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# Group-theoretical Property of Slightly Degenerate Fusion Categories of Certain Frobenius-Perron Dimensions

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**Abstract.** Let $p,q$ be odd primes, and let $d$ be an odd square-free integer such that $(pq,d)=1$. We show that slightly degenerate fusion categories of Frobenius-Perron dimensions $2p^2q^2d$, $2p^2q^3d$ and $2p^3q^3d$ are group-theoretical.

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**Key words**: Group-theoretical fusion category, slightly degenerate fusion category.

## 1 Introduction

Throughout this paper, we work over an algebraically closed field $\mathbb{K}$ of characteristic zero, $\mathbb{K}^*:=\mathbb{K}\backslash\{0\}$, $\mathbb{Z}_r:=\mathbb{Z}/r$, $r\in\mathbb{N}$. For any semisimple $\mathbb{K}$-linear finite abelian category $\mathcal{C}$, we use $\mathcal{O}(\mathcal{C})$ to denote the set of isomorphism classes of simple objects of $\mathcal{C}$.

Recall that a braided fusion category $\mathcal{C}$ is said to be slightly degenerate, if its Müger center $\mathcal{C}'$ is braided equivalent to sVec, the category of finite-dimensional super vector spaces over $\mathbb{K}$; while $\mathcal{C}$ is a non-degenerate fusion category if $\mathcal{C}'\cong$ Vec, the category of finite-dimensional vector spaces over $\mathbb{K}$. In [9], the first named author studied slightly degenerate fusion categories of various Frobenius-Perron dimensions. In particular, for odd primes $p$ and $q$, slightly degenerate fusion categories of Frobenius-Perron dimension $2p^mq^nd$ are always integral and solvable [9, Corollary 3.4, Proposition 3.14], where $d$ is an odd square-free integer such that $(pq,d)=1$. Moreover, non-degenerate fusion categories $\mathcal{C}$ of Frobenius-Perron dimensions $p^2q^3d$ and $p^3q^3d$ are group-theoretical [10], that is, $\mathcal{Z}(\mathcal{C})$ are braided equivalent to Drinfeld centers of certain pointed fusion categories.

Recently, for odd primes $p$ and $q$, let $d$ be an odd square-free integer such that $(pq,d)=1$, it was proved that a slightly degenerate fusion category $\mathcal{C}$ of Frobenius-Perron dimension $2p^mq^nd$ always contains a non-degenerate fusion subcategory $\mathcal{C}(\mathbb{Z}_d,\eta)$ [8, Corollary

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4.7], where $\mathcal{C}(\mathbb{Z}_d,\eta)$ is a non-degenerate fusion category determined by the metric group $(\mathbb{Z}_d,\eta)$ and $\eta:\mathbb{Z}_d\to\mathbb{K}^*$ is a non-degenerate quadratic form. Thus, $\mathcal{C}\cong\mathcal{C}(\mathbb{Z}_d,\eta)\boxtimes\mathcal{D}$ as braided fusion category by [2, Theorem 3.13], where $\mathcal{D}$ is a slightly degenerate fusion category of Frobenius-Perron dimensions $2p^mq^n$; in particular, it is easy to see that if $m\leq 1$ or $n\leq 1$ then $\mathcal{C}$ is nilpotent and group-theoretical by [1, Theorem 6.10, Corollary 6.8]. In this paper, we continue to study the structures of slightly degenerate fusion categories of FP-dimensions $2p^2q^2$, $2p^2q^3$ and $2p^3q^3$, we prove that they are also group-theoretical fusion categories, see Theorems 3.1, 3.2 and 3.3.

The paper is organized as follows. In Section 2, we recall some basic properties of fusion categories and braided fusion categories that we use throughout. In Section 3, we prove our main theorems: Theorems 3.1, 3.2 and 3.3, respectively.

## 2 Preliminaries

In this section, we will recall some most used results about fusion categories and braided fusion categories, we refer the readers to [1–5].

Let $G$ be a finite group. A fusion category $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is a $G$-graded fusion category, if for any $g,h\in G$, $\mathcal{C}_g$ is a $\mathbb{K}$-linear full abelian subcategories, $\mathcal{C}_g\otimes\mathcal{C}_h\subseteq\mathcal{C}_{gh}$ and $(\mathcal{C}_g)^*\subseteq\mathcal{C}_{g^{-1}}$. A $G$-grading of fusion category $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is faithful if $\mathcal{C}_g\neq 0$ for any $g\in G$. Note that the trivial component $\mathcal{C}_e$ is a fusion subcategory of $\mathcal{C}$, so when the $G$-grading $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$ is faithful, $\mathcal{C}$ is also called a $G$-extension of $\mathcal{C}_e$.

