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A Kind of Integral Representation on Complex Manifold

Teqing Chen[1,*] and Zhiwei Li[2]

[1] *School of Information Management, Minnan University of Science and Technology, Shishi 362700, China;*
[2] *School of Mathematics and Computer Science, Quanzhou Normal University, Quanzhou 362000, China.*

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Abstract. In this paper, by using the Hermitian metric and Chern connection, we study the case of a strictly pseudoconvex domain G with non-smooth boundaries in a complex manifold. By constructing a new integral kernel, we obtain a new Koppelman–Leray–Norguet formula of type (p,q) on G, and get the continuous solutions of $\bar{\partial}$–equations on G under a suitable condition. The new formula doesn't involve integrals on the boundary, thus one can avoid complex estimations of the boundary integrals, and the density of integral may be not defined on the boundary but only in the domain. As some applications, we discuss the Koppelman–Leray–Norguet formula of type (p,q) for general strictly pseudoconvex polyhedrons (unnecessarily non-degenerate) on Stein manifolds, also get the continuous solutions of $\bar{\partial}$–equations under a suitable condition.

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Key words: Complex manifold, strictly pseudoconvex domain, non-smooth boundary, Koppelman–Leray–Norguet formula, $\bar{\partial}$–equation

1 Introduction

As early as 1831, Cauchy found the famous Cauchy integral formula, which named by his name. Many mathematicians were aware of the importance of integral representation in complex analysis. Later, the integral representation method gradually became one of the main methods of complex analysis in several variables, because one of its main virtues is that it is easy to estimate like the Cauchy integral formula in one complex variable. It is well known that the integral representations and their applications for $(0,q)$ differential form in $\mathbf{C}^n$ have been deeply studied(see, for instance [1–10]). But the research for

*Corresponding author. *Email addresses:* 94453595@qq.com (Chen T), wei2785801@qztc.edu.cn (Li Z)

integral representations on complex manifolds began in 1980s. Most of the results, so far, have been concerned with Stein manifolds(see, for instance [4,5,11-13]). In the early 1990s, Berndtsson B [14] studied the theory of integral representations on general complex manifolds and gained in a quite general integral kernel under a suitable condition. Using this kernel, the Koppelman formula on complex manifolds was obtained. Based on it, Zhong T [15] got the Koppelman–Leray–Norguet formula of type (p,q) on a bounded domain G with piecewise C^1 smooth boundaries in a complex manifold, and gave the continuous solutions of $\bar{\partial}$–equations on G under a suitable condition.

In this paper, by using the Hermitian metric and Chern connection, the case of a strictly pseudoconvex domain G with non-smooth boundaries in a complex manifold is studied. By constructing a new integral kernel, a new Koppelman–Leray–Norguet formula of type (p,q) on G is obtained, and the continuous solution of $\bar{\partial}$–equations on G under a suitable condition is given. The new formula doesn't involve integrals on the boundary, thus complex estimations of the boundary integrals can be avoided, and the density of integral may be not defined on the boundary but only in the domain. As an application, $\mathbf{C}^n$ and Stein manifold are taken as examples to discuss the relationship between the conclusion of this paper and the corresponding conclusion in [4]. The Koppelman–Leray–Norguet formula of type (p,q) for general strictly pseudoconvex polyhedrons with unnecessarily non-degenerate on Stein manifolds is also discussed, and the continuous solution of $\bar{\partial}$–equations under a suitable condition is given, it implies the corresponding result in paper [13].

2 Basic knowledge and lemma

Let M be a complex manifold, $X=M\times M$, E is supposed to be a holomorphic vector bundle of rank n over X, and η is a holomorphic section to E such that

$$\{\eta=0\}=Y=\{(\zeta,z)\in X|\zeta=z\}.$$

Let ξ be any smooth section to E^* that is a dual bundle of E, which is admissible for η, i.e. For any compact set $B\subseteq X$, we have

$$|\xi|\le c_B|\eta| \text{ and } |\langle\xi,\eta\rangle|\ge C_B|\eta|^2,$$

where c_B and C_B are constants only concerned with B, e.g. ξ is dual vector of η on some measurement, then ξ is admissible for η. We consider extensional equation

$$dK=[Y]-C_n[\Theta], \tag{2.1}$$

where Θ is a formal curvature of some connections for E, $C_n[\Theta]$ is the n^{th} Chern form of Θ. Berndtsson B obtained explicit solution of (2.1)(name it Berndtsson's kernel)

$$K[\xi,\eta](z,\zeta)\Lambda=\frac{\xi\wedge D\eta}{n!(2\pi i)^n}\sum_{k=0}^{n-1}\binom{n}{k}(-1)^k\frac{(D^*\xi\wedge D\eta)^{n-k-1}}{\langle\xi,\eta\rangle^{n-k}}\wedge\widetilde{\Theta}^k, \tag{2.2}$$

where $\Lambda = e_1^* \wedge e_1 \wedge \cdots \wedge e_n^* \wedge e_n$, $e_1, \cdots, e_n$ is a local frame for E, $e_1^*, \cdots, e_n^*$ is a dual frame for E^*,

$$\left(\frac{i}{2\pi}\right)^k \frac{1}{n!} \widetilde{\Theta}^k = C_k[\Theta] e_1^* \wedge e_1 \wedge \cdots \wedge e_k^* \wedge e_k.$$

