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Besov Spaces with General Weights

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Abstract. We introduce Besov spaces with general smoothness. These spaces unify and generalize the classical Besov spaces. We establish the φ-transform characterization of these spaces in the sense of Frazier and Jawerth and we prove their Sobolev embeddings. We establish the smooth atomic, molecular and wavelet decomposition of these function spaces. A characterization of these function spaces in terms of the difference relations is given.

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1 Introduction

Function spaces have been a central topic in modern analysis, and are now of increasing applications in many fields of mathematics especially harmonic analysis and partial differential equations. The most known general scales of function spaces are the scales of Besov spaces and Triebel-Lizorkin spaces and it is known that they cover many well-known classical function spaces such as Hölder-Zygmund spaces, Hardy spaces and Sobolev spaces. For more details one can refer to Triebel's books [58–60].

In recent years many researchers have modified the classical spaces and have generalized the classical results to these modified ones. For example: Function spaces of generalized smoothness. These types of function spaces have been introduced by several authors. We refer, for instance, to Bownik [8], Cobos and Fernandez [14], Goldman [30] and [31], and Kalyabin [38]; see also Kalyabin and Lizorkin [39].

The theory of these spaces had a remarkable development in part due to its usefulness in applications. For instance, they appear in the study of trace spaces on fractals, see Edmunds and Triebel [22, 23], where they introduced the spaces $B_{p,q}^{s,\Psi}$, where Ψ is a so-called admissible function, typically of log-type near 0. For a complete treatment of these spaces we refer the readers to the work of Moura [46].

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Besov [2–4] defined function spaces of variable smoothness and obtained their characterizations by differences, interpolation, embeddings and extension. Such spaces are a special case of the so-called 2-microlocal function spaces. The concept of 2-microlocal analysis, or 2-microlocal function spaces, is due to Bony [5]. These type of function spaces have been studied in detail in [40]. We mention the papers [41, 42] and references given therein.

More general function spaces of generalized smoothness can be found in Farkas and Leopold [24], and reference therein.

Tyulenev has introduced in [62] a new family of Besov spaces of variable smoothness which cover many classes of Besov spaces, where the norm on these spaces was defined with the help of classical differences.

Based on this weighted class and the Fourier-analytical methods we introduce Besov spaces of variable smoothness consisting of tempered distributions and present their essential properties such as the φ-transforms characterization, Sobolev embeddings, atomic, molecular and wavelet decompositions.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basic facts on the Muckenhoupt classes and the weighted class of Tyulenev. Also we give some key technical lemmas needed in the proofs of the main statements. We then define the Besov spaces as follows. Let $\mathcal{S}(\mathbb{R}^n)$ be the set of all Schwartz functions φ on $\mathbb{R}^n$, i.e., φ is infinitely differentiable and

$$\big\|\varphi|\mathcal{S}_M\big\| := \sup_{\beta\in\mathbb{N}_0^n, |\beta|\le M} \sup_{x\in\mathbb{R}^n} |\partial^\beta\varphi(x)|(1+|x|)^{n+M+|\beta|} < \infty$$

for all $M\in\mathbb{N}$. Select a Schwartz function φ such that

$$\operatorname{supp}\mathcal{F}\varphi \subset \{\xi : \frac{1}{2} \le |\xi| \le 2\}$$

and

$$|\mathcal{F}\varphi(\xi)| \ge c, \qquad \text{if} \quad \frac{3}{5} \le |\xi| \le \frac{5}{3},$$

where $c>0$ and we put $\varphi_k = 2^{kn}\varphi(2^k\cdot)$, $k\in\mathbb{Z}$. Here $\mathcal{F}\varphi$ denotes the Fourier transform of φ, defined by

$$\mathcal{F}\varphi(\xi) := (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \qquad \xi\in\mathbb{R}^n.$$

Let

$$\mathcal{S}_\infty(\mathbb{R}^n) := \Big\{\varphi\in\mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} x^\beta\varphi(x)dx = 0 \text{ for all multi-indices } \beta\in\mathbb{N}_0^n\Big\}.$$

Following Triebel [58], we consider $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of $\mathcal{S}(\mathbb{R}^n)$, including the topology. Thus, $\mathcal{S}_\infty(\mathbb{R}^n)$ is a complete metric space. Equivalently, $\mathcal{S}_\infty(\mathbb{R}^n)$ can be defined as a collection of all $\varphi\in\mathcal{S}(\mathbb{R}^n)$ such that semi-norms

$$\|\varphi\|_M := \sup_{|\beta|\le M}\sup_{\xi\in\mathbb{R}^n} |\partial^\beta\varphi(\xi)|\big(|\xi|^M + |\xi|^{-M}\big) < \infty$$

for all $M \in \mathbb{N}_0$, see [7, Section 3]. The semi-norms $\{\|\cdot\|_M\}_{M\in\mathbb{N}_0}$ generate a topology of a locally convex space on $\mathcal{S}_\infty(\mathbb{R}^n)$ which coincides with the topology of $\mathcal{S}_\infty(\mathbb{R}^n)$ as a subspace of a locally convex space $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'_\infty(\mathbb{R}^n)$ be the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$. Let $0<p\leq\infty$ and $0<q\leq\infty$. Let $\{t_k\}$ be a p-admissible sequence i.e. $t_k \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, $k\in\mathbb{Z}$. The Besov-type space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ is the collection of all $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| := \Big(\sum_{k=-\infty}^{\infty}\big\|t_k(\varphi_k * f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}} < \infty$$

with the usual modifications if $q=\infty$. In this section several basic properties such as the φ-transform characterization are obtained and we extend well-known embeddings to these function spaces. The main statements are formulated in Section 4, where we give the atomic, molecular and wavelet decomposition of these function spaces. In Section 6 we study the inhomogeneous spaces $B_{p,q}(\mathbb{R}^n,\{t_k\})$, and we outline analogous results for these spaces. In addition we present an characterization of these function spaces in terms of the difference relations.

2 Some fundamental maximal inequalities

Our arguments of this paper are essentially rely on the weighted boundedness of Hardy-Littlewood maximal function. In this paper we will assume that the weight sequence $\{t_k\}$ used to define the space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ lies in the new weighted class $\dot{X}_{\alpha,\sigma,p}$ (see Definition 2.3). Therefore we need a new version of Hardy-Littlewood maximal inequality.

2.1 Notation and conventions

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f\lesssim g$ means that $f\leq cg$ for some independent constant c (and non-negative functions f and g), and $f\approx g$ means $f\lesssim g\lesssim f$. As usual for any $x\in\mathbb{R}$, $\lfloor x\rfloor$ stands for the largest integer smaller than or equal to x.

For $x\in\mathbb{R}^n$ and $r>0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. By $\mathrm{supp} f$ we denote the support of the function f, i.e., the closure of its non-zero set. If $E\subset\mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function. By c we denote generic positive constants, which may have different values at different occurrences.

A weight is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. For a measurable set $E\subset\mathbb{R}^n$ and a weight γ, $\gamma(E)$ denotes

$$\int_E \gamma(x)dx.$$

Given a measurable set $E\subset\mathbb{R}^n$ and $0<p\leq\infty$, we denote by $L_p(E)$ the space of all functions $f:E\to\mathbb{C}$ equipped with the quasi-norm

$$\big\|f|L_p(E)\big\|:=\Big(\int_E|f(x)|^p dx\Big)^{\frac{1}{p}}<\infty$$

with $0<p<\infty$ and

$$\big\|f|L_\infty(E)\big\|:=\operatorname*{ess\text{-}sup}_{x\in E}|f(x)|<\infty.$$

For a function f in $L_1^{\mathrm{loc}}(\mathbb{R}^n)$, we set

$$M_A(f):=\frac{1}{|A|}\int_A|f(x)|dx$$

for any $A\subset\mathbb{R}^n$. Furthermore, we put

$$M_{A,p}(f):=\Big(\frac{1}{|A|}\int_A|f(x)|^p dx\Big)^{\frac{1}{p}}$$

with $0<p<\infty$. Notice that if $0<p<q<\infty$, then

$$M_{A,p}(f)\leq M_{A,q}(f).$$

Further, given a measurable set $E\subset\mathbb{R}^n$ and a weight γ, we denote the space of all functions $f:\mathbb{R}^n\to\mathbb{C}$ with finite quasi-norm

$$\big\|f|L_p(\mathbb{R}^n,\gamma)\big\|:=\big\|f\gamma|L_p(\mathbb{R}^n)\big\|$$

by $L_p(\mathbb{R}^n,\gamma)$.

Let $0<p,q\leq\infty$. The space $\ell^q(L_p)$ is defined to be the set of all sequences $\{f_k\}$ of functions such that

$$\big\|\{f_k\}|\ell^q(L_p)\big\|:=\Big(\sum_{k=-\infty}^{\infty}\big\|f_k|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}<\infty$$

with the usual modifications if $q=\infty$.