Let $\mathcal{C}_{\mathrm{ad}}$ be the adjoint fusion subcategory of $\mathcal{C}$, that is, $\mathcal{C}_{\mathrm{ad}}$ is generated by simple objects $Y$ such that $Y\subseteq X\otimes X^*$ for some simple object $X$ of $\mathcal{C}$, then $\mathcal{C}$ has a faithful $U_{\mathcal{C}}$-grading with $\mathcal{C}_{\mathrm{ad}}$ be the trivial component [5, Theorem 3.5], $U_{\mathcal{C}}$ is called the universal grading group of $\mathcal{C}$. Moreover, for any other faithful $G$-grading of $\mathcal{C}=\oplus_{g\in G}\mathcal{C}_g$, there exists a surjective group homomorphism $\phi:U_{\mathcal{C}}\to G$ by [5, Corollary 3.7].

Given a fusion category $\mathcal{C}$, there exists a unique ring homomorphism FPdim(-) from the Grothendieck ring $Gr(\mathcal{C})$ to $\mathbb{K}$ [3, Theorem 8.6], which satisfies that $\mathrm{FPdim}(X)\geq 1$ is an algebraic integer for any object $X\in\mathcal{O}(\mathcal{C})$. $\mathrm{FPdim}(X)$ is called the Frobenius-Perron dimension of the object $X$ and the Frobenius-Perron dimension of fusion category $\mathcal{C}$ is defined by the following sum

$$\mathrm{FPdim}(\mathcal{C}):=\sum_{X\in\mathcal{O}(\mathcal{C})}\mathrm{FPdim}(X)^2.$$

A simple object $X\in\mathcal{C}$ is invertible if $X\otimes X^*=X^*\otimes X=I$, the unit object, equivalently $\mathrm{FPdim}(X)=1$. Recall that a fusion category $\mathcal{C}$ is pointed if all simple objects of $\mathcal{C}$ are invertible. Let $\mathcal{C}$ be a pointed fusion category, then $\mathcal{C}$ is tensor equivalent to the fusion category $\mathrm{Vec}_G^{\omega}$ of $G$-graded finite-dimensional vector spaces over $\mathbb{K}$, where $\mathcal{O}(\mathcal{C})\cong G$ is a finite group, $\omega\in Z^3(G,\mathbb{K}^*)$ is a 3-cocycle. Given an arbitrary fusion category $\mathcal{C}$, we denote by $\mathcal{C}_{\mathrm{pt}}$ the maximal pointed fusion subcategory of $\mathcal{C}$ below, that is, $\mathcal{C}_{\mathrm{pt}}$ is the fusion subcategory generated by invertible objects of $\mathcal{C}$, and $G(\mathcal{C}):=\mathcal{O}(\mathcal{C}_{\mathrm{pt}})$.

A fusion category $\mathcal{C}$ is nilpotent if there exists a nonnegative integer $n$ such that $\mathcal{C}^{(n)}=\text{Vec}$, where Vec is the category of finite-dimensional vector spaces over $\mathbb{K}$, $\mathcal{C}^{(0)}:=\mathcal{C}$, $\mathcal{C}^{(1)}:=\mathcal{C}_{\text{ad}}$, $\mathcal{C}^{(m)}:=(\mathcal{C}^{(m-1)})_{\text{ad}}$ for all $m\geq 1$ [5]. Moreover, a fusion category $\mathcal{C}$ is cyclically nilpotent [4], if there is a sequence of fusion categories $\text{Vec}=\mathcal{C}_0\subseteq\mathcal{C}_1\subseteq\cdots\subseteq\mathcal{C}_n=\mathcal{C}$ and cyclic groups $G_i$ such that fusion categories $\mathcal{C}_i$ are obtained by $G_i$-extensions of $\mathcal{C}_{i-1}$, $1\leq i\leq n$. Pointed fusion categories are nilpotent by definition, while fusion categories of prime power Frobenius-Perron dimensions are always cyclically nilpotent by [3, Theorem 8.28]. A fusion category $\mathcal{C}$ is group-theoretical if $\mathcal{C}$ is Morita equivalent to a pointed fusion category [3]. Similarly, fusion category $\mathcal{C}$ is solvable if $\mathcal{C}$ is Morita equivalent to a cyclically nilpotent fusion category $\mathcal{D}$ [4].