We denote

$$\begin{aligned} P(N) &:= \{K = (k_1, \cdots, k_l) \in \mathbf{N}^l : 1 \leq k_1, \cdots, k_l \leq N\}, \\ P'(N) &:= \{K = (k_1, \cdots, k_l) \in P(N) : 1 \leq k_1 < \cdots < k_l \leq N\}. \end{aligned}$$

Definition 2.1. *Let M be a complex manifold of complex dimension n. An open set $G \subset\subset M$ is said to be possess piecewise C^1 smooth boundaries, whenever there are finite open covering $\{U_k\}_{k=1}^N$ of neighborhood U of ∂G, and C^1 functions $\rho_k : U_k \to \mathbf{R}, (1 \leq k \leq N)$, which matches the following conditions:*
(i) $G \cap U = \{x \in U \,|\, \text{for all } 1 \leq k \leq N, \text{we have } x \notin \mathrm{U_k}, \text{ or } \rho_k(x) < 0\}$;
(ii) For all $K = (k_1, \cdots, k_l) \in P'(N)$, we have

$$\mathrm{d}\rho_{k_1}(z) \wedge \cdots \wedge \mathrm{d}\rho_{k_l}(z) \neq 0, \quad \forall z \in U_{k_1} \cap \cdots \cap U_{k_l}.$$

We call $\{U_k, \rho_k\}_{k=1}^N$ is a frame for G.

Let $S_i = \{z \in \partial G \cap U_i | \rho_i(z) = 0\}, (i = 1, \cdots, N)$, for $K = (k_1, \cdots, k_l) \in P(N)$, define

$$S_K = \begin{cases} S_{k_1} \cap \cdots \cap S_{k_l}, & \text{if integers } k_1, \cdots, k_l \text{ are different in pairs;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

Choose the orientation on S_K such that

$$\partial G = \sum_{k=1}^N S_k, \quad \partial S_K = \sum_{j=1}^N S_{Kj},$$

where $Kj := (k_1, \cdots, k_l, j)$, the orientation of ∂G and ∂S_K is induced by the orientation of G and S_K, respectively. Then the orientation on S_K is skew symmetric in the component of K. Let

$$\triangle = \left\{ \lambda = (\lambda_0, \cdots, \lambda_N) \in \mathbf{R}^{N+1} \middle| \lambda_j \geq 0, \sum_{j=0}^N \lambda_j = 1 \right\},$$

be a standard N-simplex in $\mathbf{R}^{N+1}$.For every ordered subset $J = \{j_1, \cdots, j_m\}$ of $\{0, 1, \cdots, N\}$, define

$$\triangle_J = \{\lambda \in \triangle | \sum_{j \in J} \lambda_j = 1\},$$

and the orientation of $\triangle_J$ is chosen such that

$$\partial\triangle_J=\sum_{r=1}^{m}(-1)^{r-1}\triangle_{j_1,\cdots,\widehat{j_r},\cdots,j_m},$$

where $\widehat{j_r}$ means that j_r is omitted. With this orientation, we have

$$\partial(S_K\times\triangle_J)=\partial S_K\times\triangle_J+(-1)^{|K|}S_K\times\partial\triangle_J,$$

where $|K|$ is the length of the index set K.

We set $N(\rho_k):=\{z\in\overline{U}_k|\rho_k(z)=0\}$ and suppose that $N(\rho_k)\subset\subset U_k,(k=1,\cdots,N)$, (where it's not necessary to assume $d\rho_k(\zeta)\neq 0$, $(k=1,\cdots,N)$ when $\zeta\in\partial G$), so we can use $\theta_k\subset\subset U_k$ to denote neighbour of $N(\rho_k)$.

Similar to Theorem 4.8.3 and Lemma 4.8.2 in paper [4], after shrinking θ_k, we can find numbers ε, $\alpha>0$, as well as C^1 functions $\Phi_k(z,\zeta)$ and $\widetilde{\Phi}_k(z,\zeta)$, defined for $z\in G\cup\theta_k,\zeta\in\theta_k$, such that the following conditions are fulfilled:

(i) $\Phi_k(z,\zeta)$, $\widetilde{\Phi}_k(z,\zeta)$ are holmorphic in $z\in G\cup\theta_k$, and C^1 continuous in $\zeta\in\theta_k$.
(ii) For $z\in G\cup\theta_k$, $\zeta\in\theta_k$, with $\operatorname{dist}(z,\zeta)\geq\varepsilon$, we have

$$\Phi_k(z,\zeta)\neq 0\,,\quad \widetilde{\Phi}_k(z,\zeta)\neq 0. \tag{2.3}$$

For $z\in G\cup\theta_k,\zeta\in\theta_k$ with $\operatorname{dist}(z,\zeta)\leq\varepsilon$, we have

$$|\Phi_k(z,\zeta)|\geq\alpha(\rho_k(\zeta)-\rho_k(z)+[\operatorname{dist}(z,\zeta)]^2), \tag{2.4}$$

$$|\widetilde{\Phi}_k(z,\zeta)|\geq\alpha(-\rho_k(\zeta)-\rho_k(z)+[\operatorname{dist}(z,\zeta)]^2). \tag{2.5}$$

For every $z\in\theta_k$, we have

$$\Phi_k(z,z)=0. \tag{2.6}$$

(iii) For $z\in G\cup\theta_k$, $\zeta\in N(\rho_k)$, we have

$$\widetilde{\Phi}_k(z,\zeta)=\Phi_k(z,\zeta). \tag{2.7}$$

Similar to Corollary 4.9.4 in paper [4], after shrinking θ_k, we can find $T^*(M)$-valued C^1 maps $\xi_k(z,\zeta)$ such that the following conditions are fulfilled:

(iv) $\xi_k(z,\zeta)\in T_z^*(M)$, $z\in G\cup\theta_k$, $\zeta\in\theta_k$, $(k=1,\cdots,N)$.
(v) $\xi_k(z,\zeta)$ are holmorphic in $z\in G\cup\theta_k$, $(k=1,\cdots,N)$.
(vi)

$$\Phi_k(z,\zeta)=\langle\xi_k(z,\zeta),\,\eta(z,\zeta)\rangle,\; z\in D\cup\theta_k,\;\zeta\in\theta_k,(k=1,\cdots,N). \tag{2.8}$$

Let F be a C^∞ metric over E, which induces a antilinear map $\mu:E\to E^*$, $\eta\mapsto\langle\cdot,\eta\rangle_F$, let D be a connection of E about F, D^* is a connection of E^* about F^*, where F^* is induced by the metric F over E^*. Suppose $\widehat{\eta}$ is a C^∞ section: $M\times M\to E^*(M\times M)$, which is defined by $\widehat{\eta}=\mu\circ\eta$, such that: $\forall\eta\in E(M\times M)$,we have $\langle\mu\eta,\eta\rangle_F\geq 0$, and the map $\|\eta(z,\zeta)\|_\mu:=(\langle\mu\eta,\eta\rangle_F)^{\frac{1}{2}}$, $\eta\in E(M\times M)$ define a norm on each fiber in $E(M\times M)$, denote $\langle\mu\eta,\eta\rangle_F=|\eta(z,\zeta)|_F^2$.

In $M\times M\times\triangle$, we use $\widetilde{E}^*(M\times M\times\triangle)$ to denote the pullback of dual bundle $E^*(M\times M)$ with respect to map $(z,\zeta,\lambda)\mapsto(z,\zeta)$. The metric $\widetilde{F}^*$ on $\widetilde{E}^*(M\times M\times\triangle)$ is induced by the metric F over the vector bundle $E(M\times M)$. Let ∇ be a connection of $\widetilde{E}^*(M\times M\times\triangle)$ about $\widetilde{F}^*$, which is holomorphic on the metric.

We use the notation $C_k^\infty(M\times M\times\triangle,\widetilde{E}^*)$ to mean k-order differential form space on $M\times M\times\triangle$, which is valued at $\widetilde{E}^*=\widetilde{E}^*(M\times M\times\triangle)$ and has the following decomposition

$$C_k^\infty(M\times M\times\triangle,\widetilde{E}^*)=\bigoplus_{p+q+r=k}C^\infty_{p,q,r}(M\times M\times\triangle,\widetilde{E}^*),$$

where $C^\infty_{p,q,r}(M\times M\times\triangle,\widetilde{E}^*)$ denotes differential form space of type (p,q) about varibales (z,ζ) and type r about variable λ.

The connection ∇ can be decomposed to $\nabla=\nabla'+\nabla''$, such that

$$\nabla':\ C^\infty_{p,q,r}(M\times M\times\triangle,\widetilde{E}^*)\longrightarrow C^\infty_{p+1,q,r}(M\times M\times\triangle,\widetilde{E}^*),$$

$$\nabla'':\ C^\infty_{p,q,r}(M\times M\times\triangle,\widetilde{E}^*)\longrightarrow C^\infty_{p,q+1,r}(M\times M\times\triangle,\widetilde{E}^*)\bigoplus C^\infty_{p,q,r+1}(M\times M\times\triangle,\widetilde{E}^*).$$

When $\langle\xi_k(z,\zeta),\eta(z,\zeta)\rangle\neq 0$, (z,ζ,λ) in some neighborhood of $G\times S_K\times\triangle_{0K}$, we define

$$t^*(z,\zeta,\lambda):=\lambda_0\frac{\widehat{\eta}(z,\zeta)}{|\eta(z,\zeta)|_F^2}+\sum_{k\in K}\lambda_k\frac{\xi_k(z,\zeta)}{\langle\xi_k(z,\zeta),\eta(z,\zeta)\rangle}. \tag{2.9}$$

According to the properties of η and $\xi_k(k\in K)$, the map $(z,\zeta,\lambda)\mapsto t^*(z,\zeta,\lambda)$ defines a C^1 section on a neighborhood of $G\times\partial G\times\triangle$. So this differential form

$$\widetilde{\Omega}[t^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda=\frac{t^*\wedge D\eta}{n!(2\pi i)^n}\sum_{k=0}^{n-1}\left(\begin{array}{c}n\\k\end{array}\right)(-1)^k(\nabla''t^*\wedge D\eta)^{n-k-1}\wedge\widetilde{\Theta}^k,$$

is continuous on a neighborhood of $G\times\partial G\times\triangle$,we denote

$$\widetilde{\Omega}[t^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda=\sum_{1\leq p\leq n,1\leq q\leq n-1}\Omega_{p,q}(z,\zeta,\lambda),$$

where $\Omega_{p,q}(z,\zeta,\lambda)$ is a component type (p,q) about variable z of $\widetilde{\Omega}[t^*,\hat{\eta},\eta](z,\zeta,\lambda)\Lambda$, which is abbreviated as $\widetilde{\Omega}(z,\zeta,\lambda)$). For a bounded domain with piecewise C^1 smooth boundaries in a complex manifold, the following conclusions have been proved in [15].