Let $0<p\leq\infty$. The space $\ell^p(\mathbb{Z}^n)$ is defined to be the set of all sequences $u=\{u_m\}_{m\in\mathbb{Z}^n}\subset\mathbb{C}$ such that

$$\big\|u|\ell^p(\mathbb{Z}^n)\big\|:=\Big(\sum_{m\in\mathbb{Z}^n}|u_m|^p\Big)^{\frac{1}{p}}<\infty,$$

with the usual modifications if $p=\infty$. Let $0<p,q\leq\infty$. The space $\ell^q(\ell^p(\mathbb{Z}^n))$ is defined to be the set of all sequences $a=\{a_k\}_{k\in\mathbb{Z}},a_k\in\ell^p(\mathbb{Z}^n)$ such that

$$\big\|a|\ell^q(\ell^p(\mathbb{Z}^n))\big\|<\infty,$$

where

$$\big\|a|\ell^q(\ell^p(\mathbb{Z}^n))\big\|:=\Big(\sum_{k=-\infty}^{\infty}\big\|a_k|\ell^p(\mathbb{Z}^n)\big\|^q\Big)^{\frac{1}{q}},$$

with the usual modifications if $q=\infty$ or $p=\infty$.

If $1\leq p\leq\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, then p' is called the conjugate exponent of p. In what follows, Q will denote a cube in the space $\mathbb{R}^n$ with sides parallel to the coordinate axes and $l(Q)$ will denote the side length of the cube Q. For all cubes Q and $r>0$, let rQ be the cube concentric with Q having the side length $rl(Q)$. For $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^n$, denote by $Q_{k,m}$ the dyadic cube,

$$Q_{k,m}:=2^{-k}([0,1)^n+m).$$

For the collection of all such cubes we use

$$\mathcal{Q}:=\{Q_{k,m}:k\in\mathbb{Z},m\in\mathbb{Z}^n\}.$$

For each cube Q, we denote by $x_{k,m}$ the lower left-corner $2^{-k}m$ of $Q=Q_{k,m}$. Also, we set $\chi_{k,m}=\chi_{Q_{k,m}}$,$k\in\mathbb{Z}$, $m\in\mathbb{Z}^n$.

Recall that $\eta_{k,m}(x):=2^{nk}(1+2^k|x|)^{-m}$, for any $x\in\mathbb{R}^n$, $k\in\mathbb{Z}$ and $m>0$. Note that $\eta_{k,m}\in L^1(\mathbb{R}^n)$ when $m>n$ and that $\|\eta_{k,m}\|_1=c_m$ is independent of k, where this type of function was introduced in [18] and [35].

2.2 Muckenhoupt weights

The purpose of this subsection is to review some known properties of the Muckenhoupt class.

Definition 2.1. *Let $1<p<\infty$. We say that a weight γ belongs to the Muckenhoupt class $A_p(\mathbb{R}^n)$ if there exists a constant $C>0$ such that the following inequality holds:*

$$\sup_{Q:\,cube\ in\ \mathbb{R}^n} M_Q(\gamma)M_{Q,\frac{p'}{p}}(\gamma^{-1})\leq C. \tag{2.1}$$

The smallest constant C, for which (2.1) holds, is denoted by $A_p(\gamma)$. As an example, we can take

$$\gamma(x)=|x|^\alpha,\quad \alpha\in\mathbb{R}.$$

Then $\gamma\in A_p(\mathbb{R}^n)$, $1<p<\infty$, if and only if $-n<\alpha<n(p-1)$. For $p=1$ we rewrite the above definition in the following way.

Definition 2.2. *We say that a weight γ belongs to the Muckenhoupt class $A_1(\mathbb{R}^n)$ if there exists a constant $C>0$ such that the following inequality holds:*

$$\sup_{Q:\,cube\ in\ \mathbb{R}^n} M_Q(\gamma)\big\|\gamma^{-1}|L_\infty(Q)\big\|\leq C. \tag{2.2}$$

The smallest constant C, for which (2.2) holds, will be denoted by $A_1(\gamma)$. The above classes have been first studied by Muckenhoupt [47] and use to characterize the boundedness of the Hardy-Littlewood maximal function on $L_p(\mathbb{R}^n,\gamma)$, see the monographs [28] and [32] for a complete account on the theory of Muckenhoupt weights.

We recall a few basic properties of the class $A_p(\mathbb{R}^n)$ weights, see [21, Chapter 7], [32, Chapter 7] and [57, Chapter 5].

Lemma 2.1. *Let $1 \le p < \infty$.*

- (i) *If $\gamma \in A_p(\mathbb{R}^n)$, then for any $1 \le p < q$, $\gamma \in A_q(\mathbb{R}^n)$.*
- (ii) *Let $1 < p < \infty$. $\gamma \in A_p(\mathbb{R}^n)$ if and only if $\gamma^{1-p'} \in A_{p'}(\mathbb{R}^n)$.*
- (iii) *Let $\gamma \in A_p(\mathbb{R}^n)$. There is $C > 0$ such that for any cube Q and a measurable subset $E \subset Q$*
$$\Big(\frac{|E|}{|Q|}\Big)^{p-1} M_Q(\gamma) \le C M_E(\gamma).$$
- (iv) *Suppose that $\gamma \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then there exists $1 < p_1 < p < \infty$ such that $\gamma \in A_{p_1}(\mathbb{R}^n)$.*
- (v) *Let $1 \le p < \infty$ and $\gamma \in A_p(\mathbb{R}^n)$. Then there exist $\delta \in (0,1)$ and $C > 0$ depending only on n, p, and $A_p(\gamma)$ such that for any cube Q and any measurable subset S of Q we have*
$$\frac{M_S(\gamma)}{M_Q(\gamma)} \le C \Big(\frac{|S|}{|Q|}\Big)^{\delta-1}.$$

The following theorem gives a useful property of $A_p(\mathbb{R}^n)$ weights (reverse Hölder inequality), see [32, Chapter 7] or [45, Chapter 1].

Theorem 2.1. *Let $1 \le p < \infty$ and $\gamma \in A_p(\mathbb{R}^n)$. Then there exist constants $C > 0$ and $\varepsilon_\gamma > 0$ depending only on n,p and on $A_p(\gamma)$, such that for every cube Q,*
$$M_{Q,1+\varepsilon_\gamma}(\gamma) \le C M_Q(\gamma).$$

2.3 The weight class $\dot{X}_{\alpha,\sigma,p}$

Let $0 < p \le \infty$. A weight sequence $\{t_k\}$ is called p-admissible if $t_k \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ for all $k \in \mathbb{Z}$. We mention here that
$$\Big(\int_E t_k^p(x)dx\Big)^{\frac{1}{p}} \le c(k)$$
for any $k \in \mathbb{Z}$ and any compact set $E \subset \mathbb{R}^n$ with the usual modifications if $p = \infty$. For a p-admissible weight sequence $\{t_k\}$ we set
$$t_{k,m} := \big\| t_k | L_p(Q_{k,m}) \big\|, \quad k \in \mathbb{Z},\ m \in \mathbb{Z}^n.$$

Tyulenev [61] introduced the following new weighted class and use it to study Besov spaces of variable smoothness.

Definition 2.3. *Let α_1, $\alpha_2 \in \mathbb{R}$, p, σ_1, $\sigma_2 \in (0,+\infty]$, $\alpha=(\alpha_1,\alpha_2)$ and let $\sigma=(\sigma_1,\sigma_2)$. We let $\dot{X}_{\alpha,\sigma,p}=\dot{X}_{\alpha,\sigma,p}(\mathbb{R}^n)$ denote the set of p-admissible weight sequences $\{t_k\}$ satisfying the following conditions. There exist numbers $C_1,C_2>0$ such that for any $k\leq j$ and every cube Q,*

$$M_{Q,p}(t_k)M_{Q,\sigma_1}(t_j^{-1})\leq C_1 2^{\alpha_1(k-j)}, \tag{2.3}$$

$$M_{Q,p}^{-1}(t_k)M_{Q,\sigma_2}(t_j)\leq C_2 2^{\alpha_2(j-k)}. \tag{2.4}$$

The constants $C_1,C_2>0$ are independent of both the indexes k and j.

Remark 2.1. (i) We would like to mention that if $\{t_k\}$ satisfies (2.3) with $\sigma_1=r\left(\frac{p}{r}\right)'$ and $0<r<p\leq\infty$, then $t_k^p\in A_{\frac{p}{r}}(\mathbb{R}^n)$ for any $k\in\mathbb{Z}$ with $0<r<p<\infty$ and $t_k^{-r}\in A_1(\mathbb{R}^n)$ for any $k\in\mathbb{Z}$ with $p=\infty$.
(ii) We say that $t_k\in A_p(\mathbb{R}^n)$, $k\in\mathbb{Z}$, $1<p<\infty$ have the same Muckenhoupt constant if

$$A_p(t_k)=c, \quad k\in\mathbb{Z},$$

where c is independent of k.
(iii) Definition 2.3 is different from the one used in [61, Definition 2.1] and Definition 2.7 in [62], because we used the boundedness of the maximal function on weighted Lebesgue spaces.

Example 2.1. Let $0<r<p<\infty$, a weight $\omega^p\in A_{\frac{p}{r}}(\mathbb{R}^n)$ and $\{s_k\}=\{2^{ks}\omega^p\}_{k\in\mathbb{Z}}$, $s\in\mathbb{R}$. Clearly, $\{s_k\}_{k\in\mathbb{Z}}$ lies in $\dot{X}_{\alpha,\sigma,p}$ for $\alpha_1=\alpha_2=s$, $\sigma=(r(\frac{p}{r})',p)$.

