inverse problem (2.2). It is mapping $L^2(I)$ into $L^2(0,T)$ continuously. It is also a compact operator, has a non-closed range and its inverse cannot be bounded. The goal we pursue is to assess the ill-posedness degree. We follow the methodology exposed in [8].

Tools from Fourier Series Theory enable us to know more about B. We construct the orthonormal basis $(e_k(\cdot))_{k\geq 1}$ in $L^2(I)$ of eigen-functions of the Laplace operator $\psi\mapsto -(\gamma(x)\psi')'$, defined on $H^2\cap V$. The eigenvalues $(\lambda_k)_{k\geq 1}$ are all simple and positive. Ordered increasingly, the sequence is positive and grows to infinity.

For any $\psi\in L^2(I)$, we consider Fourier's expansion

$$\psi(x)=\sum_{k\geq 1}\psi_k e_k(x), \qquad \text{in } I.$$

Inserting this series into the boundary value problem (2.1) yields the following expansion

$$(B\psi)(t)=\sum_{k\geq 1}\psi_k e_k(\pi)e^{-\lambda_k t}. \tag{2.3}$$

Remark 2.1. Some results are needed in the sequel; they are linked to the spectral decomposition of the differential operator introduced above. Following [12, Chapter VI], we have the asymptotic formula

$$\frac{\lambda_k}{(k+\beta)^2}=\tau+\frac{\epsilon_k}{k^2}, \qquad \forall k\geq 1.$$

The real-number $\tau>0$ depends on $\gamma(\cdot)$; β is a real-number and $(\epsilon_k)_{k\geq 1}$ is a bounded sequence. Moreover, the uniform bound of the sequence $(e_k(\pi))_{k\geq 1}$ is found in [28]. The proof given there is when the conductivity $\gamma(\cdot)$ is $\mathbb{C}^1$. Actually, in the very case we are involved in, it can be extended to less smooth conductivities; for instance, to those that are piecewise $\mathbb{C}^1$.

The family $(e^{-\lambda_k t})_{k\geq 1}$ is linearly independent and total in $\mathcal{R}(B)$ (see [30]). Owing to Müntz-Száz theory, we derive the following characterization of of the range subspace $\overline{\mathcal{R}(B)}$.

Proposition 2.1. *The kernel of B is reduced to the null-subspace, i.e. $\mathcal{N}(B)=\{0\}$. The co-dimension of $\mathcal{R}(B)$ in $L^2(0,T)$ is infinite. Moreover, all the functions in the closure $\overline{\mathcal{R}(B)}$ are analytic in $]0,T]$ and we have*

$$\overline{\mathcal{R}(B)}=\Big\{v=\sum_{k\geq 1}v_k e^{-\lambda_k t};\quad \sum_{k\geq 1}\sum_{m\geq 1}\frac{1-e^{-(\lambda_k+\lambda_m)T}}{\lambda_k+\lambda_m}v_k v_m<\infty\Big\}. \tag{2.4}$$

Proof. Notice that for any k, we have $e_k(\pi)\neq 0$; otherwise $e_k(\cdot)\equiv 0$. The kernel of B contains therefore only $\psi=0$. Then, given that $(1/\lambda_k)_{k\geq 1}$ is summable ; according to the Müntz theorem, the sequence $(e^{-\lambda_k t})_{k\geq 1}$ is not dense in $L^2(0,T)$. Another theorem by Clarkson and Erdös [11] (see also [10, Theorem 6.4]) brings about more information on the closure space $\overline{\mathcal{R}(B)}$. In fact, any function v in $\overline{\mathcal{R}(B)}$ is analytic in $]0,T]$ and the representation (2.4) holds true ; the bound is a consequence of the fact that $v \in L^2(0,T)$. Moreover, an infinite number of functions e^{-mt} are necessarily outside $\overline{\mathcal{R}(B)}$. Otherwise it would coincide with $L^2(0,T)$. As a result, any finite dimensional subspace spanned by any sub-family, taken among those functions, belongs to a supplementary subspace of $\overline{\mathcal{R}(B)}$. This yields that the co-dimension of $\overline{\mathcal{R}(B)}$ in $L^2(0,T)$ is infinite. The proof is complete. $\square$

Now, to step further on the path of the singular values analysis for the operator B, we need the adjoint operator B^* which maps $L^2(0,T)$ into $L^2(I)$ continuously and compactly. It is easy to check out the formula

$$(B^*h)(x) = \sum_{k\geq 1}\left(\int_{(0,T)} h(t)e^{-\lambda_k t}\,dt\right) e_k(\pi)e_k(x). \tag{2.5}$$

In view of Proposition 2.1, the kernel of B^* is an infinite dimensional subspace. Indeed, its dimension is equal to the co-dimension of $\mathcal{R}(B)$.