Lemma 2.1. ([15]) *Let M be a a complex manifold of complex dimension n, G is a bounded domain with piecewise C^1 smooth boundaries in M, $f\in C_{(p,q)}(\overline{G})$, $\bar\partial f$ is also continuous on $\overline{G}$,*

$0\le p\le n$, $1\le q\le n$, then for all $z\in G$, we have

$$
\begin{aligned}
(-1)^{p+q}f(z)=\bar\partial_z\Big[&\sum_{|K|\le n-q}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}f(\zeta)\wedge\Omega_{p,q-1}(z,\zeta,\lambda)\\
&+\int_{G\times\triangle_0}f(\zeta)\wedge\Omega_{p,q-1}(z,\zeta,\lambda)\Big]\\
&-\Big[\sum_{|K|\le n-q-1}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}\bar\partial_\zeta f(\zeta)\wedge\Omega_{p,q}(z,\zeta,\lambda)\\
&+\int_{G\times\triangle_0}\bar\partial_\zeta f(\zeta)\wedge\Omega_{p,q}(z,\zeta,\lambda)\Big]\\
&+(-1)^{p+q+1}\sum_{|K|\le n-q}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}f(\zeta)\wedge Q_{p,q}(z,\zeta,\lambda)\\
&+\int_{G\times\triangle_0}f(\zeta)\wedge C_n[\Theta]_{p,q}(z,\zeta),
\end{aligned}
$$

where $Q_{p,q}(z,\zeta,\lambda)$ and $C_n[\Theta]_{p,q}(z,\zeta)$ are the component type (p,q) about variable z of $\mathrm{d}\widetilde\Omega(z,\zeta,\lambda)=(\bar\partial_{z,\zeta}+\mathrm{d}_\lambda)\widetilde\Omega(z,\zeta,\lambda)$ and $C_n[\Theta](z,\zeta)$, respectively.

In particular, if we add the condition $\Theta e=D^2e=0$, $\bar\partial f=0$, then

$$
\begin{aligned}
g(z)=(-1)^{p+q}\Big[&\sum_{|K|\le n-q}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}f(\zeta)\wedge\Omega_{p,q-1}(z,\zeta,\lambda)\\
&+\int_{G\times\triangle_0}f(\zeta)\wedge\Omega_{p,q-1}(z,\zeta,\lambda)\Big]
\end{aligned}
$$

is the continuous solution of $\bar\partial$–equation $\bar\partial g=f$ in G.

3 New Koppelman–Leray-Norguet formula

For $K=(k_1,\cdots,k_l)\in P(N)$, define

$$
U^K_{\overline G}:=\begin{cases}\{\zeta\in U_{\overline G}|\rho_{k_1}(\zeta)=\cdots=\rho_{k_l}(\zeta)\}, & \text{if integers } k_1,\cdots,k_l \text{ are different in pairs;}\\ \emptyset, & \text{otherwise.}\end{cases}
$$

By the definition of local q-convex wedge (see [16]), we can be sure that $U^K_{\overline G}$ is a closed C^2 submanifold of $U_{\overline G}$, and denote $\rho_K(\zeta)=\rho_{k_r}(\zeta)$ $(\zeta\in U^K_{\overline G},\ r=1,\cdots,l)$. For $K\in P(N)$, we define

$$
\Gamma_K:=\{\zeta\in U^K_{\overline G}|\rho_j(\zeta)\le\rho_K(\zeta)\le 0,\ j=1,\cdots,N\}.
$$

It is not difficult to verify that Γ_K is a C^2 submanifold with piecewise C^2 smooth boundaries of $\overline G$. We can also verify the following conclusion:

$$
\overline G=\Gamma_1\cup\cdots\cup\Gamma_N,\qquad \partial\Gamma_K=S_K\cup\Gamma_{K1}\cup\cdots\cup\Gamma_{KN},\ K\in P(N),
$$

Select the orientation on Γ_K such that the component of the orientation with respect to K is skew symmetric and that the following conditions hold: $\Gamma_1,\cdots,\Gamma_N$ carry the orientation on $\mathbf{C}^n$; If $K\in P(N)$, $1\le j\le N$, $j\notin K$, then the orientation on Γ_{Kj} is consistent with that on $-\partial\Gamma_K$.

Lemma 3.1. ([16]) *The following identity holds*

$$\sum_{K\in P'(N)}(-1)^{|K|}\partial(\Gamma_K\times\triangle_{0K})=\overline{G}\times\triangle_0+\sum_{K\in P'(N)}(-1)^{|K|}S_K\times\triangle_{0K}-\sum_{K\in P'(N)}\Gamma_K\times\triangle_K.$$

Selecting $\chi_k\in C_0^\infty(\theta_k)$ $(k=1,\cdots,N)$, such that $\chi_k(\zeta)=1$ on a neighborhood of $N(\rho_k)$. From formula (2.3) and (2.5), for each $z\in G$, the neighborhood $V_k\subseteq\theta_k$ of $N(\rho_k)$ exists that makes for $\zeta\in(G\cap\theta_k)\cup V_k$, we have $\widetilde{\Phi}_k(z,\zeta)\neq 0$. Since $\mathrm{supp}(\chi_k)\subset\subset\theta_k$, for each fixed $z\in G$, the map $\chi_k(\zeta)\xi_k(z,\zeta)/\widetilde{\Phi}_k(z,\zeta)$ is C^1 continuous with respect to $\zeta\in G\cup V_k$. We denote

$$\widetilde{t}^*(z,\zeta,\lambda):=\lambda_0\frac{\widehat{\eta}(z,\zeta)}{|\eta(z,\zeta)|_F^2}+\sum_{k\in K}\lambda_k\frac{\chi_k(\zeta)\xi_k(z,\zeta)}{\widetilde{\Phi}_k(z,\zeta)}. \tag{3.1}$$