Remark 2.2. Let $0<\theta\leq p<\infty$. Let α_1, $\alpha_2\in\mathbb{R}$, $\sigma_1,\sigma_2\in(0,+\infty]$, $\sigma_2\geq p$, $\alpha=(\alpha_1,\alpha_2)$ and let $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2)$. Let $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence. Then

$$\alpha_2\geq\alpha_1.$$

Indeed, applying Hölder's inequality we find that for any cube Q

$$1=\Big(\frac{1}{|Q|}\int_Q t_j^{-\theta}(y)t_j^{\theta}(y)dy\Big)^{\frac{1}{\theta}}\leq M_{Q,p}(t_j)M_{Q,\sigma_1}(t_j^{-1}), \quad j\in\mathbb{Z}.$$

Hence

$$M_{Q,\sigma_1}^{-1}(t_j^{-1})\leq M_{Q,p}(t_j)$$

and since $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$, we get for any $k\leq j$

$$M_{Q,p}^{-1}(t_k)M_{Q,\sigma_1}^{-1}(t_j^{-1})\leq C2^{\alpha_2(j-k)}.$$

Therefore, for some positive constant c independent of k and j the following estimate is valid:

$$2^{(\alpha_2-\alpha_1)(j-k)}\geq c,$$

that obviously implies $\alpha_2\geq\alpha_1$.

Lemma 2.2. *Let $0<\theta\leq p<\infty$ and $\{t_k\}$ be a p-admissible weight sequence such that $t_k^p\in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k\in\mathbb{Z}$ with*

$$A_{\frac{p}{\theta}}(t_k^p)\leq C,\quad k\in\mathbb{Z},$$

where C is a positive constant independent of k. Then

$$\big\|t_k\eta_{j,N}|L_p(\mathbb{R}^n)\big\|\leq c\,2^{jn(1-\frac{1}{p})}M_{B(0,2^{-j}),p}(t_k)$$

holds for any $N>\frac{n}{\theta}$ and any $j,k\in\mathbb{Z}$ where the positive constant c is independent of k and j. In addition and under the same assumption on N, we have

$$\big\|t_k\eta_{j,N}|L_p(\mathbb{R}^n)\big\|\leq c\,2^{jn(1-\frac{1}{\theta})}M_{B(0,1),p}(t_k),\quad j\leq 0,\, k\in\mathbb{Z},$$

where the positive constant c is independent of k and j.

Proof. This is probably a known result but we include the proof of the first estimate for convenience. We can write

$$2^{jn(1-p)}\big\|t_k\eta_{j,N}|L_p(\mathbb{R}^n)\big\|^p\leq M_{B(0,2^{-j})}(t_k^p)+\int_{\mathbb{R}^n\setminus B(0,2^{-j})}t_k^p(x)\eta_{j,Np}(x)dx.$$

The second integral can be rewritten as

$$\begin{aligned}&\sum_{i=0}^{\infty}\int_{2^{i-j}\leq|x|<2^{i-j+1}}t_k^p(x)\eta_{j,Np}(x)dx\\&\leq\sum_{i=0}^{\infty}2^{-iNp+jn}\int_{2^{i-j}\leq|x|<2^{i-j+1}}t_k^p(x)dx\lesssim\sum_{i=0}^{\infty}2^{-i(N-\frac{n}{\theta})p}M_{B(0,2^{-j})}(t_k^p)\\&\lesssim M_{B(0,2^{-j})}(t_k^p),\end{aligned}$$

since $N>\frac{n}{\theta}$ where in the second estimate we have used Lemma 2.1 (iii). □

Further notation will be properly introduced whenever needed.

2.4 Auxiliary results

In this subsection we present some results which are useful for us. As usual, we put

$$\mathcal{M}(f)(x):=\sup_{Q}\frac{1}{|Q|}\int_{Q}|f(y)|dy,\quad f\in L_1^{\text{loc}}(\mathbb{R}^n),$$

where the supremum is taken over all cubes with sides parallel to the axis and $x\in Q$. Also we set

$$\mathcal{M}_\sigma(f):=\big(\mathcal{M}(|f|^\sigma)\big)^{\frac{1}{\sigma}},\quad 0<\sigma<\infty.$$

Theorem 2.2. *Let* $1<p\leq\infty$. *Then*

$$\big\|\mathcal{M}(f)|L_p(\mathbb{R}^n)\big\|\lesssim\big\|f|L_p(\mathbb{R}^n)\big\|$$

holds for all $f\in L_p(\mathbb{R}^n)$.

For the proof see [32, Chapter]. We need the following version of the Calderón-Zygmund covering lemma, see [15, Lemma 3.3], [16, Appendix A] and [50, Chapter 7].

Lemma 2.3. *Let* f *be a measurable function such that* $\frac{1}{|Q|}\int_Q|f|\to 0$ *as* $|Q|\to\infty$. *Given* $a>2^{n+1}$, *for each* $i\in\mathbb{Z}$ *there exists a disjoint collection of maximal dyadic cubes* $\{Q^{i,h}\}_h$ *such that for each* h,

$$a^i\leq\frac{1}{|Q^{i,h}|}\int_{Q^{i,h}}|f(x)|dx\leq 2^n a^i,$$
$$\Omega_i:=\{x\in\mathbb{R}^n:\mathcal{M}(f)(x)>4^n a^i\}\subset\cup_h 3Q^{i,h}.$$

Let

$$E^i:=\cup_h Q^{i,h},\qquad E^{i,h}:=Q^{i,h}\backslash(Q^{i,h}\cap E^{i+1}).$$

Then $E^{i,h}\subset Q^{i,h}$, *there exists a constant* $\beta>1$, *depending only on* a, *such* $|E^{i,h}|\beta\geq|Q^{i,h}|$ *and the sets* $E^{i,h}$ *are pairwise disjoint for all* i *and* h.

Now we state the main result of this subsection.

Lemma 2.4. *Let* $1<\theta\leq p<\infty$. *Let* $\{t_k\}$ *be a p-admissible weight sequence such that* $t_k^p\in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k\in\mathbb{Z}$. *Assume that* t_k^p, $k\in\mathbb{Z}$, *have the same Muckenhoupt constant,* $A_{\frac{p}{\theta}}(t_k^p)=C, k\in\mathbb{Z}$. *Then*

$$\big\|\mathcal{M}(f_k)|L_p(\mathbb{R}^n,t_k)\big\|\leq c\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\| \tag{2.5}$$

for all sequences of functions $f_k\in L_p(\mathbb{R}^n,t_k)$, $k\in\mathbb{Z}$, *where* $c>0$ *is independent of* k.

Proof. By duality the left-hand side of (2.5) can be estimated by

$$\sup\int_{\mathbb{R}^n}t_k(x)\mathcal{M}(f_k)(x)|g_k(x)|dx=\sup T_k,$$

where the supremum is taken over all sequences of functions $g_k\in L_{p'}(\mathbb{R}^n)$ with

$$\big\|g_k|L_{p'}(\mathbb{R}^n)\big\|\leq 1,\quad k\in\mathbb{Z},$$

where p' is the conjugate exponent of p. Let Q be a cube. By Hölder's inequality,

$$M_Q(f_k)\leq\frac{1}{|Q|}\big\|t_kf_k|L_p(Q)\big\|\big\|t_k^{-1}|L_{p'}(Q)\big\|\leq\frac{c}{|Q|}\big\|t_k^{-1}|L_{p'}(Q)\big\|,\quad k\in\mathbb{Z}.$$

Since $t_k^p \in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k\in\mathbb{Z}$, by Lemma 2.1 (i), $t_k^p \in A_p(\mathbb{R}^n)$, $k\in\mathbb{Z}$ and

$$\frac{1}{|Q|}\big\|t_k^{-1}|L_{p'}(Q)\big\| \le C\big\|t_k|L_p(Q)\big\|^{-1},$$

where the positive constant C is independent of k. Moreover $\big\|t_k|L_p(Q)\big\|\to\infty$ as $|Q|\to\infty$ for any $k\in\mathbb{Z}$, by Lemma 2.1 (v). Hence, we can apply Lemma 2.3. Let

$$\Omega_k^i := \{x\in\mathbb{R}^n : \mathcal{M}(f_k)(x) > 4^n\lambda^i\}, \quad k,i\in\mathbb{Z},$$

with $\lambda > 2^{n+1}$ and

$$H_k^i := \{x\in\mathbb{R}^n : 4^n\lambda^i < \mathcal{M}(f_k)(x) \le 4^n\lambda^{i+1}\}, \quad k,i\in\mathbb{Z}.$$

We have

$$T_k = \sum_{i=-\infty}^{\infty}\int_{H_k^i} t_k(x)\mathcal{M}(f_k)(x)|g_k(x)|dx \le 4^n\sum_{i=-\infty}^{\infty}\lambda^{i+1}\int_{\Omega_k^i} t_k(x)|g_k(x)|dx.$$

Let $\{Q^{i,k,h}\}_h$ be the collection of maximal dyadic cubes as in Lemma 2.3 with

$$\Omega_k^i \subset \cup_h 3Q^{i,k,h},$$

which implies that

$$T_k \le 4^n\sum_{i=-\infty}^{\infty}\sum_{h=0}^{\infty}\lambda^{i+1}\int_{3Q^{i,k,h}} t_k(x)|g_k(x)|dx, \quad k\in\mathbb{Z}. \tag{2.6}$$