2.2 Ill-posedness degree

The purpose is now to clarify the compactness degree of B. Investigating its singular values can be undertaken after deriving a closed form of the B^*B and determine it as an infinite dimensional matrix. We have that

$$\begin{aligned}
(B^*B\psi)(x) &= \sum_{k\geq 1}\left(\int_{(0,T)} (B\psi)(t)e^{-\lambda_k t}\,dt\right) e_k(\pi)e_k(x)\\
&= \sum_{k\geq 1}\left(\int_{(0,T)}\left(\sum_{m\geq 1}\psi_m e_m(\pi)e^{-\lambda_m t}\right)e^{-\lambda_k t}\,dt\right) e_k(x)\\
&= \sum_{k\geq 1}\sum_{m\geq 1}\psi_m\left(\int_{(0,T)} e^{-(\lambda_k+\lambda_m)t}\,dt\right) e_m(\pi)e_k(\pi)e_k(x)\\
&= \sum_{k\geq 1}\sum_{m\geq 1} e_m(\pi)\frac{1-e^{-(\lambda_k+\lambda_m)T}}{\lambda_k+\lambda_m} e_k(\pi)\psi_m e_k(x).
\end{aligned}$$

Then considering the eigenvalue problem: *find* $(\psi,\lambda)\in L^2(I)\times]0,\infty[$ *with* $\psi\neq 0$ *such that*

$$B^*B\psi = \lambda\psi, \tag{2.6}$$

we come up with the (infinite) collection of (infinite dimensional) algebraic equations,

$$\sum_{m\geq 1} e_m(\pi)\frac{1-e^{-(\lambda_k+\lambda_m)T}}{\lambda_k+\lambda_m} e_k(\pi)\psi_m = \lambda\psi_k, \qquad \forall k\geq 1.$$

Let us denote by $\boldsymbol{\psi} = (\psi_m) \in \ell^2(\mathbb{R})$, and let the symbol $\mathcal{C}_T \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ be used for the symmetric matrix whose entries are

$$(c_T)_{km} = e_m(\pi)\frac{1-e^{-(\lambda_k+\lambda_m)T}}{\lambda_k+\lambda_m}e_k(\pi), \qquad \forall k,m\geq 1.$$

These new notations allow for a condensed form of equation (2.6) and to derive formally that

$$\mathcal{C}_T\boldsymbol{\psi} = \lambda\boldsymbol{\psi}. \tag{2.7}$$

The target now is to exhibit asymptotics of the eigenvalues of the infinite-dimensional matrix $\mathcal{C}_T = ((c_T)_{km})_{k,m\geq 1}$. The infinite matrix $\mathcal{C}_T$ should be seen as a linear operator on $\ell^2(\mathbb{R})$, that inherits the properties of the operator B^*B. It is symmetric and positive definite, though not elliptic. As it is compact, the Hilbert-Schmidt theorem applies. This matrix can be then put under a diagonal form. The countable sequence of its eigenvalues is denoted $((\nu_T)_k)_{k\geq 1}$. It is positive and when ordered decreasingly, it decays towards zero. The singular values $((\sigma_T)_k)_{k\geq 1}$ of the operator B are therefore obtained as the square-roots of $((\nu_T)_k)_{k\geq 1}$. Finding out how these sequences decrease toward zero can be achieved after stating that $\mathcal{C}_T$ is spectrally equivalent to the Pick matrix $\mathcal{P}_T$ whose entries are of the form

$$(p_T)_{km} = \frac{\tanh(\lambda_k T/2)+\tanh(\lambda_m T/2)}{\lambda_k+\lambda_m}, \qquad \forall k,m\geq 1.$$