So the differential form

$$\overline{\Omega}[\widetilde{t}^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda:=\frac{\widetilde{t}^*\wedge D\eta}{n!(2\pi i)^n}\sum_{k=0}^{n-1}\left(\begin{array}{c}n\\k\end{array}\right)(-1)^k(\nabla''\widetilde{t}^*\wedge D\eta)^{n-k-1}\wedge\widetilde{\Theta}^k, \tag{3.2}$$

is continuous on a neighborhood of $G\times\partial G\times\triangle$. We denote

$$\overline{\Omega}[t^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda=\sum_{1\le p\le n,1\le q\le n-1}\overline{\Omega}_{p,q}(z,\zeta,\lambda),$$

where $\overline{\Omega}_{p,q}(z,\zeta,\lambda)$ is the component type (p,q) about variable z of $\overline{\Omega}[t^*,\hat{\eta},\eta](z,\zeta,\lambda)\Lambda$, which is abbreviated as $\overline{\Omega}(z,\zeta,\lambda)$. We denote

$$\widehat{\Omega}[\widetilde{t}^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda:=\mathrm{d}\overline{\Omega}[\widetilde{t}^*,\widehat{\eta},\eta](z,\zeta,\lambda)\Lambda,$$

which is abbreviated as $\widehat{\Omega}(z,\zeta,\lambda)$. So we have

$$\begin{aligned}&\mathrm{d}_{\zeta,\lambda}[f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)]\\=&\,\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)+(-1)^{p+q}f(\zeta)\wedge\widehat{\Omega}(z,\zeta,\lambda)-\bar{\partial}_z[f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)].\end{aligned} \tag{3.3}$$

If $\lambda=(\lambda_0,\cdots.\lambda_N)\in\triangle_0$, then

$$\lambda_0=1,\qquad\widetilde{t}^*=t^*=\frac{\widehat{\eta}(z,\zeta)}{|\eta(z,\zeta)|_F^2},$$

when denote

$$\Omega^0[\widehat{\eta},\eta](z,\zeta)\Lambda=\frac{\widehat{\eta}\wedge D\eta}{n!(2\pi i)^n}\sum_{k=0}^{n-1}\left(\begin{array}{c}n\\k\end{array}\right)(-1)^k\frac{(\nabla''\widehat{\eta}\wedge D\eta)^{n-k-1}}{|\eta|_F^{2(n-k)}}\wedge\widetilde{\Theta}^k,$$

we obtain the following lemmas.

Lemma 3.2. $\overline{\Omega}(z,\zeta,\lambda)|_{\triangle_0}=\Omega^0[\widehat{\eta},\eta](z,\zeta)\Lambda=\widetilde{\Omega}(z,\zeta,\lambda)|_{\triangle_0}$.

Lemma 3.3. *When $\zeta\in\partial G$, we have*

$$\overline{\Omega}(z,\zeta,\lambda)=\widetilde{\Omega}(z,\zeta,\lambda).$$

Proof. If $\zeta\in\partial G$, we have $\chi_k(\zeta)=1$, $\widetilde{\Phi}_k(z,\zeta)=\Phi_k(z,\zeta)$. Then by formula (2.8) , we can get

$$\frac{\chi_k(\zeta)\xi_k(z,\zeta)}{\widetilde{\Phi}_k(z,\zeta)}=\frac{\xi_k(z,\zeta)}{\Phi_k(z,\zeta)}=\frac{\xi_k(z,\zeta)}{\langle\xi_k(z,\zeta),\eta(z,\zeta)\rangle}.$$

So $\widetilde{t}^*|_{\zeta\in\partial G}=t^*|_{\zeta\in\partial G}$, thus this Lemma holds. □

Lemma 3.4. *For continuous bounded (p,q) differential form f on $\overline{G}$, the following conclusions are valid.*

$$(i)\quad \int_{\Gamma_K\times\triangle_K}f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)=0,\qquad (z\in G,\quad q\neq 1); \tag{3.4}$$

$$(ii)\quad \bar{\partial}_z\int_{\Gamma_K\times\triangle_K}f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)=0,\qquad (z\in G). \tag{3.5}$$

Proof. (i) If $(\zeta,\lambda)\in\Gamma_K\times\triangle_K$, then

$$\widetilde{t}^*=\sum_{k\in K}\lambda_k\chi_k(\zeta)\xi_k(z,\zeta)/\widetilde{\Phi}_k(z,\zeta),$$

Since $\xi_k(z,\zeta)$, $\widetilde{\Phi}_k(z,\zeta)$ $(k=1,\cdots,N)$ are holomorphic with respect to the variable z, $\overline{\Omega}(z,\zeta,\lambda)|_{(\zeta,\lambda)\in(\Gamma_K\times\triangle_K)}$ is zero-order about $\bar{\partial}_z$, so the order sum, with respect to the variable ζ and λ, of $f(\zeta)\wedge\overline{\Omega}(z,\zeta,\lambda)$ is equal to $2n+1-q$, however $\dim_{\mathbf{R}}(\Gamma_K\times\triangle_K)=2n$, thus the formula (3.4) holds when $z\in G$, $q\neq 1$.