A braided fusion category $\mathcal{C}$ is a fusion category admitting a braiding $c$. More precisely, for arbitrary objects $X,Y,Z\in\mathcal{C}$, there exists a natural isomorphism $c_{X,Y}:X\otimes Y\xrightarrow{\sim}Y\otimes X$, which satisfies $c_{X,I}=c_{I,X}=\text{id}_X$, and

$$c_{X\otimes Y,Z}=(c_{X,Z}\otimes\text{id}_Y)(\text{id}_X\otimes c_{Y,Z}),\quad c_{Z,X\otimes Y}=(\text{id}_X\otimes c_{Z,Y})(c_{Z,X}\otimes\text{id}_Y),$$

here we suppress the associativity isomorphism of $\mathcal{C}$.

Given a braided fusion subcategory $\mathcal{D}\subseteq\mathcal{C}$, the centralizer $\mathcal{D}'_{\mathcal{C}}\subseteq\mathcal{C}$ of $\mathcal{D}$ is the fusion subcategory generated by all simple objects $X$ of $\mathcal{C}$ satisfying

$$c_{Y,X}c_{X,Y}=\text{id}_{X\otimes Y},\quad\forall Y\in\mathcal{D}.$$

In particular, we call $\mathcal{C}':=\mathcal{C}'_{\mathcal{C}}$ the Müger center of $\mathcal{C}$. A braided fusion category $\mathcal{C}$ is symmetric if $\mathcal{C}'=\mathcal{C}$. Braided fusion category $\mathcal{C}$ is non-degenerate if $\mathcal{C}'=\text{Vec}$, while $\mathcal{C}$ is slightly degenerate if $\mathcal{C}'=\text{sVec}$, the category of finite-dimensional super-vector spaces over $\mathbb{K}$. A slightly degenerate fusion category $\mathcal{C}$ is split, if $\mathcal{C}\cong\text{sVec}\boxtimes\mathcal{D}$, where $\mathcal{D}$ is a non-degenerate fusion category.

A symmetric fusion category $\mathcal{E}$ is Tannakian, if $\mathcal{E}\cong\text{Rep}(G)$, where $G$ is a finite group, the braiding of $\text{Rep}(G)$ is given by standard reflection of vector spaces

$$c_{X,Y}(x\otimes y)=y\otimes x,\quad\forall x\in X,\ y\in Y,\quad\forall X,Y\in\text{Rep}(G).$$

We can deduce from [2, Corollary 2.50] that symmetric fusion categories of odd FP-dimensions are Tannakian categories. Let $\mathcal{E}=\text{Rep}(G)$ be a Tannakian subcategory of braided fusion category $\mathcal{C}$. Then there exists a fusion category $\mathcal{C}_G$ such that $\mathcal{C}\cong(\mathcal{C}_G)^G$ as braided fusion category. $\mathcal{C}_G$ is called the de-equivariantization of $\mathcal{C}$, and $\mathcal{C}$ is called the equivariantization of $\mathcal{C}_G$. In general, $\mathcal{C}_G$ is a $G$-crossed braided fusion category, and the $G$-grading of $\mathcal{C}_G$ is faithful if $\mathcal{C}'\cap\mathcal{E}=\text{Vec}$ [2, Proposition 4.56]. If $\text{Rep}(G)=\mathcal{E}\subseteq\mathcal{C}'$, then $\mathcal{C}_G$ is also a braided fusion category, see [2, section 4] for details.

In the last, we recall the classification of pointed non-degenerate fusion categories. An abelian group $G$ is a metric group, if there is a quadratic form $\eta:G\to\mathbb{K}^*$ such that $\eta(g)=\eta(g^{-1})$, and the bicharacter $\beta:G\times G\to\mathbb{K}^*$ determined by $\eta$ is non-degenerate,

$$\beta(g,h):=\frac{\eta(gh)}{\eta(g)\eta(h)},\quad\forall g,h\in G.$$

Pointed non-degenerate fusion categories are in bijective correspondence with metric groups, see [2, subsection 2.11] for details. We use $\mathcal{C}(G,\eta)$ to denote the non-degenerate pointed fusion category determined by the metric group $(G,\eta)$ below.

