Zeros of Primitive Characters

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Abstract. Let G be a finite group. An irreducible character χ of G is said to be primitive if $\chi \neq \vartheta^G$ for any character ϑ of a proper subgroup of G. In this paper, we consider about the zeros of primitive characters. Denote by $\mathrm{Irr}_{\mathrm{pri}}(G)$ the set of all irreducible primitive characters of G. We proved that if $g \in G$ and the order of gG' in the factor group G/G' does not divide $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, then there exists $\varphi \in \mathrm{Irr}_{\mathrm{pri}}(G)$ such that $\varphi(g) = 0$.

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1 Introduction

Let G be a finite group and $\mathrm{Irr}(G)$ be the set of all irreducible characters of G. For an element g of G, g is called a vanishing element if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. In [3], W. Burnside proved that for any nonlinear irreducible character χ, there always exists $g \in G$ such that $\chi(g) = 0$, which means that there exists at least a vanishing element for any nonlinear irreducible character χ. It is interesting to investigate when an element of a finite group can be a vanishing element. In [1], G. Chen obtained a sufficient condition to determine when an element is a vanishing element. More precisely, suppose that $g \in G - G'$ and the order of gG' in the factor group G/G' is coprime to $|\mathrm{Irr}(G)|$, then there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. In [4], H. Wang, X. Chen and J. Zeng showed a similar sufficient condition about the Brauer characters. In [2], X. Chen and G. Chen investigated the monomial Brauer characters. An irreducible character χ of G is said to be primitive if $\chi \neq \vartheta^G$ for any character ϑ of a proper subgroup of G. In this paper, we consider about the zeros of primitive characters.

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2   Main results and proofs

First of all, we give the proof of the following formula about induced characters for the readers' convenience.

Lemma 2.1. *[3, Problem 5.3] Let $H \subseteq G$ and suppose φ is a class function of H and ψ is a class function of G. Then $(\varphi \psi_H)^G = \varphi^G \psi$.*

Proof. For any $g \in G$, we have $(\varphi \psi_H)^G(g) = \frac{1}{|H|} \sum\limits_{x \in G} (\varphi \psi_H)^o(xgx^{-1})$.

If $xgx^{-1} \in H$, then

$$(\varphi \psi_H)^o(xgx^{-1}) = (\varphi \psi_H)(xgx^{-1}) = \varphi(xgx^{-1})\psi_H(xgx^{-1}) = \varphi^o(xgx^{-1})\psi(g).$$

If $xgx^{-1} \notin H$, then

$$(\varphi \psi_H)^o(xgx^{-1}) = 0 = \varphi^o(xgx^{-1})\psi(g).$$

It follows that

$$(\varphi \psi_H)^G(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^o(xgx^{-1})\psi(g) = [\frac{1}{|H|} \sum_{x \in G} \varphi^o(xgx^{-1})]\psi(g) = (\varphi^G \psi)(g)$$

and we are done.   □

Denote by $\mathrm{Lin}(G)$ the set of all linear irreducible characters of G. Now we consider the product of an irreducible primitive character and a linear irreducible character.

Lemma 2.2. *Let $\chi \in \mathrm{Irr}(G)$ be a primitive character of G and λ be a linear irreducible character of G. Then $\lambda \chi$ is an irreducible primitive character.*

Proof. Since $\chi \in \mathrm{Irr}(G)$ and $\lambda(1) = 1$, it follows that $\lambda \chi$ is an irreducible character. By way of contradiction, suppose that $\lambda \chi$ is not a primitive character, then there exists a character φ of a proper subgroup H of G such that $\lambda \chi = \varphi^G$. It follows that $\chi = \overline{\lambda} \varphi^G$. Then we have

$$\chi = \overline{\lambda} \varphi^G = (\overline{\lambda}_H \varphi)^G$$

by Lemma 2.1. Since φ is a character of H, we may assume that $\varphi = a_1 \psi_1 + a_2 \psi_2 + \cdots + a_t \psi_t$, where $a_i \in \mathbb{Z}^+$ and $\psi_i \in \mathrm{Irr(H)}$. Since $\overline{\lambda}_H(1) = 1$ and $\psi_i \in \mathrm{Irr(H)}$, we have $\overline{\lambda}_H \psi_i \in \mathrm{Irr(H)}$ and

$$\overline{\lambda}_H \varphi = \overline{\lambda}_H \left(\sum_{i=1}^{t} a_i \psi_i \right) = \sum_{i=1}^{t} a_i (\overline{\lambda}_H \psi_i)$$

with $a_i \in \mathbb{Z}^+$, which implies that $\overline{\lambda}_H \varphi$ is a character of H. It follows that $\chi = \overline{\lambda} \varphi^G = (\overline{\lambda}_H \varphi)^G$ is induced by a character $\overline{\lambda}_H \varphi$ of a proper subgroup H of G, a contradiction.   □

Denote the set of all irreducible primitive characters of G by $\mathrm{Irr}_{\mathrm{pri}}(G)$. We shall use the action of linear irreducible characters on the set of irreducible primitive characters to get some interesting results. The proof of the following Theorem 2.1 is an adaptation of the proof of Theorem 2.1 in [1].