Applying Hölder's inequality,

$$\begin{aligned}\int_{3Q^{i,k,h}} t_k(x)|g_k(x)|dx &\le \Big(\int_{3Q^{i,k,h}} t_k^{\tau}(x)dx\Big)^{\frac{1}{\tau}}\Big(\int_{3Q^{i,k,h}} |g_k(x)|^{\tau'}dx\Big)^{\frac{1}{\tau'}} \\ &= |3Q^{i,k,h}|M_{3Q^{i,k,h},\tau}(t_k)M_{3Q^{i,k,h},\tau'}(g_k)\end{aligned}$$

with $\tau>1$. Put $\tau=p(1+\varepsilon_{t_k^p})$ with $\varepsilon_{t_k^p}$ as in Theorem 2.1, which is possible since $t_k^p\in A_{\frac{p}{\theta}}(\mathbb{R}^n)$ for any $k\in\mathbb{Z}$. Obviously, we have

$$M_{3Q^{i,k,h},\tau}(t_k) = M_{3Q^{i,k,h},p(1+\varepsilon_{t_k^p})}(t_k) \le c\, M_{3Q^{i,k,h},p}(t_k), \quad k\in\mathbb{Z}.$$

Since t_k^p, $k\in\mathbb{Z}$, have the same Muckenhoupt constant and from the proof of Theorem 7.2.2 in [32] the constant c in (2.7) is independent of k. Therefore,

$$\int_{3Q^{i,k,h}} t_k(x)|g_k(x)|dx \lesssim |Q^{i,k,h}|M_{3Q^{i,k,h},p}(t_k)M_{3Q^{i,k,h},\tau'}(g_k).$$

We deduce from the above that

$$\lambda^i \int_{3Q^{i,k,h}} t_k(x)|g_k(x)|dx \lesssim |Q_k^{i,k,h}| M_{3Q^{i,k,h},p}(t_k) M_{Q_k^{i,k,h}}(f_k) M_{3Q^{i,k,h},\tau'}(g_k).$$

By Hölder's inequality,

$$\begin{aligned} &M_{Q^{i,k,h}}(f_k) \leq M_{3Q^{i,k,h},\frac{\sigma_1}{\theta}}(t_k^{-1}) M_{3Q^{i,k,h},\frac{p}{\theta}}(t_k f_k), \\ &M_{3Q^{i,k,h},\frac{\sigma_1}{\theta}}(t_k^{-1}) \leq M_{3Q^{i,k,h},\sigma_1}(t_k^{-1}) \end{aligned}$$

with $\sigma_1 = \theta(\frac{p}{\theta})'$. Hence

$$\lambda^i \int_{3Q^{i,k,h}} t_k(x)|g_k(x)|dx \lesssim |Q^{i,k,h}| M_{3Q^{i,k,h},\frac{p}{\theta}}(t_k f_k) M_{3Q^{i,k,h},\tau'}(g_k).$$

Since $|Q^{i,k,h}| \leq \beta |E^{i,k,h}|$, with $E^{i,k,h} = Q^{i,k,h} \backslash (Q^{i,k,h} \cap (\cup_h Q^{i+1,k,h}))$ and the family $E^{i,k,h}$ are pairwise disjoint, the last expression is bounded by

$$c\int_{E^{i,k,h}} M_{3Q^{i,k,h},\frac{p}{\theta}}(t_k f_k) M_{3Q^{i,k,h},\tau'}(g_k) dx \lesssim \int_{\mathbb{R}^n} \mathcal{M}_{\frac{p}{\theta}}(t_k f_k)(x) \mathcal{M}_{\tau'}(g_k)(x) \chi_{E^{i,k,h}}(x) dx.$$

Therefore, (2.6) does not exceed

$$c\sum_{i=-\infty}^{\infty}\sum_{h=0}^{\infty}\int_{\mathbb{R}^n} \mathcal{M}_{\frac{p}{\theta}}(t_k f_k)(x) \mathcal{M}_{\tau'}(g_k)(x) \chi_{E^{i,k,h}}(x) dx \lesssim \int_{\mathbb{R}^n} \mathcal{M}_{\frac{p}{\theta}}(t_k f_k)(x) \mathcal{M}_{\tau'}(g_k)(x) dx.$$

This implies that

$$T_k \lesssim \int_{\mathbb{R}^n} \mathcal{M}_{\frac{p}{\theta}}(t_k f_k)(x) \mathcal{M}_{\tau'}(g_k)(x) dx$$

for any $k \in \mathbb{Z}$. By Hölder's inequality T_k can be estimated by

$$c\big\|\mathcal{M}_{\frac{p}{\theta}}(t_k f_k)|L_p(\mathbb{R}^n)\big\| \big\|\mathcal{M}_{\tau'}(g_k)|L_{p'}(\mathbb{R}^n)\big\| \lesssim \big\|t_k f_k|L_p(\mathbb{R}^n)\big\|,$$

where we used Theorem 2.2 and the fact that

$$\big\|g_k|L_{p'}(\mathbb{R}^n)\big\| \leq 1, \quad k \in \mathbb{Z}.$$

The proof is complete. □

Remark 2.3. (i) We would like to mention that the result of this lemma is true if we assume that $t_k \in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, $1 < \theta < p < \infty$ with

$$A_{\frac{p}{\theta}}(t_k^p) \leq c, \quad k \in \mathbb{Z},$$

where c is a positive constant independent of k.

(ii) The property (2.5) can be generalized in the following way. Let $1 < \theta \leq p < \infty$ and $\{t_k\}$ be a p-admissible sequence such that $t_k^p \in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k \in \mathbb{Z}$.

- If t_k^p, $k\in\mathbb{Z}$ satisfies (2.3), then

$$\big\|\mathcal{M}(f_j)|L_p(\mathbb{R}^n,t_k)\big\|\leq c\,2^{\alpha_1(k-j)}\big\|f_j|L_p(\mathbb{R}^n,t_j)\big\|$$

holds for all sequences of functions $f_j\in L_p(\mathbb{R}^n,t_j)$, $j\in\mathbb{Z}$ and $j\geq k$, where $c>0$ is independent of k and j. Indeed, we use the same schema as in the proof of Lemma 2.4, we arrive at the inequality

$$\lambda^i\int_{3Q^{i,j,h}}t_k(x)|g_k(x)|dx\lesssim|Q_k^{i,j,h}|M_{3Q^{i,j,h},p}(t_k)M_{Q_k^{i,j,h}}(f_j)M_{3Q^{i,j,h},\tau'}(g_k). \tag{2.7}$$

By Hölder's inequality and (2.3), we obtain

$$\begin{aligned}M_{Q^{i,j,h}}(f_j)&\leq M_{3Q^{i,j,h},\sigma_1}(t_j^{-1})M_{3Q^{i,j,h},\frac{p}{\theta}}(t_jf_j)\\&\lesssim 2^{\alpha_1(k-j)}\big(M_{3Q^{i,j,h},p}(t_k)\big)^{-1}M_{3Q^{i,j,h},\frac{p}{\theta}}(t_jf_j),\quad j\geq k.\end{aligned} \tag{2.8}$$

Plug (2.8) into (2.7). Then this gives

$$\lambda^i\int_{3Q^{i,j,h}}t_k(x)|g_k(x)|dx\lesssim 2^{\alpha_1(k-j)}M_{3Q^{i,j,h},\frac{p}{\theta}}(t_jf_j)M_{3Q^{i,j,h},\tau'}(g_k).$$

The remaining arguments are similar to those in the proof of Lemma 2.4.

- If t_k^p, $k\in\mathbb{Z}$ satisfies (2.4) with $\sigma_2\geq p$, then

$$\big\|\mathcal{M}(f_j)|L_p(\mathbb{R}^n,t_k)\big\|\leq c\,2^{\alpha_2(k-j)}\big\|f_j|L_p(\mathbb{R}^n,t_j)\big\|$$

holds for all sequences of functions $f_j\in L_p(\mathbb{R}^n,t_j)$, $j\in\mathbb{Z}$ and $k\geq j$, where $c>0$ is independent of k and j. Indeed, we use again the same schema as in the proof of Lemma 2.4, we arrive at the estimate (2.7). By (2.4), we obtain

$$M_{3Q^{i,j,h},p}(t_k)\lesssim 2^{\alpha_2(k-j)}M_{3Q^{i,j,h},p}(t_j).$$

The remaining part of the argument is quite similar to those used in the proof of Lemma 2.4.

(iii) A proof of this result for $t_k^p=\omega$, $k\in\mathbb{Z}$ may be found in [47].

(iv) In view of Lemma 2.1 (iv) we can assume that $t_k^p\in A_p(\mathbb{R}^n)$, $k\in\mathbb{Z}$, $1<p<\infty$ with

$$A_p(t_k^p)\leq c,\quad k\in\mathbb{Z},$$

where $c>0$ is independent of k.

We need the following lemma, which is a discrete convolution inequality, see for example [6, p. 135].