# 3 Group-theoretical property of some slightly degenerate fusion categories

In this section, for odd primes $p\neq q$, we show that slightly degenerate fusion categories of Frobenius-Perron dimensions $2p^2q^2d$, $2p^2q^3d$ and $2p^3q^3d$ are group-theoretical, where $d$ is a square-free odd integer such that $(pq,d)=1$.

Let $\mathcal{D}$ be a slightly degenerate fusion category, then we deduce from [2, Theorem 3.14] and [2, Proposition 3.29] that

$$\mathrm{FPdim}(\mathcal{D}_{\mathrm{ad}})\mathrm{FPdim}(\mathcal{D}_{\mathrm{pt}})=\mathrm{FPdim}(\mathcal{D})\mathrm{FPdim}(\mathcal{D}_{\mathrm{ad}}\cap\mathcal{D}')$$

and

$$(\mathcal{D}_{\mathrm{ad}})'_{\mathcal{D}}=\mathcal{D}_{\mathrm{pt}},\qquad \mathcal{D}_{\mathrm{ad}}\vee\mathcal{D}'=(\mathcal{D}_{\mathrm{pt}})'_{\mathcal{D}},$$

where $\mathcal{D}_{\mathrm{ad}}\vee\mathcal{D}'$ is the fusion subcategory generated by $\mathcal{D}_{\mathrm{ad}}$ and $\mathcal{D}'$, that is, $\mathcal{D}_{\mathrm{ad}}\vee\mathcal{D}'$ is the smallest braided fusion subcategory of $\mathcal{D}$ containing both $\mathcal{D}_{\mathrm{ad}}$ and $\mathcal{D}'$. Also, notice that $(\mathcal{D}_{\mathrm{ad}})_{\mathrm{pt}}$ is always a symmetric fusion subcategory of $\mathcal{D}$. In addition, since the universal grading of $\mathcal{D}$ is faithful, [3, Proposition 8.20] means that we have the following equation

$$\mathrm{FPdim}(\mathcal{D})=\mathrm{FPdim}(\mathcal{D}_{\mathrm{ad}})|U_{\mathcal{D}}|.$$

Therefore,

$$|U_{\mathcal{D}}|=\frac{\mathrm{FPdim}(\mathcal{D})}{\mathrm{FPdim}(\mathcal{D}_{\mathrm{ad}})}=\frac{\mathrm{FPdim}(\mathcal{D}_{\mathrm{pt}})}{\mathrm{FPdim}(\mathcal{D}_{\mathrm{ad}}\cap\mathcal{D}')}. \tag{3.1}$$

We begin with the following lemmas.

**Lemma 3.1.** *Let $n$ be an odd integer. Assume that $\mathcal{C}$ is a slightly degenerate fusion category such that FPdim$(\mathcal{C})=2n$. If $\mathcal{C}_{ad}\cap\mathcal{C}'=$ Vec, then $\mathcal{C}$ splits.*

*Proof.* Notice that $U_{\mathcal{C}}$ is an abelian group, Eq. (3.1) shows that 2 divides $|U_{\mathcal{C}}|$ since $\mathcal{C}_{\mathrm{ad}}\cap\mathcal{C}'=\mathrm{Vec}$, thus $\mathcal{C}$ admits a faithful $\mathbb{Z}_2$-grading $\mathcal{C}=\oplus_{g\in\mathbb{Z}_2}\mathcal{C}_g$ by [5, Corollary 3.7]. [3, Proposition 8.20] says that $\mathrm{FPdim}(\mathcal{C}_e)=n$, so $\mathcal{C}_e\cap\mathcal{C}'=\mathrm{Vec}$ as $n$ is an odd integer, hence the Frobenius-Perron dimension of the centralizer of $\mathcal{C}_e$ in $\mathcal{C}$ is 2 by [2, Theorem 3.14]. Thus, the centralizer of $\mathcal{C}_e$ in $\mathcal{C}$ must be $\mathcal{C}'$, which implies that $\mathcal{C}_e$ is a non-degenerate fusion category, as Müger center $(\mathcal{C}_e)'=\mathcal{C}_e\cap\mathcal{C}'=\mathrm{Vec}$, and $\mathcal{C}\cong\mathrm{sVec}\boxtimes\mathcal{C}_e$ as braided fusion category by [2, Theorem 3.13], as desired. □

**Lemma 3.2.** *Let $p,q$ be distinct odd primes, and let $a,b$ be positive integers. Assume that $\mathcal{D}$ is a slightly degenerate fusion category of Frobenius-Perron dimension $2p^aq^b$. Then $\mathcal{D}_{pt}\not\supseteq\mathcal{D}'$. In particular, $U_{\mathcal{D}}$ is non-trivial.*