This matrix has a finite Frobenius norm. Thus, it is a bounded operator on $\ell^2(\mathbb{R})$. Next, setting $\mathcal{O}_T$ the diagonal matrix

$$\mathcal{O}_T = \text{diag}\ ((o_T)_{kk})_{k\geq 1} = \frac{1}{\sqrt{2}}\text{diag}\ (e_k(\pi)(1+e^{-\lambda_k T}))_{k\geq 1},$$

it is easily seen that $\mathcal{C}_T = \mathcal{O}_T\mathcal{P}_T\mathcal{O}_T$. The symmetry of $\mathcal{P}_T$ and the non-negative definiteness are the first consequences. The compactness is ensured if $\mathcal{O}_T$ is an isomorphism on $\ell^2(\mathbb{R})$. The diagonal entries $\mathcal{O}_T$ are doubly bounded. The uniform boundedness (away from infinity) of it is discussed in Remark 2.1, see also [28, Theorem 1.2], while the uniform boundedness away from zero is shown in [29, Theorem 2.4]. The positive eigenvalues of $\mathcal{P}_T$ are denoted $((\mu_T)_k)_{k\geq 1}$. We will also need

$$o_m = \inf_{k\geq 1}((o_T)_{kk})^2, \qquad o_M = \sup_{k\geq 1}((o_T)_{kk})^2.$$

The following holds

Lemma 2.2. *The matrices $\mathcal{C}_T$ and $\mathcal{P}_T$ are spectrally equivalent, that is*

$$\frac{(\mu_T)_k}{o_M} \leq (\nu_T)_k \leq \frac{(\mu_T)_k}{o_m}, \qquad \forall k\geq 1.$$

Proof. First setting $\boldsymbol{\varphi}=\mathcal{O}_T\boldsymbol{\psi}$, for all $\boldsymbol{\psi}\in\ell^2(\mathbb{R})$, we get that

$$\frac{1}{o_M}\frac{(\mathcal{P}_T\boldsymbol{\varphi},\boldsymbol{\varphi})_{\ell^2}}{\|\boldsymbol{\varphi}\|^2_{\ell^2}}\le\frac{(\mathcal{C}_T\boldsymbol{\psi},\boldsymbol{\psi})_{\ell^2}}{\|\boldsymbol{\psi}\|^2_{\ell^2}}\le\frac{1}{o_m}\frac{(\mathcal{P}_T\boldsymbol{\varphi},\boldsymbol{\varphi})_{\ell^2}}{\|\boldsymbol{\varphi}\|^2_{\ell^2}}.$$

The result of the lemma is therefore derived by Courant-Fisher Min-max formula. The proof is complete. $\square$

Now, recall that the objective we are assigned to is the derivation of asymptotic formulas for the eigenvalues $((\nu_T)_k)_{k\ge1}$ of $\mathcal{C}_T$. According to Lemma 2.2, it sufficient to exhibit the asymptotics for those $((\mu_T)_k)_{k\ge1}$ of $\mathcal{P}_T$ first and then extend them to $\mathcal{C}_T$. Actually, we focus on $\mathcal{P}_T$. As it has a displacement rank equal to two (see [20]), exhibiting asymptotics of it is possible. In reality, this will be done for the principal sub-matrix of arbitrary finite order N. Then, we use an error approximation result of $\mathcal{P}_T$, we state as a preliminary. To do so we denote by $\mathcal{P}_N=((p_T)_{km})_{1\le k,m\le N}$.

Lemma 2.3. *We have that*

$$\|\mathcal{P}_T-\mathcal{P}_N\|_{\mathcal{L}(\ell^2,\ell^2)}\le\frac{C}{\sqrt{N}}.$$

The constant C is independent upon N.

Proof. The convergence is proven with respect to the stronger Frobenius norm. It is straightforward that

$$\begin{aligned}\|\mathcal{P}_T-\mathcal{P}_N\|^2_{\mathcal{L}(\ell^2,\ell^2)}&=\sum_{1\le m\le N}\sum_{k\ge N}[(p_T)_{km}]^2+\sum_{m>N}\sum_{k\ge1}[(p_T)_{km}]^2\\&\le\sum_{1\le m\le N}\sum_{k>N}\frac{1}{\lambda_k\lambda_m}+\sum_{m>N}\sum_{k\ge1}\frac{1}{\lambda_k\lambda_m}\le\frac{C'}{N}.\end{aligned}$$

The last bound comes from the fact that the function $\tanh(\cdot)$ is lower than one, together with the fact that λ_k grows like $(k+\beta)^2$. The proof is complete. $\square$

Now, let us first introduce the new notation $((\mu_N)_k)_{1\le k\le N}$ for the eigenvalues of the truncated matrix $\mathcal{P}_N$ ordered decreasingly. The following result holds

Lemma 2.4. *The largest eigenvalue of $\mathcal{P}_N$ is uniformly bounded, that is*

$$(\mu_N)_1\le\mu.$$

μ is positive real-number independent upon N.