Lemma 2.5. *Let* $0<a<1$ *and* $0<q\leq\infty$. *Let* $\{\kappa_k\}$ *be a sequences of positive real numbers and denote*

$$\delta_k:=\sum_{j=-\infty}^{k}a^{k-j}\kappa_j,\quad k\in\mathbb{Z},\qquad \eta_k:=\sum_{j=k}^{\infty}a^{j-k}\kappa_j,\quad k\in\mathbb{Z}.$$

Then there exists constant $c>0$ *depending only on* a *and* q *such that*

$$\Big(\sum_{k=-\infty}^{\infty}\delta_k^q\Big)^{\frac{1}{q}}+\Big(\sum_{k=-\infty}^{\infty}\eta_k^q\Big)^{\frac{1}{q}}\leq c\,\Big(\sum_{k=-\infty}^{\infty}\kappa_k^q\Big)^{\frac{1}{q}}.$$

The next lemma is important for the study of our function spaces.

Lemma 2.6. *Let* $K\geq 0,1<\theta\leq p<\infty,1<q<\infty$ *and* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a* p-*admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$. *Then for all sequences* $\{t_kf_k\}\in\ell_q(L_p)$,

$$\Big(\sum_{k=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{k}2^{(j-k)K}\|\mathcal{M}(f_j)|L_p(\mathbb{R}^n,t_k)\|\Big)^q\Big)^{\frac{1}{q}}\lesssim\Big(\sum_{k=-\infty}^{\infty}\|f_k|L_p(\mathbb{R}^n,t_k)\|^q\Big)^{\frac{1}{q}}$$

if $K>\alpha_2$ *and*

$$\Big(\sum_{k=-\infty}^{\infty}\Big(\sum_{j=k}^{\infty}2^{(j-k)K}\|\mathcal{M}(f_j)|L_p(\mathbb{R}^n,t_k)\|\Big)^q\Big)^{\frac{1}{q}}\lesssim\Big(\sum_{k=-\infty}^{\infty}\|f_k|L_p(\mathbb{R}^n,t_k)\|^q\Big)^{\frac{1}{q}}$$

if $K<\alpha_1$.

Proof. This is a direct consequence of Remark 2.3 and Lemma 2.5. □

3 The space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$

In this section we present the Fourier analytical definition of Besov spaces of variable smoothness and we prove their basic properties in analogy to the classical Besov spaces.

3.1 Definition and some properties

Select a pair of Schwartz functions φ and ψ satisfy

$$\text{supp}\mathcal{F}\varphi,\text{supp}\mathcal{F}\psi\subset\Big\{\xi:\frac{1}{2}\leq|\xi|\leq 2\Big\},\tag{3.1}$$

$$|\mathcal{F}\varphi(\xi)|,|\mathcal{F}\psi(\xi)|\geq c\qquad\text{if}\quad\frac{3}{5}\leq|\xi|\leq\frac{5}{3},\tag{3.2}$$

$$\sum_{k=-\infty}^{\infty}\overline{\mathcal{F}\varphi(2^{-k}\xi)}\mathcal{F}\psi(2^{-k}\xi)=1\qquad\text{if}\quad\xi\neq 0,\tag{3.3}$$

where $c>0$. Throughout the paper, for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$, we put $\varphi_k(x):=2^{kn}\varphi(2^kx)$ and $\tilde{\varphi}(x):=\overline{\varphi(-x)}$.

Now, we define the spaces under consideration.

Definition 3.1. *Let $0<p\leq\infty$ and $0<q\leq\infty$. Let $\{t_k\}$ be a p-admissible weight sequence. Let ψ and $\varphi\in\mathcal{S}$ be a function satisfying* (3.1) *and* (3.2).*The Besov space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ is the collection of all $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ such that*

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|:=\Big(\sum_{k=-\infty}^{\infty}\big\|t_k(\varphi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}<\infty$$

with the usual modifications if $q=\infty$.

Remark 3.1. (i) We would like to mention that the elements of the spaces $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ are not distributions but equivalence classes of distributions, i.e., we have to calculate modulo polynomials. If f is a polynomial then $\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|=0$.

(ii) In [61–63] Tyulenev introduced and investigated function spaces of generalised smoothness. These spaces are defined in terms of the difference relations. Let $M\in\mathbb{N},0<p,r<\infty$ and $0<q\leq\infty$. Let $\{t_k\}$ be a p-admissible weight sequence. The class $\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})$ of Tyulenev is defined to be the collection of all $f\in L_r^{\mathrm{loc}}(\mathbb{R}^n)$ satisfying

$$\big\|f|\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})\big\|<\infty,$$

where

$$\big\|f|\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})\big\|:=\big\|f|\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})\big\|^{\bullet}+\big\|t_0\big\|f|L_r(\cdot+I^n)\big\|\,|L_p(\mathbb{R}^n)\big\|$$

with

$$\big\|f|\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})\big\|^{\bullet}:=\Big(\sum_{k=1}^{\infty}t_k^q\big\|\delta_r^M(\cdot+2^{-k}I^n)f\,|\,L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}},$$

where

$$\delta_r^M(x+2^{-k}I^n)f:=\Big(\frac{1}{2^{-2nk}}\int_{2^{-k}I^n}\int_{x+2^{-k}I^n}\Big|\Delta_h^Mf(y)\Big|^r dydh\Big)^{\frac{1}{r}},$$

$x\in\mathbb{R}^n,k\in\mathbb{N}$ and $I^n:=(-1,1)^n$. Here Δ_h^M is the differences of order M, see Section 5.

(iii) The Triebel-Lizorkin version of $\tilde{B}^M_{p,q,r}(\mathbb{R}^n,\{t_k\})$ is given in [20], see [64] for some properties of the inhomogeneous counterparts of $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Using the system $\{\varphi_k\}$ we can define the quasi-norms

$$\big\|f|\dot{B}^s_{p,q}(\mathbb{R}^n)\big\|:=\Big(\sum_{k=-\infty}^{\infty}2^{ksq}\big\|\varphi_k*f|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$

for constants $s\in\mathbb{R}$ and $0<p,q\leq\infty$, with the usual modifications if $q=\infty$. The Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ consist of all distributions $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ for which

$$\big\|f|\dot{B}^s_{p,q}(\mathbb{R}^n)\big\|<\infty.$$

It is well-known that these spaces do not depend on the choice of the systems $\{\varphi_k\}$ (up to equivalence of quasinorms). Further details on the classical theory of these spaces, included the nonhomogeneous case, can be found in [25-27, 52, 58-60].

One recognizes immediately that if $\{t_k\}=\{2^{sk}\}$, $s\in\mathbb{R}$, then

$$\dot{B}_{p,q}(\mathbb{R}^n,\{2^{sk}\})=\dot{B}^s_{p,q}(\mathbb{R}^n).$$

Moreover, for $\{t_k\}=\{2^{sk}w\}$, $s\in\mathbb{R}$ with a weight w we re-obtain the weighted Besov spaces; we refer, in particular, to the papers [10-13, 36, 51, 53, 54] for a comprehensive treatment of the weighted spaces. See the papers [1, 17, 19, 42]for more information about Besov and Triebel-Lizorkin spaces of variable smoothness and integrability.

A basic tool to study the above function spaces is the following Calderón reproducing formula, see [67, Lemma 2.1].

Lemma 3.1. *Suppose φ, $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying* (3.1) *through* (3.3). *If $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$, then*

$$f=\sum_{k=-\infty}^{\infty}2^{-kn}\sum_{m\in\mathbb{Z}^n}\widetilde{\varphi}_k*f(2^{-k}m)\psi_k(\cdot-2^{-k}m), \tag{3.4}$$

where $\widetilde{\varphi}(x)\equiv\overline{\varphi(-x)}$ for all $x\in\mathbb{R}^n$.

Let φ, $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) through (3.3). Recall that the φ-transform S_φ is defined by setting

$$(S_\varphi f)_{k,m}=\langle f,\varphi_{k,m}\rangle,\quad k\in\mathbb{Z},m\in\mathbb{Z}^n,$$

where $\varphi_{k,m}(x)=2^{\frac{kn}{2}}\varphi(2^kx-m)$, $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^n$. The inverse φ-transform T_ψ is defined by

$$T_\psi\lambda:=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\psi_{k,m},$$

where $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\subset\mathbb{C}$, see [26].

Now we introduce the corresponding sequence spaces of $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Definition 3.2. *Let $0<p\leq\infty$ and $0<q\leq\infty$. Let $\{t_k\}$ be a p-admissible weight sequence. Then for all complex valued sequences $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\subset\mathbb{C}$ we define*

$$\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}):=\Big\{\lambda:\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|<\infty\Big\},$$

where

$$\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|:=\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\Big\|\sum_{m\in\mathbb{Z}^n}t_k\lambda_{k,m}\chi_{k,m}|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}$$

with the usual modifications if $q=\infty$.

Allowing the smoothness t_k, $k\in\mathbb{Z}$ to vary from point to point will raise extra difficulties to study these function spaces. But by the following lemma the problem can be reduced to the case of fixed smoothness. The proof is obvious.

Proposition 3.1. *Let $0<p\le\infty$ and $0<q\le\infty$. Let $\{t_k\}$ be a p-admissible weight sequence. Then*

$$\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|^* := \Big(\sum_{k=-\infty}^{\infty} 2^{\frac{knq}{2}}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{k,m}|^p t_{k,m}^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}},$$

is an equivalent quasi-norm in $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$.

The following important properties of the sequence spaces will be required in what follows.