*Proof.* If $\mathcal{D}$ is pointed, there is nothing to prove. Assume that $\mathcal{D}$ is not pointed below. Notice that $\mathcal{D}$ is solvable by [9, Proposition 3.14], then $\mathcal{D}$ contains a non-trivial Tannakian subcategory $\mathcal{E}$ by [7, Theorem 3.1], which is also solvable by [4, Theorem 1.6]. Then [4, Proposition 4.5] says that $\mathcal{E}_{\mathrm{pt}}$ is non-trivial, so $\mathcal{D}_{\mathrm{pt}}\not\supseteq\mathcal{D}'$. □

The following proposition is based on the classification results of [10], and it will play a key role in the classification theorems below.

**Proposition 3.1.** *Let $\mathcal{C}$ be a non-degenerate fusion category of Frobenius-Perron dimension $p^2q^2$, $p^2q^3$ or $p^3q^3$, where $p\neq q$ are odd primes. Then $\mathcal{C}$ is pointed or $\mathcal{C}$ contains a Tannakian fusion category of Frobenius-Perron dimension $pq$.*

*Proof.* It follows from [10, Corollary 3.5] that $\mathcal{C}$ is a group-theoretical fusion category. Assume that $\mathcal{C}$ is not pointed below, let $\mathcal{E}$ be a maximal Tannakian subcategory of $\mathcal{C}$, the existence of $\mathcal{E}$ is guaranteed by [6, Theorem 7.2]. Then $\mathrm{FPdim}(\mathcal{E})\in\{p,q,pq\}$, as $\mathrm{FPdim}(\mathcal{E})^2$ divides $\mathrm{FPdim}(\mathcal{C})$ [2, Theorem 3.13]. If $\mathrm{FPdim}(\mathcal{E})$ is a prime, then $\mathcal{E}$ is a pointed fusion category, thus [1, Corollary 4.14] says that $\mathcal{E}'_{\mathcal{C}}$ is nilpotent as $(\mathcal{E}'_{\mathcal{C}})_{\mathrm{ad}}\subseteq\mathcal{E}$, which then implies that $\mathcal{E}'_{\mathcal{C}}$ is braided tensor equivalent to a Deligne tensor product of fusion subcategories of prime power Frobenius-Perron dimensions by [1, Theorem 6.10]. Obviously, this also means that $\mathcal{C}$ is a nilpotent fusion category, then $\mathcal{C}$ must be pointed, it is a contradiction. Hence $\mathrm{FPdim}(\mathcal{E})=pq$. □

**Lemma 3.3.** *Let $\mathcal{C}$ be a slightly degenerate fusion category of Frobenius-Perron dimension $2p^2q^2$, $2p^2q^3$, or $2p^3q^3$, where $p,q$ are distinct odd primes. If $\mathcal{C}$ contains a Tannakian subcategory of Frobenius-Perron dimension $pq$, then $\mathcal{C}$ is group-theoretical.*

*Proof.* By [9, Theorem 2.8], we know that $\mathcal{C}$ is an integral fusion category. Let $\mathcal{E}=\mathrm{Rep}(G)\subseteq\mathcal{C}$ be a Tannakian subcategory of Frobenius-Perron dimension $pq$, then $\mathcal{C}_G$ is a faithfully $G$-graded fusion category by [2, Proposition 4.56]. Notice that $\mathcal{C}_G^0$ is a slightly degenerate fusion category of square-free Frobenius-Perron dimension by [2, Proposition 4.30], which must be pointed by [9, Corollary 3.4]. Since $(\mathcal{C}_G)_{\mathrm{ad}}\subseteq\mathcal{C}_G^0$ by [5, Corollary 3.7], $\mathcal{C}_G$ is nilpotent. Therefore, we deduce from [5, Corollary 5.3] that $\mathcal{C}_G$ is also pointed, hence $\mathcal{C}$ is a group-theoretical fusion category by [6, Theorem 7.2]. □

Hence, in order to prove the group-theoretical property of slightly degenerate fusion categories $\mathcal{C}$ of Frobenius-Perron dimensions $2p^2q^2$, $2p^2q^3$, or $2p^3q^3$, it suffices to show that $\mathcal{C}$ contains a Tannakian fusion subcategory of Frobenius-Perron dimension $pq$.