(ii) Since $\xi_k(z,\zeta)$, $\widetilde{\Phi}_k(z,\zeta)$ $(k=1,\cdots,N)$ are holomorphic with respect to the variable z, so the formula (3.5) holds. □

Theorem 3.1. *Suppose G is a strictly pseudoconvex domain with non-smooth boundaries in a n-dimensional complex manifold M, and there are holomorphic support functions $\widetilde{\Phi}_k(z,\zeta)$, $\Phi_k(z,\zeta)$, $(k=1,\cdots,N)$ that satisfy formula (2.3) to (2.8) in G, $f\in C_{(p,q)}(\overline{G})$, $\bar{\partial}f$ is also continuous on $\overline{G}$, $0\leq p\leq n$, $1\leq q\leq n$, $\Theta e=D^2e=0$, then for all $z\in G$, we have*

$$f(z)=\sum_{|K|\leq n-q+1}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}}f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)$$

$$+\sum_{|K|\leq n-q}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}f(\zeta)\wedge\widehat{\Omega}_{p,q}(z,\zeta,\lambda). \tag{3.6}$$

In particular, if $\bar{\partial} f = 0$ *, then*

$$g(z)=\sum_{|K|\le n-q+1}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda),$$

is the continuous solution of $\bar{\partial}$*-equation* $\bar{\partial} g = f$ *in* G*. Where* $\widehat{\Omega}_{p,q}(z,\zeta,\lambda)$ *is the component type* (p,q) *about variable* z *of* $\widehat{\Omega}(z,\zeta,\lambda)$*.*

Proof. Let us first prove the special case when $\zeta\in\partial G$, $d\rho_k(\zeta)\neq 0$ $(k=1,\cdots,N)$. f and $\bar{\partial} f$ are continuous on $\overline{G}$, we apply Stokes Formula to the differential form $f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)$ on $\sum_{K\in P'(N)}(-1)^{|K|}\Gamma_K\times\triangle_{0K}$,

$$\begin{aligned}&\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}} d_{\zeta,\lambda}[f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)]\\ =&\sum_{K\in P'(N)}(-1)^{|K|}\int_{\partial(\Gamma_K\times\triangle_{0K})} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda).\end{aligned}$$

By Lemma 3.1 and formula (3.3), we have

$$\begin{aligned}&\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\ &\quad+(-1)^{p+q}\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)\\ &\quad+\sum_{K\in P'(N)}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-2}(z,\zeta,\lambda)\\ =&\int_{\overline{G}\times\triangle_0} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\ &\quad-\sum_{K\in P'(N)}\int_{\Gamma_K\times\triangle_K} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda).\end{aligned}$$

Then we have

$$\begin{aligned}&\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)\\ =&(-1)^{p+q}\Big[\int_{\overline{G}\times\triangle_0} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\Big]\\ &\quad+(-1)^{p+q+1}\Big[\sum_{K\in P'(N)}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-2}(z,\zeta,\lambda)\\ &\quad+\sum_{K\in P'(N)}\int_{\Gamma_K\times\triangle_K} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\ &\quad+\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\Big].\end{aligned}\tag{3.7}$$

Using $\bar{\partial}_z$ to act on both sides of formula (3.7), we get

$$\begin{aligned}
&\sum_{K\in P'(N)}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)\\
=&(-1)^{p+q}\bar{\partial}_z\Big[\int_{\overline{D}\times\triangle_0} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\Big]\\
&+(-1)^{p+q+1}\bar{\partial}_z\Big[\sum_{K\in P'(N)}\int_{\Gamma_K\times\triangle_K} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\
&+\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\Big], \qquad (3.8)
\end{aligned}$$

Substitute $\bar{\partial}_\zeta f(\zeta)$ for $f(\zeta)$ in formula (3.7), we have

$$\begin{aligned}
&\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\widehat{\Omega}_{p,q}(z,\zeta,\lambda)\\
=&(-1)^{p+q+1}\Big[\int_{\overline{G}\times\triangle_0}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)\Big]\\
&+(-1)^{p+q}\Big[\sum_{K\in P'(N)}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\
&+\sum_{K\in P'(N)}\int_{\Gamma_K\times\triangle_K}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)\Big]. \qquad (3.9)
\end{aligned}$$

From Lemma 3.4, for any $K\in P'(N)$, we have

$$\begin{aligned}
&\int_{\Gamma_K\times\triangle_K}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)=0,\quad (\because q\geq 1),\\
&\bar{\partial}_z\int_{\Gamma_K\times\triangle_K} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)=0.
\end{aligned}$$

Moreover, add formula (3.8) to (3.9) , and we can get

$$\begin{aligned}
&\sum_{K\in P'(N)}(-1)^{|K|}\bar{\partial}_z\int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)\\
&\quad+\sum_{K\in P'(N)}(-1)^{|K|}\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\widehat{\Omega}_{p,q}(z,\zeta,\lambda)\\
=&(-1)^{p+q}\bar{\partial}_z\Big[\int_{\overline{D}\times\triangle_0} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\\
&\quad+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}} f(\zeta)\wedge\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)\Big]\\
&\quad+(-1)^{p+q+1}\Big[\int_{\overline{D}\times\triangle_0}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)
\end{aligned}$$

$$+\sum_{K\in P'(N)}(-1)^{|K|}\int_{S_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\overline{\Omega}_{p,q}(z,\zeta,\lambda)\Big]. \tag{3.10}$$