Theorem 2.1. *Let G be a finite group, G' the derived subgroup of G. Suppose that $g \in G$ and the order of gG' in the factor group G/G' does not divide $|\mathrm{Irr}_{\mathrm{pri}}(G)|$. Then there exists $\varphi \in \mathrm{Irr}_{\mathrm{pri}}(G)$ such that $\varphi(g) = 0$.*

Proof. Suppose that $\lambda \in \mathrm{Lin}(G)$ and $\mathrm{Irr}_{\mathrm{pri}}(G) = \{\varphi_1, \varphi_2, \cdots, \varphi_l\}$. By Lemma 2.2, we have $\mathrm{Irr}_{\mathrm{pri}}(G) = \{\lambda\varphi_1, \lambda\varphi_2, \cdots, \lambda\varphi_l\}$. Assume that $g \in G$ and $o(gG')$ does not divide $|\mathrm{Irr}_{\mathrm{pri}}(G)| = l$. Working for a contradiction, suppose that there is no irreducible primitive character vanishing on g. It follows that $\varphi_i(g) \neq 0$ for all i and $(\varphi_1\varphi_2\cdots\varphi_l)(g) \neq 0$. Since

$$\begin{aligned}(\varphi_1\varphi_2\cdots\varphi_l)(g) &= \varphi_1(g)\varphi_2(g)\cdots\varphi_l(g) \\ &= (\lambda\varphi_1)(g)(\lambda\varphi_2)(g)\cdots(\lambda\varphi_l)(g) \\ &= \lambda(g)^l(\varphi_1\varphi_2\cdots\varphi_l)(g),\end{aligned}$$

which implies that $\lambda(g)^l = 1$. Since $\lambda(g^l) = \lambda(g)^l$, we have $g^l \in \ker\lambda$ for any $\lambda \in \mathrm{Lin}(G)$. Hence, $g^l \in \bigcap\ker\lambda = G'$, where λ runs over $\mathrm{Lin}(G)$. Since $(gG')^l = g^lG' = G'$, we have $o(gG')$ divides $|\mathrm{Irr}_{\mathrm{pri}}(G)| = l$, a contradiction. □

As analogs of results in [1], we can obtain the following results.

Corollary 2.1. *Let G be a finite nonabelian group. Suppose that $g \in G - G'$ and the order of gG' in the factor group G/G' is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, then there exists $\chi \in \mathrm{Irr}_{\mathrm{pri}}(G)$ such that $\chi(g) = 0$.*

Proof. If $o(gG') = 1$, then $g \in G'$, a contradiction. Thus $o(gG') \neq 1$. Suppose that $o(gG')$ is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, it follows that $o(gG')$ does not divide $|\mathrm{Irr}_{\mathrm{pri}}(G)|$. By Theorem 2.1, there exists $\chi \in \mathrm{Irr}_{\mathrm{pri}}(G)$ such that $\chi(g) = 0$. □

Lemma 2.3. *Let G be a nonabelian finite group, $Z(G)$ be the center of G. If $|Z(G)/(Z(G)\bigcap G')|$ is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, then $Z(G) \leq G'$.*

Proof. If $G' = G$, the conclusion holds. So we assume that $G' < G$.

By way of contradiction, suppose there exists $g \in Z(G) - G'$, which implies that $o(gG') \neq 1$. Since $o(gG')$ divides $|Z(G)G'/G'| = |Z(G)/(Z(G)\bigcap G')|$ which is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, it follows that $o(gG')$ does not divide $|\mathrm{Irr}_{\mathrm{pri}}(G)|$. By Theorem 2.1, there exists $\chi \in \mathrm{Irr}_{\mathrm{pri}}(G)$ such that $\chi(g) = 0$. Notice that $g \in Z(G)$, which is a contradiction, and we are done. □

Corollary 2.2. *Let G be a finite group and $|G/G'|$ is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$. Then $Z(G) \leq G'$.*

Proof. If G is abelian, we have $|\mathrm{Irr}_{\mathrm{pri}}(G)| = |\mathrm{Lin}(G)| = |G/G'|$. Since $|G/G'|$ is coprime to $|\mathrm{Irr}_{\mathrm{pri}}(G)|$, we have $|\mathrm{Lin}(G)| = |\mathrm{Irr}(G)| = 1$ and $|G| = 1$, the conclusion holds. So we may assume that G is nonabelian.

Since $Z(G)G'/G'$ is a subgroup of G/G' and $(|G/G'|, |\mathrm{Irr}_{\mathrm{pri}}(G)|) = 1$, we have

$$(|Z(G)G'/G'|, |\mathrm{Irr}_{\mathrm{pri}}(G)|) = 1.$$

By Lemma 2.3, we have $Z(G) \leq G'$. □

Theorem 2.2. *Let G be a finite group. Denote by Ψ the sum of all irreducible primitive characters of G. Then $\Psi(x) = 0$ for any $x \in G - G'$.*

Proof. Suppose $\mathrm{Irr}_{\mathrm{pri}}(G) = \{\varphi_1, \varphi_2, \cdots, \varphi_l\}$, then $\Psi = \varphi_1 + \varphi_2 + \cdots + \varphi_l$.

Working for a contradiction, suppose that there exists an element $x \in G - G'$ such that $\Psi(x) \neq 0$. Let $\lambda \in \mathrm{Lin}(G)$, by Lemma 2.2, it follows that

$$\lambda \Psi = \lambda(\varphi_1 + \varphi_2 + \cdots + \varphi_l) = \lambda\varphi_1 + \lambda\varphi_2 + \cdots + \lambda\varphi_l$$
$$= \varphi_1 + \varphi_2 + \cdots + \varphi_l = \Psi.$$

So we have $(\lambda\Psi)(x) = \Psi(x)$ and $\lambda(x) = 1$, which implies that $x \in \ker \lambda$ for any $\lambda \in \mathrm{Lin}(G)$. It follows that $x \in G'$, a contradiction. $\qquad\square$

Acknowledgments

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