**Theorem 3.1.** *Let $p<q$ be odd primes. Assume that $\mathcal{D}$ is a slightly degenerate fusion category of Frobenius-Perron dimension $2p^2q^2$. Then $\mathcal{D}$ is a group-theoretical fusion category.*

*Proof.* If $\mathcal{D}_{\rm ad}\cap\mathcal{D}'=\text{Vec}$, then we deduce from Lemma 3.1 that $\mathcal{D}$ is split, so $\mathcal{D}$ is group-theoretical by [10, Theorem 3.4]. Thus, we only need to consider the case when $\mathcal{D}'\subseteq\mathcal{D}_{\rm ad}$. We assume that $\mathcal{D}$ is not pointed below, and note that $\mathcal{D}$ is not nilpotent. Hence

$$\text{FPdim}(\mathcal{D}_{\rm ad})\in\{2p,2q,2pq,2p^2,2q^2,2p^2q,2pq^2,2p^2q^2\}.$$

Obviously, $\text{FPdim}(\mathcal{D}_{\rm ad})\neq 2p^2q^2$ by Lemma 3.1 and equation (3.1). Since $(\mathcal{D}_{\rm ad})'=\mathcal{D}_{\rm pt}$ by [2, Proposition 3.29], $\mathcal{D}_{\rm ad}$ can not pointed, particularly, $\mathcal{D}_{\rm ad}$ is not symmetric. While notice that if $\text{FPdim}(\mathcal{D}_{\rm ad})=2p$, $2q$, $2p^2$ or $2q^2$, then the Müger center of $\mathcal{D}_{\rm ad}$ is $(\mathcal{D}_{\rm ad})_{\rm pt}\cong\mathcal{D}'=\text{sVec}$, this is impossible, as $\mathcal{D}_{\rm ad}$ is slightly degenerate, which must be pointed by [9, Lemma 3.5].

If $\text{FPdim}(\mathcal{D}_{\rm ad})=2pq$, again by [9, Lemma 3.5] $\mathcal{D}_{\rm ad}$ can not be slightly degenerate, so the Müger center of $\mathcal{D}_{\rm ad}$ contains a non-trivial Tannakian subcategory of Frobenius-Perron dimension $p$ or $q$, as $\mathcal{D}_{\rm ad}$ is not symmetric. Consequently, we have the following braided tensor equivalences

$$\mathcal{D}_{\rm ad}\cong(\text{sVec}\boxtimes\mathcal{C}(\mathbb{Z}_p,\eta_1))^{\mathbb{Z}_q}\ \text{ or }\ \mathcal{D}_{\rm ad}\cong(\text{sVec}\boxtimes\mathcal{C}(\mathbb{Z}_q,\eta_2))^{\mathbb{Z}_p},$$

thus, as a fusion category, we obtain that

$$\mathcal{D}_{\rm ad}\cong\text{sVec}^{\mathbb{Z}_q}\vee\mathcal{C}(\mathbb{Z}_p,\eta_1)^{\mathbb{Z}_q}\ \text{ or }\ \mathcal{D}_{\rm ad}\cong\text{sVec}^{\mathbb{Z}_q}\vee\mathcal{C}(\mathbb{Z}_q,\eta_2)^{\mathbb{Z}_p},$$

however, each of these two tensor equivalences implies that $\mathcal{D}_{\rm ad}$ is pointed, since $\text{sVec}^{\mathbb{Z}_q}\cong\text{sVec}\boxtimes\text{Rep}(\mathbb{Z}_q)$ as symmetric fusion category, and for odd primes $p\neq q$,

$$\mathcal{C}(\mathbb{Z}_p,\eta_1)^{\mathbb{Z}_q}\cong\mathcal{C}(\mathbb{Z}_p,\eta_1)\boxtimes\text{Rep}(\mathbb{Z}_q),\ \mathcal{C}(\mathbb{Z}_q,\eta_2)^{\mathbb{Z}_p}\cong\mathcal{C}(\mathbb{Z}_q,\eta_2)\boxtimes\text{Rep}(\mathbb{Z}_q)$$

as braided fusion categories, it is a contradiction.