Proof. The bound on $(\mu_N)_1$ of $\mathcal{P}_N$ may be established owing to the Gershgorin-Hadamard circle theorem (see [27, Chapter 8]),

$$(\mu_N)_1\le\max_{1\le k\le N}\sum_{1\le m\le N}(p_T)_{km}.$$

We have that ($\tanh(\cdot)$ is bounded by one)

$$\sum_{1\le m\le N}(p_T)_{km}\le \sum_{1\le m\le N}\frac{2}{\lambda_k+\lambda_m}\le C\int_{(0,\infty)}\frac{1}{(k+\beta)^2+(z+\beta)^2}\,dz=\frac{C}{|k+\beta|}.$$

The lemma is thus proved after switching to the max on k. □

Lemma 2.5. *Assume that N is large enough. The eigenvalues $((\mu_N)_k)_{1\le k\le N}$ satisfy the following bound*

$$(\mu_N)_k\le \mu\exp\left(-\frac{bk}{\log(N)}\right),\qquad 1\le k\le N.$$

The constant b does not depend upon N.

Proof. Owing to [7, Corollary 4.1] we have that

$$\frac{(\mu_N)_k}{(\mu_N)_1}\le 4\exp\left(-\frac{\pi^2 k}{4\log(4\lambda_N/\lambda_1)}\right).$$

The result is derived after observing that $\lambda_N=\mathcal{O}(N^2)$. The proof is complete. □

Proposition 2.2. *The eigenvalues $((\mu_T)_k)_{1\le k}$ of the infinite Pick matrix $\mathcal{P}_T$ satisfy the following bound*

$$(\mu_T)_k\le \exp(-b'\sqrt{k}),\qquad \forall k\ge 1.$$

Proof. Starting from the expansion

$$\mathcal{P}_T=\mathcal{P}_N+(\mathcal{P}_T-\mathcal{P}_N),$$

and applying the Courant-Fisher Min-max formula we obtain the bound

$$(\mu_T)_k\le (\mu_N)_k+\|\mathcal{P}_T-\mathcal{P}_N\|_{\mathcal{L}(\ell^2,\ell^2)}\le \mu_T\exp\left(-\frac{bk}{\log(N)}\right)+\frac{C}{\sqrt{N}}.$$

Being aware that b and C are not dependent on N is so important. Next, choose $N=e^{\sqrt{k}}$, we obtain that

$$(\mu_T)_k\le \mu e^{-b\sqrt{k}}+Ce^{-\frac{1}{2}\sqrt{k}}.$$

This achieves the proof. □

Proposition 2.3. *The singular values of the operator B are such that*

$$(\sigma_T)_k\le \exp(-b''\sqrt{k}),\qquad \forall k\ge 1.$$

The data completion problem (2.2) *is therefore severely ill-posed.*

Remark 2.2. Proposition 2.3 is stated in [9], for the conductivity $\gamma(\cdot)=1$. The proof proposed there is deeply different. It is based on the Jacobi Theta Functions and the Jacobi Imaginary Transformation. All these tools are lost for varying conductivities. The proof developed here is an alternative one with a higher degree of generality.

3 Boundary data completion

We pursue an exploration of the inverse boundary data completion problem. This consists in recovering the data at one extremity of I from redundant boundary data on the other extremity accessible to measurements. Let $h=h(t)$ be given in $L^2(0,T)$. The following heat equation is the focus

$$\begin{aligned}
&\partial_t y-(\gamma y')'=0, && \text{in } Q,\\
&y(0,t)=h(t), \quad (\gamma y')(0,t)=0, && \forall t\in(0,T),\\
&y(x,0)=0, && \forall x\in I.
\end{aligned} \tag{3.1}$$