Also pay attention to

$$\int_{S_K\times\triangle_{0K}}f(\zeta)\wedge\bar{\partial}_z\overline{\Omega}_{p,q-1}(z,\zeta,\lambda)=0,\ \ |K|>n-q, \tag{3.11}$$

this is because if $|K|>n-q$, then $\dim_{\mathbf{R}}S_K=2n-|K|<n+q$, but the order with respect to the variable ζ of $f(\zeta)\wedge\bar{\partial}_z\overline{\Omega}_{p,q-1}$ is no less than $n+q$. Hence this integral of (3.11) is equal to zero when $|K|>n-q$. In the same way, we can draw the following conclusions:

$$\int_{S_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\Omega_{p,q}(z,\zeta,\lambda)=0,\qquad |K|>n-q-1, \tag{3.12}$$

$$\int_{\Gamma_K\times\triangle_{0K}}f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda)=0,\qquad |K|>n-q+1, \tag{3.13}$$

$$\int_{\Gamma_K\times\triangle_{0K}}\bar{\partial}_\zeta f(\zeta)\wedge\widehat{\Omega}_{p,q}(z,\zeta,\lambda)=0,\qquad |K|>n-q. \tag{3.14}$$

If $\Theta e=D^2e=0$, then we have $Q_{p,q}(z,\zeta,\lambda)=0$, $C_n[\Theta]_{p,q}(z,\zeta)=0$. From Lemma 3.2, Lemma 3.3, formula (3.10) to (3.14) and Koppelman–Leray–Norguet formula (Lemma 2.1) , the formula (3.6) holds when $\zeta\in\partial G$ and $\mathrm{d}\rho_k(\zeta)\neq 0$.

Second, we consider the general case that we do not have to assume $\mathrm{d}\rho_k(\zeta)\neq 0$, $(k=1,\cdots,N)$, when $\zeta\in\partial G$. In this case, we prove the theorem by considering a sequence of strictly pseudoconvex domains G_m which are infinitely close to G . And all constructions in this section are uniform convergence with respect to m . From this we complete the proof of the Theorem 3.1 . □

Example 3.1. If $M=\mathbf{C}^n$, then the vector bundle E is chosen as an trivial bundle of order n, the section η is usually chosen as $\eta=\zeta-z$, and ξ_k is chosen as the section of the Cauchy–Leray vector bundle. It is clear that the corresponding results for strictly pseudoconvex domains with non-smooth boundaries in $\mathbf{C}^n$ can be obtained from Theorem 3.1, it contains the Theorem 3.1.3 in paper [4].

Example 3.2. If M is a Stein manifold of complex dimension n, G is a strictly pseudoconvex domain with non-smooth boundary in M, $s(z,\zeta)$, $\varphi(z,\zeta)$, κ is the same as Lemma 4.2.4 in paper [4] , $T(M)$ and $T^*(M)$ are the complex tangent bundle and the complex cotangent bundle of M, respectively, $\widetilde{T}(M)$ and $\widetilde{T}^*(M)$ are pullback maps of $T(M)$ and $T^*(M)$ with respect to projection $M\times M\to M$, $(z,\zeta)\mapsto z$. E and E^* are chosen as $\widetilde{T}(M)$ and $\widetilde{T}^*(M)$, respectively. The sections ξ_1, $\cdots$, ξ_N of E^*, which are taken as s_1^*, $\cdots$, s_N^*, respectively. We choose holomorphic cross section $\eta=s(z,\zeta)$ such that $\{\eta=0\}=Y\cup P$, where P is a closed set disconnected to Y, the exceptional zero point of η leads to new difficulty. To overcome this difficulty, we introduce a holomorphic function φ, which has the property $\varphi(z,z)=1$. From Theorem 3.1, we obtain Koppelman–leray–norguet formula with weight factor $\varphi^\nu(z,\zeta)$ $(\nu\geq max\{n\kappa^*,n\kappa\})$. This is the corresponding result of strictly pseudoconvex domain with non-smooth boundary in Stein manifolds. It contains Theorem 4.10.4 in paper [4].

4 New formula for general strictly pseudoconvex polyhedron on Stein manifold

Definition 4.1. *Let M be a Stein manifold of complex dimension n, an open set $G \subset\subset M$ is said to be a strictly pseudoconvex polyhedron, that is, if there is a neighborhood $U_{\overline{G}}$ of $\overline{G}$, finitely many Stein manifolds $M_1, \cdots, M_N$ whose complex dimensions are no more than n, holomorphic maps $F_k: U_{\overline{G}} \to M_k$, $(k=1,\cdots,N)$, as well as strictly pseudoconvex open sets $G_k \subset\subset M_k$, $(k=1,\cdots,N)$, such that*

$$G = F_1^{-1}(G_1) \cap \cdots \cap F_1^{-1}(G_N).$$

If $\rho_1, \cdots, \rho_N$ are strictly plurisubharmonic C^2 functions in some neighborhoods $\theta_1, \cdots, \theta_N$ of $\partial G_1, \cdots, \partial G_N$, respectively, such that

$$G_k \cap \theta_k = \{z \in \theta_k | \rho_k(z) < 0\}, \quad (k=1,\cdots,N),$$

then $\partial G \subseteq F_1^{-1}(\theta_1) \cup \cdots \cup F_N^{-1}(\theta_N)$, and a point $z \in F_1^{-1}(\theta_1) \cup \cdots \cup F_N^{-1}(\theta_N)$ belongs to G, if and only if $\forall k \in \{1 \le k \le N | z \in F_k^{-1}(\theta_k)\}$, we have $\rho_k(F_k(z)) < 0$.