If $\text{FPdim}(\mathcal{D}_{\rm ad})=2p^2q$, then $\mathcal{D}_{\rm pt}\subseteq\mathcal{D}_{\rm ad}$ and $\mathcal{D}_{\rm pt}$ contains a Tannakian category of Frobenius-Perron dimension $q$. In fact, if not, $\mathcal{D}_{\rm pt}\cong\text{sVec}\boxtimes\mathcal{C}(\mathbb{Z}_q,\eta)$, then $\mathcal{D}\cong\mathcal{C}(\mathbb{Z}_q,\eta)\boxtimes\mathcal{C}(\mathbb{Z}_q,\eta)'_{\mathcal{C}}$ as braided fusion category, and $\mathcal{D}$ must be pointed by [9, Corollary 3.4]. Thus,

$$\mathcal{D}_{\rm ad}\cong(\text{sVec}\boxtimes\mathcal{C}(G,\eta))^{\mathbb{Z}_q}\cong\text{sVec}^{\mathbb{Z}_q}\vee\mathcal{C}(G,\eta)^{\mathbb{Z}_q}\supseteq\mathcal{C}(G,\eta)^{\mathbb{Z}_q},$$

where $(G,\eta)$ is a metric group of order $p^2$. Thus the proof of [10, Theorem 3.3] shows that $\mathcal{C}(G,\eta)^{\mathbb{Z}_q}$ can be embedded into a non-degenerate fusion category of Frobenius-Perron dimension $p^2q^2$, moreover, since we have assumed that $\mathcal{C}_{\rm ad}$ is not pointed, $\mathcal{C}(G,\eta)^{\mathbb{Z}_q}$ contains a Tannakian subcategory of Frobenius-Perron dimension $pq$ by Proposition 3.1 and [10, Theorem 3.3], so $\mathcal{D}$ is a group-theoretical fusion category Lemma 3.3. The subcase $\text{FPdim}(\mathcal{D}_{\rm ad})=2pq^2$ can be proved in the same way. □

**Theorem 3.2.** *Let p and q be odd primes, then slightly degenerate fusion categories of Frobenius-Perron dimension* $2p^2q^3$ *are group-theoretical.*

*Proof.* Let $\mathcal{C}$ be a slightly degenerate fusion category of FP-dimensions $2p^2q^3$. We assume that $\mathcal{C}$ is not pointed below. Same as Theorem 3.1, we only need to prove the theorem when $\mathcal{C}'\subseteq\mathcal{C}_{\rm ad}$. Then we have that

$$\mathrm{FPdim}(\mathcal{C}_{\rm ad})\in\{2,2p,2q,2pq,2p^2,2q^2,2p^2q,2pq^2,2q^3,2p^2q^2,2pq^3,2p^2q^3\}.$$

The arguments of Theorem 3.1 show that

$$\mathrm{FPdim}(\mathcal{C}_{\rm ad})\notin\{2,2p,2q,2pq,2p^2,2q^2,2q^3,2p^2q^3\}.$$

If $\mathrm{FPdim}(\mathcal{C}_{\rm ad})=2p^2q^2$ and $\mathrm{FPdim}(\mathcal{C}_{\rm pt})=2q$, then $\mathrm{FPdim}((\mathcal{C}_{\rm ad})_{\rm pt})=2q$ by Lemma 3.2, which them implies that $(\mathcal{C}_{\rm ad})_{\mathbb{Z}_q}$ is slightly degenerate pointed fusion category by [2, Proposition 4.30]. Thus, the same argument of Theorem 3.1 implies that $\mathcal{C}_{\rm ad}$ contains a pointed fusion subcategory $\mathcal{C}(\mathbb{Z}_q,\eta)^{\mathbb{Z}_q}$ of Frobenius-Perron dimension $q^2$, it is impossible.

If $\mathrm{FPdim}(\mathcal{C}_{\rm ad})=2pq^2$ and $\mathrm{FPdim}(\mathcal{C}_{\rm pt})=2pq$, then we obtain $(\mathcal{C}_{\rm ad})_{\rm pt}=2p$ or $2pq$. In fact, if $\mathrm{FPdim}((\mathcal{C}_{\rm ad})_{\rm pt})=2q$, then previous argument shows that $\mathcal{C}_{\rm ad}$ contains a pointed fusion subcategory of Frobenius-Perron dimension $q^2$, impossible. If $\mathrm{FPdim}((\mathcal{C}_{\rm ad})_{\rm pt})=2p$, then

$$\mathcal{C}_{\rm ad}\cong(\mathrm{sVec}\boxtimes\mathcal{C}(G,\eta))^{\mathbb{Z}_p}\cong\mathrm{sVec}^{\mathbb{Z}_p}\vee\mathcal{C}(G,\eta)^{\mathbb{Z}_p},$$