The Cauchy conditions are at $x=0$ and the aim is to recover the missing boundary condition at $x=\pi$. Here again the aimed issue has to do with the ill-posedness. We provide a proof of the severe ill-posedness of the problem. A condensation argument is applied to problem (3.1) : the incomplete boundary data $\eta\in L^2(0,T)$ is taken as the very unknown of the problem. Would it be known, the full temperature field could be reconstructed at least in different 'numerical' ways. Therefore we consider the heat problem

$$\begin{aligned}
&\partial_t y_\eta-(\gamma y'_\eta)'=0 && \text{in } Q,\\
&(\gamma y'_\eta)(0,t)=0, \quad (\gamma y_\eta)'(\pi,t)=\eta(t) && \forall t\in(0,T),\\
&y_\eta(x,0)=0 && \forall x\in I.
\end{aligned}$$

The data completion problem reduces then to: *find $\eta\in L^2(0,T)$ fulfilling*

$$y_\eta(0,t)=h(t), \qquad \forall t\in(0,T). \tag{3.2}$$

Before tackling the severe ill-posedness we conduct a brief discussion about the exact data $h(\cdot)$, those for which a of the inverse problem is secured.

3.1 Identifiability

The unique continuation theorem allows us to state the identifiability. The proof comes from, see, e.g., [21].

Lemma 3.1. *The reduced data completion problem* (3.2) *has at most one solution.*

The sequel is focussed on the study of the linear operator defining problem (3.2). It is defined on $L^2(0,T)$ and is bounded

$$D:\ \eta\mapsto y_\eta(0,\cdot).$$

An interesting issue is related to the space of the exact data $h\in L^2(0,T)$ for (3.2), the range $\mathcal{R}(D)$.

Lemma 3.2. *The range $\mathcal{R}(D)$ of exact data is dense in $L^2(0,T)$.*

Proof. It is enough to show that the kernel of the adjoint operator D^* is reduced to $\{0\}$. It can be stated that D^* is defined by

$$(D^*h)(t) = -z_h(\pi,t), \qquad \forall t \in (0,T).$$

The function $z_h(\cdot,\cdot)$ is the unique solution of the backward heat equation

$$\begin{aligned}
&-\partial_t z_h - (\gamma z_h')' = 0 && \text{in } Q,\\
&(\gamma z_h')(0,t) = h(t), \quad (\gamma z_h')(\pi,t) = 0 && \forall t \in (0,T),\\
&z_h(x,T) = 0 && \forall x \in I.
\end{aligned}$$

Calling for the unique continuation theorem results in the fact that $\mathcal{N}(D^*)$ contains only zero. The range $\mathcal{R}(D)$ is then dense in $L^2(0,T)$. □

3.2 Ill-posedness degree

The compactness of the operator D is not hard to check out using a suitable compact Sobolev embedding. Thus, Hilbert-Schmidt's theorem, applied D, yields that the set of its singular values is discrete with the origin as the only possible accumulative point. Stating that the sequence of the singular values decreases rapidly towards zero passes by writing the operator D as a convolution operator with a kernel flat at the origin.

To step forth, we use the orthonormal basis $(e_k(\cdot))_{k\geq 0}$ in $L^2(I)$ of eigen-functions of the Laplace operator operator $-(\gamma(\cdot)')'$ defined in Section 2. Carrying out the necessary calculation to come up with y_η (*spanned on this particular basis*) provides the following form of the operator D,

$$(D\eta)(t) = (\eta \star g)(t) = \int_{(0,t)} \eta(s) g(t-s)\, ds$$

with

$$g(t) = \sum_{k\geq 0} e_k(0) e_k(\pi) e^{-\lambda_k t} = \frac{1}{\pi} + \sum_{k\geq 1} e_k(0) e_k(\pi) e^{-\lambda_k t}, \qquad \forall t \geq 0. \tag{3.3}$$

As a result equation (3.2) is a Volterra equation of the first kind with the kernel $g(\cdot)$. Appropriate results picked up in the specialized literature link the ill-posedness of this equation to the flatness of the kernel $g(\cdot)$ in the vicinity of the origin $t=0$ (see [13, 24, 35]). We use an indirect approach to derive the result. First, let us notice that applying d'Alembert rule, the series (3.3) is normally convergent in $[\epsilon,\infty[$, for ϵ arbitrary small. This is a consequence of $\lambda_k \approx \tau(k+\beta)^2$ together with the bound $\|e_k\|_\infty \leq C$. Therefore, $g(\cdot)$ is indefinitely smooth in $]0,\infty[$ and bounded at infinity. Its behavior at vicinity of $t=0^+$ is given in the following proposition. The proof is postponed to the appendix and is based on the Laplace transform.