Lemma 3.2. *Let $0<\theta\le p<\infty$ and $0<q\le\infty$. Let $\{t_k\}$ be a p-admissible weight sequence satisfying* (2.3) *with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $j=k$. Let $k\in\mathbb{Z}, m\in\mathbb{Z}^n$ and $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. Then there exists $c>0$ independent of k and m such that*

$$|\lambda_{k,m}|\le c\,2^{-\frac{kn}{2}}t_{k,m}^{-1}\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Proof. Let $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}), k\in\mathbb{Z}$ and $m\in\mathbb{Z}^n$. Since $\{t_k\}$ is a p-admissible sequence satisfying (2.3) with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$, we get by Hölder's inequality

$$\begin{aligned}|\lambda_{k,m}| &= \Big(\frac{1}{|Q_{k,m}|}\int_{Q_{k,m}}|\lambda_{k,m}|^\theta dy\Big)^{\frac{1}{\theta}}\\ &\le M_{Q_{k,m},p}(\lambda_{k,m}t_k)M_{Q_{k,m},\sigma_1}(t_k^{-1})\\ &\le c\,2^{-\frac{kn}{2}}t_{k,m}^{-1}\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|,\end{aligned}$$

where $c>0$ is independent of $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^n$. □

Lemma 3.3. *Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\le p<\infty$ and $0<q\le\infty$. Let $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\ge p)$. Let $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfy* (3.1) *and* (3.2). *Then for all $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$*

$$T_\psi\lambda := \sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\psi_{k,m},$$

converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, $T_\psi:\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\to\mathcal{S}'_\infty(\mathbb{R}^n)$ is continuous.

Proof. Let $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ and $\varphi\in\mathcal{S}_\infty(\mathbb{R}^n)$. We see that

$$\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{k,m}||\langle\psi_{k,m},\varphi\rangle| = I_1+I_2,$$

where

$$I_1 := \sum_{k=-\infty}^{0} \sum_{m\in\mathbb{Z}^n} |\lambda_{k,m}||\langle \psi_{k,m},\varphi\rangle| \quad \text{and} \quad I_2 := \sum_{k=1}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{k,m}||\langle \psi_{k,m},\varphi\rangle|.$$

It suffices to show that both I_1 and I_2 are dominated by

$$c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

(1) **Estimate of I_1.** Let us recall the following estimate, see (3.18) in [8]. For any $L>0$, there exists a positive constant $M\in\mathbb{N}$ such that for all φ, $\psi\in\mathcal{S}_\infty(\mathbb{R}^n)$,$i,k\in\mathbb{Z}$ and $m,h\in\mathbb{Z}^n$,

$$|\langle \varphi_{k,m},\psi_{i,h}\rangle| \lesssim \|\varphi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\Big(1+\frac{|2^{-k}m-2^{-i}h|^n}{\max(2^{-kn},2^{-in})}\Big)^{-L}\min\Big(2^{(i-k)nL},2^{(k-i)nL}\Big).$$

Therefore,

$$|\langle \psi_{k,m},\varphi\rangle| \lesssim \|\varphi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\Big(1+\frac{|2^{-k}m|^n}{\max(1,2^{-kn})}\Big)^{-L}2^{-|k|nL}.$$

Our estimate employs partially some decomposition techniques already used in [26] and [43]. For each $j\in\mathbb{N}$ we define

$$\Omega_j := \{m\in\mathbb{Z}^n : 2^{j-1}<|m|\le 2^j\}, \quad \Omega_0 := \{m\in\mathbb{Z}^n : |m|\le 1\}.$$

Thus,

$$\begin{aligned} I_1 &\lesssim \sum_{k=-\infty}^{0} 2^{knL} \sum_{m\in\mathbb{Z}^n} \frac{|\lambda_{k,m}|}{\big(1+|m|\big)^{nL}} \\ &= c\sum_{k=-\infty}^{0} 2^{knL} \sum_{j=0}^{\infty}\sum_{m\in\Omega_j} \frac{|\lambda_{k,m}|}{\big(1+|m|\big)^{nL}} \\ &\lesssim \sum_{k=-\infty}^{0} 2^{knL} \sum_{j=0}^{\infty} 2^{-nLj} \sum_{m\in\Omega_j} |\lambda_{k,m}|. \end{aligned}$$

We claim that there exists $0<\varrho<\min(1,\theta)$ such that $\frac{1}{\varrho}=\frac{1}{\tau}+\frac{1}{\sigma_1}$ with $0<\tau<p$. Indeed, if $\theta>1$, then $\sigma_1>1$ and we choose

$$0<\tau<\min\big(\frac{1}{1-\frac{1}{\sigma_1}},p\big).$$

Now, if $0<\theta\le 1$, then we choose $0<\tau<p$ and we obtain that

$$\frac{1}{\varrho}=\frac{1}{\tau}+\frac{1}{\sigma_1}>\frac{1}{\theta},$$

which proves the above claim. We have

$$I_1 \lesssim \sum_{k=-\infty}^{0} 2^{knL} \sum_{j=0}^{\infty} 2^{-nLj} \Big(\sum_{m\in\Omega_j} |\lambda_{k,m}|^{\varrho} \Big)^{\frac{1}{\varrho}}.$$

$$= c \sum_{k=-\infty}^{0} 2^{knL} \sum_{j=0}^{\infty} 2^{(\frac{1}{\varrho}-L)nj} \Big(2^{(k-j)n} \int_{\cup_{z\in\Omega_j} Q_{k,z}} \sum_{m\in\Omega_j} |\lambda_{k,m}|^{\varrho} \chi_{k,m}(y) dy \Big)^{\frac{1}{\varrho}}.$$

Let $y \in \cup_{z\in\Omega_j} Q_{k,z}$ and $x \in Q_{0,0}$. Then $y \in Q_{k,z}$ for some $z \in \Omega_j$ and $2^{j-1} < |z| \le 2^j$. From this it follows that

$$|y-x| \le |y-2^{-k}z| + |x-2^{-k}z| \le \sqrt{n}\, 2^{-k} + |x| + 2^{-k}|z| \le 2^{j-k+\delta_n}, \quad \delta_n \in \mathbb{N},$$

which implies that y is located in the ball $B(x,2^{j-k+\delta_n})$. In addition, from the fact that

$$|y| \le |y-x| + |x| \le 2^{j-k+\delta_n} + 1 \le 2^{j-k+c_n}, \quad c_n \in \mathbb{N},$$

we have that y is located in the ball $B(0,2^{j-k+c_n})$. Therefore, by Hölder's inequality

$$\begin{aligned} &\Big(2^{(k-j)n} \int_{\cup_{z\in\Omega_j} Q_{k,z}} \sum_{m\in\Omega_j} |\lambda_{k,m}|^{\varrho} \chi_{k,m}(y) dy \Big)^{\frac{1}{\varrho}} \\ &\le \Big(2^{(k-j)n} \int_{B(x,2^{j-k+c_n})} \sum_{m\in\Omega_j} |\lambda_{k,m}|^{\tau} t_k^{\tau} \chi_{k,m}(y) dy \Big)^{\frac{1}{\tau}} M_{B(0,2^{j-k+c_n}),\sigma_1}(t_k^{-1}) \\ &\lesssim \mathcal{M}_\tau \Big(\sum_{m\in\mathbb{Z}^n} t_k |\lambda_{k,m}| \chi_{k,m} \Big)(x) M_{B(0,2^{j-k+c_n}),\sigma_1}(t_k^{-1}). \end{aligned}$$

Since $t_k^{-\sigma_1} \in A_{(\frac{p}{\theta})'}(\mathbb{R}^n)$, $k \in \mathbb{Z}$, by Lemma 2.1/(iii), (2.3) and (2.4) we obtain

$$\begin{aligned} M_{B(0,2^{j-k+c_n}),\sigma_1}(t_k^{-1}) &\lesssim 2^{(j-k)\frac{n}{p}} M_{B(0,1),\sigma_1}(t_k^{-1}) \\ &\lesssim 2^{(j-k)\frac{n}{p}} \Big(M_{B(0,1),p}(t_k) \Big)^{-1} \\ &\lesssim 2^{(j-k)\frac{n}{p} - k\alpha_2} \Big(M_{B(0,1),\sigma_2}(t_0) \Big)^{-1} \end{aligned}$$

for any $k \le 0$ and any $j \in \mathbb{N}_0$. Hence, for any L large enough,

$$I_1 \lesssim \sum_{k=-\infty}^{0} 2^{k(nL-\alpha_2-\frac{n}{p})} \mathcal{M}_\tau \Big(\sum_{m\in\mathbb{Z}^n} t_k |\lambda_{k,m}| \chi_{k,m} \Big)(x), \quad x \in Q_{0,0}.$$

The last term is bounded in the $L_p(Q_{0,0})$-quasi-norm by

$$c \big\| \lambda | \dot{b}_{p,q}(\mathbb{R}^n, \{t_k\}) \big\|$$

with the help of Theorem 2.2.