where $\mathcal{C}(G,\eta)$ is a non-degenerate fusion category of Frobenius-Perron dimension $q^2$, the arguments of Theorem 3.1 and [10, Theorem 3.3] imply that $\mathcal{C}$ contains a Tannakian fusion subcategory of FP-dimension $pq$. If $\mathrm{FPdim}((\mathcal{C}_{\rm ad})_{\rm pt})=2pq$, then $\mathcal{C}_{\rm pt}$ is symmetric. Thus $\mathcal{C}$ is group-theoretical by Lemma 3.3. If $\mathrm{FPdim}(\mathcal{C}_{\rm ad})=2p^2q$, group-theoretical property of $\mathcal{C}$ can be proved similarly.

If $\mathrm{FPdim}(\mathcal{C}_{\rm ad})=2pq^3$, then $\mathcal{C}_{\rm pt}\subseteq\mathcal{C}_{\rm ad}$ is a symmetric fusion category of Frobenius-Perron dimension $2p$, and $\mathcal{C}_{\rm ad}\cong\mathrm{sVec}^{\mathbb{Z}_p}\vee\mathcal{D}^{\mathbb{Z}_p}$, where $\mathcal{D}$ is a non-degenerate pointed fusion category of Frobenius-Perron dimension $q^3$. By [10, Theorem 3.3], $\mathcal{D}^{\mathbb{Z}_p}$ (then $\mathcal{C}$) contains a Tannakian fusion subcategory of Frobenius-Perron dimension $pq$, so $\mathcal{C}$ is group-theoretical by Lemma 3.3. □

**Theorem 3.3.** *Let p and q be odd primes, then slightly degenerate fusion categories of Frobenius-Perron dimension* $2p^3q^3$ *are group-theoretical.*

*Proof.* We always assume $\mathcal{C}$ is not pointed below. In particular, $\mathcal{C}$ is not nilpotent. If $\mathcal{C}_{\rm ad}\cap\mathcal{C}'=\mathrm{sVec}$, then it is easy to see that

$$\mathrm{FPdim}(\mathcal{C}_{\rm ad})\notin\{2,2p,2q,2pq,2p^2,2q^2,2p^3,2q^3,2p^2q^3,2p^3q^2,2p^3q^3\}.$$

Therefore,

$$\mathrm{FPdim}(\mathcal{C}_{\rm ad})\in\{2p^2q,2pq^2,2p^3q,2p^2q^2,2pq^3\}.$$

Using the same arguments of Theorems 3.1 and 3.2, one can prove that $\mathcal{C}$ contains a Tannakian fusion subcategory of Frobenius-Perron dimension $pq$. Therefore, $\mathcal{C}$ is a group-theoretical fusion category by Lemma 3.3. □

The following corollary is deduced from [8, Corollary 4.7] and [9, Theorem 2.8].

**Corollary 3.1.** *Let $d$ be an odd square-free integer such that $(pq,d)=1$. Then slightly degenerate fusion categories of Frobenius-Perron dimensions $2p^2q^2d$, $2p^2q^3d$ and $2p^3q^3d$ are group-theoretical.*

*Proof.* Let $\mathcal{C}$ be a slightly degenerate fusion category with the assumed Frobenius-Perron dimension. Then $\mathcal{C}$ is an integral fusion category by [9, Theorem 2.8], and $(\mathrm{FPdim}(X),d)=1$ for all simple objects $X$ of $\mathcal{C}$ by [9, Corollary 3.4]. It follows from [8, Corollary 4.7] that $\mathcal{C}$ contains a non-degenerate fusion subcategory $\mathcal{C}(\mathbb{Z}_d,\eta)$, thus $\mathcal{C}\cong\mathcal{C}(\mathbb{Z}_d,\eta)\boxtimes\mathcal{C}(\mathbb{Z}_d,\eta)'_{\mathcal{C}}$ as braided fusion category by [2, Theorem 3.13], then the proposition follows. □

**Remark 3.1.** Let $\mathcal{C}$ be a slightly degenerate fusion category of Frobenius-Perron dimension $2p^2q^2$, $2p^2q^3$ or $2p^3q^3$. Is $\mathcal{C}$ always split? Notice that $\mathcal{C}$ is split if and only if $\mathcal{C}'\cap\mathcal{C}_{\mathrm{ad}}=$ Vec by Lemma 3.1.

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