Proposition 3.1. *There holds that*

$$\lim_{t\to 0^+} g^{(m)}(t) = 0.$$

The singular values $(\sigma_k)_{k\geq 0}$ *of the operator* D *decay faster than* k^{-m} *for all* $m\in\mathbb{N}$. *The ill-posedness degree of problem* (3.2) *is then infinity.*

Proof. It is direct from (5.1) and the second formula in Lemma 5.2. □

Remark 3.1. The series (3.3) converges for $t=0$. Be aware that in view of [29, Theorem 2.4], the sequence $(e_k(0)e_k(\pi))_{k\geq 1}$ fails to decrease toward zero, it however enjoys some alterning sign property which explains the convergence of the series. See Section 4 when γ is constant to get an insight.

4 Numerical discussion

To assess the severe ill-posedness of the Initial data recovery from Cauchy's data, studied in Section 2, we provide some numerical illustrations. Therefore, we run some MATLAB computations to check out the estimates provided in Proposition 2.3. The singular values $((\sigma_N)_k)_{1\leq k\leq N}$ of the operator B are calculated as $(\sqrt{(\nu_N)_k})_{1\leq k\leq N}$, the eigenvalues of $\mathcal{C}_N$, the N-dimensional truncation of $\mathcal{C}_T$. They are represented in Fig. 1, for a range of $N\in[5,25]$. An important fraction of the singular values are concentrated at the origin for high values of N. The smallest ones goe towards zero with at a high speed. Observe that the singular values lower than 10^{-8} suffer from inaccuracy. In fact, MATLAB fails to provide them and sends back incorrect values. Another feature to point out consists in the fact that if the final instant T is short, then only few singular values are not numerically zero (larger than 10^{-8}).

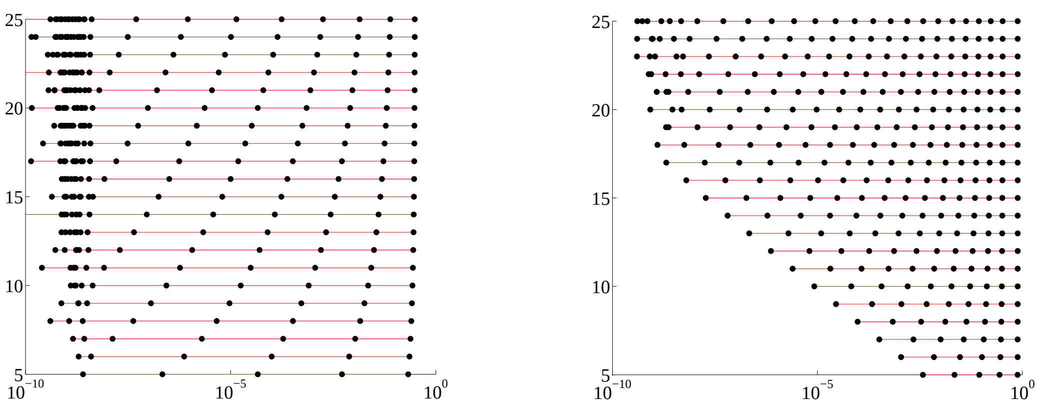

Figure 1: The singular values for $T=0.01$ (left), $T=1$ (right).

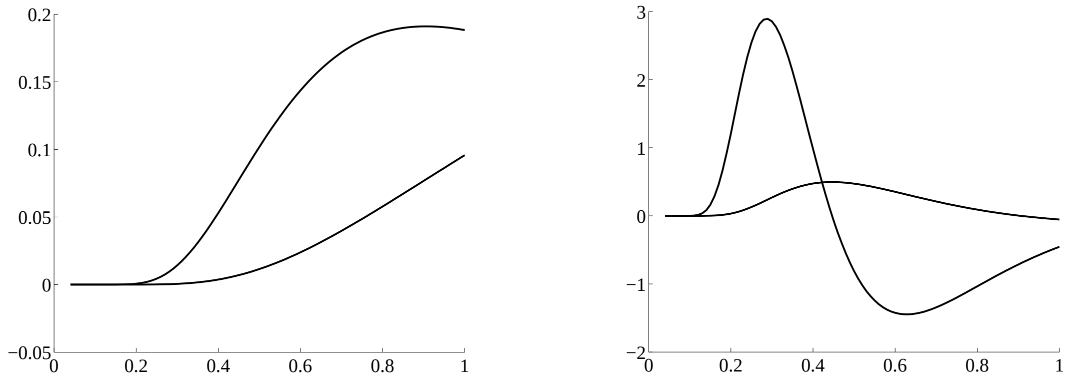

Figure 2: The representative curves of the kernel and its first derivatives, g,g' (left) and g'',g''' (right). They are all flat at the vicinity of zero.