(2) Estimate of I_2. We have

$$|\langle\psi_{k,m},\varphi\rangle|\lesssim 2^{-knL}\|\varphi\|_{\mathcal{S}_{M+1}}\|\psi\|_{\mathcal{S}_{M+1}}\left(1+2^{-kn}|m|^n\right)^{-L},\quad k\geq 1.$$

For each $j,k\in\mathbb{N}$, define

$$\Omega_{j,k}:=\{m\in\mathbb{Z}^n:2^{j+k-1}<|m|\leq 2^{j+k}\},$$
$$\Omega_{0,k}:=\{m\in\mathbb{Z}^n:|m|\leq 2^k\}.$$

Then we find

$$\begin{aligned}I_2&\lesssim\sum_{k=1}^{\infty}2^{-knL}\sum_{m\in\mathbb{Z}^n}\frac{|\lambda_{k,m}|}{\left(1+2^{-k}|m|\right)^{nL}}\\&=c\sum_{k=1}^{\infty}2^{-knL}\sum_{j=0}^{\infty}\sum_{m\in\Omega_{j,k}}\frac{|\lambda_{k,m}|}{\left(1+2^{-k}|m|\right)^{nL}}\\&\leq c\sum_{k=1}^{\infty}2^{-knL}\sum_{j=0}^{\infty}2^{-nLj}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|.\end{aligned}$$

Let $0<\varrho<\min(1,\theta)$ be such that $\frac{1}{\varrho}=\frac{1}{\tau}+\frac{1}{\sigma_1}$ with $0<\tau<p$. Using the embedding $\ell_\varrho\hookrightarrow\ell_1$ we find that

$$\begin{aligned}I_2&\lesssim\sum_{k=1}^{\infty}2^{-knL}\sum_{j=0}^{\infty}2^{-nLj}\Big(\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\Big)^{\frac{1}{\varrho}}.\\&=c\sum_{k=1}^{\infty}2^{-knL}\sum_{j=0}^{\infty}2^{(\frac{n}{\varrho}-nL)j}\Big(2^{(k-j)n}\int_{\cup_{z\in\Omega_{j,k}}Q_{k,z}}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\chi_{k,m}(y)dy\Big)^{\frac{1}{\varrho}}.\end{aligned}$$

Let $y\in\cup_{z\in\Omega_{j,k}}Q_{k,z}$ and $x\in Q_{0,0}$. Then $y\in Q_{k,z}$ for some $z\in\Omega_{j,k}$ and $2^{j-1}<2^{-k}|z|\leq 2^j$. From this it follows that

$$|y-x|\leq|y-2^{-k}z|+|x-2^{-k}|\leq\sqrt{n}\,2^{-k}+|x|+2^{-k}|z|\leq 2^{j+\delta_n},\quad\delta_n\in\mathbb{N},$$

which implies that y is located in the ball $B\left(x,2^{j+\delta_n}\right)$. In addition, from the fact that

$$|y|\leq|y-x|+|x|\leq 2^{j+\delta_n}+1\leq 2^{j+c_n},\quad c_n\in\mathbb{N},$$

we have that y is located in the ball $B(0,2^{j+c_n})$. Therefore,

$$\begin{aligned}&\Big(2^{(k-j)n}\int_{\cup_{z\in\Omega_{j,k}}Q_{k,z}}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\chi_{k,m}(y)dy\Big)^{\frac{1}{\varrho}}\\&\leq 2^{k\frac{n}{\varrho}}\Big(2^{-jn}\int_{B(x,2^{j+\delta_n})}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\tau}t_k^{\tau}\chi_{k,m}(y)dy\Big)^{\frac{1}{\tau}}M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1})\\&\lesssim 2^{k\frac{n}{\varrho}}\mathcal{M}_\tau\Big(\sum_{m\in\mathbb{Z}^n}t_k\lambda_{k,m}\chi_{k,m}\Big)(x)M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1}).\end{aligned}$$

By (2.3) and Lemma 2.1 (v) we obtain

$$M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1}) \lesssim 2^{-k\alpha_1}\big(M_{B(0,2^{j+c_n}),p}(t_0)\big)^{-1} \lesssim 2^{j(\frac{n}{p}-\frac{n\delta}{p})-k\alpha_1}\big(M_{B(0,1),p}(t_0)\big)^{-1}.$$

Therefore

$$I_2 \lesssim \sum_{k=1}^{\infty} 2^{-k(nL-\frac{n}{\varrho}+\alpha_1)} \mathcal{M}_\tau\Big(t_k \sum_{m\in\mathbb{Z}^n} \lambda_{k,m}\chi_{k,m}\Big)(x), \quad x\in Q_{0,0} \tag{3.5}$$

for any L large enough. Now we take the $L_p(Q_{0,0})$-quasi-norm of both sides of (3.5) and then use Theorem 2.2, we obtain

$$I_2 \lesssim \big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

The proof is finished. □

For a sequence $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\subset\mathbb{C}, 0<r\le\infty$ and a fixed $d>0$, set

$$\lambda^*_{k,m,r,d} := \Big(\sum_{h\in\mathbb{Z}^n} \frac{|\lambda_{k,h}|^r}{(1+2^k|2^{-k}h-2^{-k}m|)^d}\Big)^{\frac{1}{r}}$$

and $\lambda^*_{r,d} := \{\lambda^*_{k,m,r,d}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\subset\mathbb{C}$, with the usual modifications if $q=\infty$.

Lemma 3.4. *Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\le p<\infty, 0<q\le\infty, \gamma\in\mathbb{Z}$ and $d>n$. Let $\{t_k\}$ be a p-admissible weight sequence satisfying* (2.3) *with $\sigma_1=\theta\big(\frac{p}{\theta}\big)'$ and $\alpha_1\in\mathbb{R}$. Then*

$$\big\|\lambda^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\| \approx \big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\|. \tag{3.6}$$

In addition if $\{t_k\}$ satisfies (2.4) *with $\sigma_2\ge p$ and $\alpha_2\in\mathbb{R}$, then*

$$\big\|\lambda^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\| \lesssim A\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|, \tag{3.7}$$

where

$$A=\begin{cases} 2^{-\gamma\alpha_2}, & \text{if} \quad \gamma<0,\\ 2^{-\gamma\alpha_1}, & \text{if} \quad \gamma\ge 0.\end{cases}$$

Proof. First we prove (3.6). Obviously,

$$\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\| \le \big\|\lambda^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\|.$$

Let $\frac{np}{d}<a<p$. For each $j\in\mathbb{N}, m\in\mathbb{Z}^n$ define

$$\digamma_{j,m} := \{h\in\mathbb{Z}^n : 2^{j-1}<|h-m|\le 2^j\},$$
$$\digamma_{0,m} := \{h\in\mathbb{Z}^n : |h-m|\le 1\}.$$

Thus,

$$\sum_{h\in\mathbb{Z}^n}\frac{|\lambda_{k,h}|^p}{\left(1+|h-m|\right)^d}=\sum_{j=0}^{\infty}\sum_{h\in F_{j,m}}\frac{|\lambda_{k,h}|^p}{\left(1+|h-m|\right)^d}\lesssim\sum_{j=0}^{\infty}2^{-dj}\sum_{h\in F_{j,m}}|\lambda_{k,h}|^p,$$

which is bounded by

$$c\sum_{j=0}^{\infty}2^{-dj}\Big(\sum_{h\in F_{j,m}}|\lambda_{k,h}|^a\Big)^{\frac{p}{a}},$$

which can be rewritten as

$$c\sum_{j=0}^{\infty}2^{(\frac{np}{a}-d)j}\Big(2^{(k-j)n}\int_{\cup_{z\in F_{j,m}}Q_{k,z}}\sum_{h\in F_{j,m}}|\lambda_{k,h}|^a\chi_{k,h}(y)dy\Big)^{\frac{p}{a}}. \tag{3.8}$$

Let $y\in\cup_{z\in F_{j,m}}Q_{k,z}$ and $x\in Q_{k,m}$. Therefore $y\in Q_{k,z}$ for some $z\in F_{j,m}$ and $2^{j-1}<|z-m|\le 2^j$. From this it follows that

$$\begin{aligned}|y-x|&\le|y-2^{-k}z|+|x-2^{-k}z|\\&\le\sqrt{n}\,2^{-k}+|x-2^{-k}m|+2^{-k}|z-m|\\&\le 2^{j-k+\delta_n},\quad \delta_n\in\mathbb{N},\end{aligned}$$

which implies that y is located in the ball $B(x,2^{j-k+\delta_n})$. Therefore, (3.8) can be estimated by

$$c\big(\mathcal{M}_a\big(\sum_{h\in\mathbb{Z}^n}\lambda_{k,h}\chi_{k,h}\big)(x)\big)^p,$$

where the positive constant c is independent of x and k. Consequently

$$\big\|\lambda^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\| \tag{3.9}$$

does not exceed

$$\begin{aligned}&c\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|t_{k-\gamma}\mathcal{M}_a\big(\sum_{h\in\mathbb{Z}^n}\lambda_{k,h}\chi_{k,h}\big)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}\\&=c\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|\mathcal{M}_a\big(\sum_{h\in\mathbb{Z}^n}\lambda_{k,h}\chi_{k,h}\big)|L_p(\mathbb{R}^n,t_{k-\gamma})\big\|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Applying Lemma 2.4 we estimate (3.9) by

$$c\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|\sum_{h\in\mathbb{Z}^n}\lambda_{k,h}\chi_{k,h}|L_p(\mathbb{R}^n,t_{k-\gamma})\big\|^q\Big)^{\frac{1}{q}}=c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_{k-\gamma}\})\big\|.$$

To prove (3.7) we use again Lemma 2.4 combined with Remark 2.3 (ii)-(iii). □

Now we have the following result which is called the φ-transform characterization in the sense of Frazier and Jawerth. It will play an important role in the rest of the paper.