G is called a real non-degenerate strictly pseudoconvex polyhedron, when we can choose functions F_k and ρ_k such that: for every index set $(k_1,\cdots,k_l) \in P'(N)$, as well as every point z which satisfies $z \in \partial G$ with $\rho_{k_1}(F_{k_1}(z)) = \cdots = \rho_{k_l}(F_{k_l}(z)) = 0$, we have

$$\mathrm{d}(\rho_{k_1} \circ F_{k_1})(z) \wedge \cdots \wedge \mathrm{d}(\rho_{k_l} \circ F_{k_l})(z) \neq 0.$$

Remark 4.1. The boundary of a real non-degenerate strictly pseudoconvex polyhedron is piecewise C^1 in the sense of Definition 2.1.

For general strictly pseudoconvex polyhedron (it's not necessary to assume real non-degeneracy) which possess holomorphic support functions $\widetilde{\Phi}_k(z,\zeta)$, from Corollary 4.9.4 in paper [4], there are $T^*(M)$-valued C^1 maps $h_k^*(z,\zeta)$ such that

$$\varphi(z,\zeta)\widetilde{\Psi}_k(z,\zeta) = \langle h_k^*(z,\zeta), s(z,\zeta)\rangle,$$

where $\widetilde{\Psi}_k(z,\zeta) = \widetilde{\Phi}_k(F_k(z),F_k(\zeta))$, $z \in F_k^{-1}(G_k \cup \theta_k)$, $\zeta \in F_k^{-1}(\theta_k)$, $(k=1,\cdots,N)$. By the property of $\widetilde{\Psi}_k$, we have $\widetilde{\Phi}_k(F_k(z),F_k(\zeta)) \neq 0$, then

$$\frac{\varphi(z,\zeta)h_k^*(z,\zeta)}{\langle h_k^*(z,\zeta), s(z,\zeta)\rangle} = \frac{\chi_k(\zeta)h_k^*(z,\zeta)}{\widetilde{\Phi}_k(F_k(z),F_k(\zeta))}$$

are C^1 for any $z \in F_k^{-1}(G_k \cup \theta_k)$, $\zeta \in F_k^{-1}(\theta_k)$, $(k=1,\cdots,N)$. Therefore $(h_1^*,\cdots,h_N^*,1)$ is a Leray–Norguet section for G.

If we set

$$\overline{t}^*_{(h_1^*,\cdots,h_N^*,\widehat{s},s)}(z,\zeta,\lambda) := \lambda_0 \frac{\widehat{s}(z,\zeta)}{|s(z,\zeta)|_F^2} + \sum_{k\in K} \lambda_k \frac{\chi_k(\zeta) h_k^*(z,\zeta)}{\widetilde{\Phi}_k(F_k(z),F_k(\zeta))},$$

$$\overline{\Omega}[\overline{t}^*,\widehat{s},s](z,\zeta,\lambda)\Lambda := \frac{\varphi^\nu(z,\zeta)\overline{t}^*\wedge Ds}{n!(2\pi i)^n} \sum_{k=0}^{n-1} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^k (\nabla''\overline{t}^*\wedge Ds)^{n-k-1}\wedge \widetilde{\Theta}^k,$$

then the following theorem can be obtained.

Theorem 4.1. *Let M be a Stein manifold, and $G\subset\subset M$ is a strictly pseudoconvex polyhedron (unnecessarily real non-degeneracy), which possess holomorphic support functions $\widetilde{\Phi}_k(z,\zeta)$, $\nu \geq 2n\kappa$, $f\in C_{(p,q)}(\overline{G})$, $\bar{\partial}f$ is also continuous on $\overline{G}$, $0\le p\le n, 1\le q\le n, \Theta e = D^2 e = 0$, then for all $z\in G$, we have*

$$f(z) = \sum_{|K|\le n-q+1} (-1)^{|K|}\bar{\partial}_z \int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda) + \sum_{|K|\le n-q} (-1)^{|K|} \int_{\Gamma_K\times\triangle_{0K}} \bar{\partial}f(\zeta)\wedge\widehat{\Omega}_{p,q}(z,\zeta,\lambda),$$

where we need to be aware of $\widehat{\Omega}_{p,q}(z,\zeta,\lambda)$ is the component type (p,q) about variable z of

$$\widehat{\Omega}(z,\zeta,\lambda) := \mathrm{d}\overline{\Omega}[\overline{t}^*,\widehat{s},s](z,\zeta,\lambda)\Lambda.$$

In particular, if $\bar{\partial}f = 0$, then

$$g(z) = \sum_{|K|\le n-q+1} (-1)^{|K|} \int_{\Gamma_K\times\triangle_{0K}} f(\zeta)\wedge\widehat{\Omega}_{p,q-1}(z,\zeta,\lambda),$$

is the continuous solution of $\bar{\partial}$-equation $\bar{\partial}g = f$ in G .

Remark 4.2. When we only choose the first term in the expansion of Berndtsson's kernel (i.e. formula (2.2)), it is exactly the kernel of type (p,q) on the Stein manifold. From Example 3.2 and Theorem 4.1 , it implies the corresponding result in paper [13] .

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