The second illustration deals with the data completion problem adressed in Section 3. We show the flat behavior for the kernal g, at the heart of the analysis of the ill-posedness degree of our inverse problem. For constant conductivities, let us say when $\gamma(\cdot)=1$, it is expressed as follows

$$g(t)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{k\geq 1}(-1)^k e^{-k^2 t}.$$

The behavior of g is depicted in Fig. 2. The representative curves of g, g', g'' and g'''. The kernel is hence strongly flat and problem (3.2) is severely ill-posed.

In this case of constant $\gamma(\cdot)$, Jacobi's theta functions actually provides an elegant insight on how g and its derivatives varies at the vicinity of the origin. The kernel takes the following forms

$$g(t)=\frac{1}{\pi}+\frac{2}{\pi}\sum_{k\geq 1}(-1)^k e^{-k^2 t}=\frac{1}{\sqrt{\pi t}}e^{-\frac{\pi^2}{4t}}\left(1+2\sum_{k\geq 1}e^{-\frac{k^2\pi^2}{t}}\cosh(\frac{k\pi^2}{t})\right).$$

As a result, we have that for all $m\geq 0$

$$\lim_{t\to 0^+} g^{(m)}(t)=0,$$

in agreement with the curves depicted in Fig. 2.

5 Conclusion

The focus of the current work is the ill-posedness degree of the inverse problems (2.2), of the recovery of the initial state and (3.2), of the missing boundary data reconstruction from one sided Cauchy boundary data. The difficulties to compute a good quality solution of these problems is well known. This is a reliable indication of their severe ill-posedness. The novelty here is mainly echnical: the generality of the heat equation we work with; the conductivity is space dependent. The case where this parameter is constant has been already successfully studied (see [8,9]). Many issues related with both problems are worth to be deeply investigated, especially those related to a safe numerical approximation. Some have doubtless been treated in the specialized literature (see, eg, [2,3]) and some others remain to be studied. We think particularly about the regularization strategies. What are the most efficient method to use and the best rule to fix the related parameters?

Appendix: Technical Lemmas

We are inerested in the convolution kernel $g(\cdot)$, governing the Volterra equation exhibited from the Boundary data completion adressed in Section 3. We focus on its behavior at the origin $t=0$. We follow an indirect argument, based on the Laplace transform. In fact, the behavior of $\hat{g}(\cdot)$ at infinity reflects that of $g(\cdot)$ at the origin. Following the first Heaviside's rule, we have in particular that

$$\left(\lim_{p\to+\infty} p^{m+1}\hat{g}(p)=0\right) \Longrightarrow \left(\lim_{t\to 0^+} g^{(k)}(t)=0, \quad \forall k\le m\right). \tag{5.1}$$

When η belongs to $L^2(0,\infty)$, a-priori estimates show that at least $y_\eta\in L^2((0,\infty)\times I)$. The Laplace transform $\hat{y}_\eta(\cdot,x)$, with respect to t, of $y_\eta(\cdot,x)$ is well defined, for alemmaost every x. Transforming problem (3.1), we derive that $\hat{y}_\eta(\cdot,p)$ is solution of the elliptic problem, for all $p\ge 0$,

$$\begin{aligned} &p\hat{y}_\eta(\cdot,p)-(\gamma\hat{y}_\eta')'(\cdot,p)=0, \qquad \text{in } I,\\ &(\gamma\hat{y}_\eta')(0,p)=0, \quad (\gamma\hat{y}_\eta')(\pi,p)=\hat{\eta}(p). \end{aligned} \tag{5.2}$$

The convolution theorem of the Laplace transform provides

$$\widehat{D\eta}(p)=\hat{y}_\eta(0,p)=\hat{g}(p)\hat{\eta}(p).$$

It comes out that $\hat{g}(p)=z(0,p)$ where z is solution of problem (5.2), where $\hat{\eta}$ is changed into 1, that is

$$\begin{aligned} &pz(\cdot,p)-(\gamma z')'(\cdot,p)=0, \qquad \text{in } I,\\ &(\gamma z')(0,p)=0, \quad (\gamma z')(\pi,p)=1. \end{aligned} \tag{5.3}$$

Owing to the maximum principle, we have that $z(\cdot,p)\ge 0$. The rod is heated by injecting a heat flux at the right extremity $x=\pi$; then the temperature remains above zero. This is already an indication that $\hat{g}(p)\ge 0$. The aim to recall is to investigate the behavior of $\hat{g}(p)$

for large p. We will use a comparison argument. Recall that $\gamma_\infty = \|\gamma\|_{L^\infty}$, we consider the problem

$$\begin{aligned} &p\zeta(\cdot,p) - \gamma_\infty \zeta''(\cdot,p) = 0 \qquad \text{in } I, \\ &(\gamma_\infty \zeta')(0,p) = 0, \quad (\gamma_\infty \zeta')(\pi,p) = 1. \end{aligned} \tag{5.4}$$

The preliminary comparison result is given in

Lemma 5.1. *There holds that*

$$0 \le \gamma(x) z'(p,x) \le \gamma_\infty \zeta'(p,x), \qquad \forall (x,p) \in I \times (0,\infty[.$$

Proof. Set $w = \gamma z'(p,\cdot) \in H^1(I)$. It is easily seen that

$$\begin{aligned} &\frac{p}{\gamma} w(\cdot,p) - w''(\cdot,p) = 0 \qquad \text{in } I, \\ &w(0,p) = 0, \quad w(\pi,p) = 1. \end{aligned}$$

Analogously, if $\omega = \gamma_\infty \zeta'(p,\cdot) \in H^1(I)$, we obtain the new boundary value equation

$$\begin{aligned} &\frac{p}{\gamma_\infty} \omega(\cdot,p) - \omega''(\cdot,p) = 0 \qquad \text{in } I, \\ &\omega(0,p) = 0, \quad \omega(\pi,p) = 1. \end{aligned}$$

The maximum principle applied to both equations results in $w(\cdot,p) \ge 0$ and $\omega(\cdot,p) \ge 0$. Now considering the difference function $\epsilon = (\omega - w)$; it is solution to problem

$$\begin{aligned} &\frac{p}{\gamma_\infty} \epsilon(\cdot,p) - \epsilon''(\cdot,p) = p\Big(\frac{1}{\gamma} - \frac{1}{\gamma_\infty}\Big) w(\cdot,p) \qquad \text{in } I, \\ &\epsilon(0,p) = 0, \quad \epsilon(\pi,p) = 0. \end{aligned}$$

The right hand side in the equation is non-negative. Another use of the maximum principle yields that $\epsilon(\cdot,p) \ge 0$ and $0 \le w(\cdot,p) \le \omega(\cdot,p)$. The proof is complete. □

Lemma 5.2. *We have for all $m \ge 0$,*

$$\lim_{p \to +\infty} p^m \hat{g}(p) = 0.$$

Proof. Integrate equations (5.3) and (5.4), we find that for all $x \in I$

$$p \int_0^x z(u,p)\, du = \gamma z'(p,x), \qquad p \int_0^x \zeta(u,p)\, du = \gamma_\infty \zeta'(p,x).$$

Using Lemma 5.1, we come out with the bound

$$0 \le \int_0^x z(u,p)\, du \le \int_0^x \zeta(u,p)\, du,$$

valid for all $x \in I$. Dividing by x, using the Mean Value Theorem and passing to the limit $x{=}0$, yields that $\hat{g}(p){=}z(0,p){\le}\zeta(0,p)$. Noticing that $\zeta(\cdot,p)$ has a closed form by explicitly solving (5.4), this provides the desired result. The proof is complete. □

Remark 5.1. The bound of $\hat{g}(\cdot)$ given in Lemma 5.2 cannot be improved. Indeed, denoting $\gamma_m = \min_{x\in I}\gamma(x)$. Following the same lines, one derives that

$$\hat{g}(p) \geq \frac{1}{\gamma_m\sqrt{q}\sinh(\pi\sqrt{q})},$$

where $p = q\gamma_m$.

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