Theorem 3.1. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\leq p<\infty$ *and* $0<q\leq\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)', \sigma_2\geq p)$. *Let* φ, $\psi\in\mathcal{S}(\mathbb{R}^n)$ *satisfying* (3.1) *through* (3.3). *The operators*

$$S_\varphi:\ \dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\to\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}),$$
$$T_\psi:\ \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\to\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$$

are bounded. Furthermore, $T_\psi\circ S_\varphi$ *is the identity on* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Proof. The proof is a straightforward adaptation of [26, Theorem 2.2]. For any $f\in\mathcal{S}'_\infty(\mathbb{R}^n)$ we put $\sup(f):=\{\sup_{k,m}(f)\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ where

$$\sup_{k,m}(f):=2^{-\frac{kn}{2}}\sup_{y\in Q_{k,m}}|\widetilde{\varphi_k}*f(y)|,\quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n.$$

For any $\gamma\in\mathbb{N}_0$, we define the sequence $\inf_\gamma(f):=\{\inf_{k,m,\gamma}(f)\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ by setting

$$\inf_{k,m,\gamma}(f):=2^{-\frac{kn}{2}}\max\left\{\inf_{y\in\tilde{Q}}|\widetilde{\varphi_k}*f(y)|:\tilde{Q}\subset Q_{k,m}, l(\tilde{Q})=2^{-k-\gamma}\right\},\quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n,$$

where $\widetilde{\varphi_k}:=2^{kn}\overline{\varphi(-2^k\cdot)}$.

Step 1. In this step we prove that

$$\big\|\inf_\gamma(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Define a sequence $\lambda=\{\lambda_{j,h}\}_{j\in\mathbb{Z},h\in\mathbb{Z}^n}$ by

$$\lambda_{j,h}=2^{-\frac{jn}{2}}\inf_{y\in Q_{j,h}}|\widetilde{\varphi_{j-\gamma}}*f(y)|,\quad j\in\mathbb{Z},h\in\mathbb{Z}^n.$$

Then for all $0<r<\infty$, any $k\in\mathbb{Z},m\in\mathbb{Z}^n$ and a fixed $\lambda>n$, we have

$$\inf_{k,m,\gamma}(f)\chi_{k,m}\lesssim 2^{\gamma(\frac{d}{r}+\frac{n}{2})}\sum_{h\in\mathbb{Z}^n,Q_{k+\gamma,h}\subset Q_{k,m}}\lambda^*_{k+\gamma,h,r,d}\chi_{k+\gamma,h}.$$

Picking $r=p$, we obtain

$$\begin{aligned}&\big\|\inf_\gamma(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\\ &\lesssim 2^{\gamma(\frac{d}{p}+\frac{n}{2})}\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|\sum_{h\in\mathbb{Z}^n}t_k\lambda^*_{k+\gamma,h,r,d}\chi_{k+\gamma,h}|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}\\ &\lesssim 2^{\gamma\frac{d}{p}}\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|\sum_{h\in\mathbb{Z}^n}t_{k-\gamma}\lambda^*_{k,h,r,d}\chi_{k,h}|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Applying Lemma 3.4, we estimate the last expression by

$$
\begin{aligned}
&c2^{\gamma\frac{d}{p}}\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\Big\|\sum_{h\in\mathbb{Z}^n}t_{k-\gamma}\lambda_{k,h}\chi_{k,h}|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}\\
\lesssim\, &2^{\gamma\frac{d}{p}}\Big(\sum_{k=-\infty}^{\infty}\big\|t_k(\widetilde{\varphi_k}*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}\\
\lesssim\, &2^{\gamma\frac{d}{p}}\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.
\end{aligned}
$$

Step 2. We will prove that

$$
\big\|\inf{}_\gamma(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\big\|\sup(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|. \tag{3.10}
$$

Applying Lemma A.4 of [26], see also Lemma 8.3 of [8], to the function $(\widetilde{\varphi_k}*f)(2^{-j}x)$ we obtain

$$
\inf{}_\gamma(f)^*_{p,d}\approx\sup(f)^*_{p,d}.
$$

Hence for $\gamma>0$ sufficiently large we obtain by applying Lemma 3.4,

$$
\begin{aligned}
\big\|\inf{}_\gamma(f)^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|&\approx\big\|\inf{}_\gamma(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|,\\
\big\|\sup(f)^*_{p,d}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|&\approx\big\|\sup(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.
\end{aligned}
$$

Therefore,

$$
\big\|\inf{}_\gamma(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\big\|\sup(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|. \tag{3.11}
$$

From the definition of the spaces $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ it follows that

$$
\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\sup(f)|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.
$$

Consequently (3.11) and Step 1 yield (3.10).

Step 3. In this step we prove the boundedness of S_φ and T_ψ. We have

$$
|(S_\varphi f)_{k,m}|=|\langle f,\varphi_{k,m}\rangle|=2^{-\frac{kn}{2}}|f*\widetilde{\varphi_k}(2^{-k}m)|\le\sup_{k,m}(f).
$$

Step 2 yields that

$$
\big\|S_\varphi f|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.
$$

To prove the boundedness of T_ψ suppose $\lambda=\{\lambda_{j,h}\}_{j\in\mathbb{Z},h\in\mathbb{Z}^n}$. Then

$$
T_\psi\lambda:=\sum_{j=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\lambda_{j,h}\psi_{j,h}.
$$

Obviously

$$
\widetilde{\varphi_k}*T_\psi\lambda=\sum_{j=k-1}^{k+1}\sum_{h\in\mathbb{Z}^n}\lambda_{j,h}\widetilde{\varphi_k}*\psi_{j,h}.
$$

Since $\widetilde{\varphi}$ and ψ belong to $\mathcal{S}(\mathbb{R}^n)$ we obtain

$$|\widetilde{\varphi_k}*\psi_{j,h}(x)|\lesssim 2^{\frac{jn}{2}}(1+2^j|x-2^{-j}h|)^{-\frac{d}{\min(1,p)}}, \quad d>n,$$

where the implicit constant is independent of j,k,h and x. Therefore, if $x\in Q_{k+1,z}\subset Q_{k,m}\subset Q_{k-1,l}$, $z,l\in\mathbb{Z}^n$, then we obtain

$$|\widetilde{\varphi_k}*T_\psi\lambda(x)|\lesssim 2^{\frac{kn}{2}}\sum_{j=k-1}^{k+1}\sum_{h\in\mathbb{Z}^n}\frac{|\lambda_{j,h}|}{(1+2^j|x-2^{-j}h|)^{\frac{d}{\min(1,p)}}}.$$

Assume that $0<p\leq 1$. Using the inequality

$$\Big(\sum_{h\in\mathbb{Z}^n}|a_h|\Big)^p\leq\sum_{h\in\mathbb{Z}^n}|a_h|^p, \qquad \{a_h\}_{h\in\mathbb{Z}^n}\subset\mathbb{C},$$

we obtain

$$|\widetilde{\varphi_k}*T_\psi\lambda(x)|\lesssim 2^{\frac{kn}{2}}\sum_{j=k-1}^{k+1}\Big(\sum_{h\in\mathbb{Z}^n}\frac{|\lambda_{j,h}|^p}{(1+2^j|x-2^{-j}h|)^d}\Big)^{\frac{1}{p}}. \tag{3.12}$$

Now if $p>1$, then by the Hölder inequality and the fact that

$$\sum_{h\in\mathbb{Z}^n}\frac{1}{(1+2^j|x-2^{-j}h|)^d}\lesssim 1,$$

we also have (3.12) with $p>1$. Hence if $x\in Q_{k+1,z}\subset Q_{k,m}\subset Q_{k-1,l}$, then we have

$$|\widetilde{\varphi_k}*T_\psi\lambda(x)|\lesssim 2^{\frac{kn}{2}}(\lambda^*_{k-1,l,p,d}+\lambda^*_{k,m,p,d}+\lambda^*_{k+1,z,p,d}).$$

Consequently

$$\big\|T_\psi\lambda|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\sum_{i=-1}^{1}I_i,$$

where

$$I_i=\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\big\|\sum_{h\in\mathbb{Z}^n}t_{k+i}\lambda^*_{k,h,p,d}\chi_{k,h}|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}, \qquad i=-1,0,1.$$

Applying (3.7) we obtain

$$\big\|T_\psi\lambda|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

The proof is complete. □

This theorem can then be exploited to obtain a variety of results for the $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ spaces, where arguments can be equivalently transferred to the sequence space, which is often more convenient to handle. More precisely, under the same hypothesis of the last theorem,

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\big\|\{\langle f,\varphi_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|. \tag{3.13}$$

From Theorem 3.1, we obtain the next important property of the spaces $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Corollary 3.1. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\le p<\infty$ *and* $0<q\le\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\ge p)$. *The definition of the spaces* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *is independent of the choices of* $\varphi\in\mathcal{S}(\mathbb{R}^n)$ *satisfying* (3.1) *and* (3.2).

The proof of the completeness of $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ is based on the following lemma.

Lemma 3.5. *Let* $0<\theta\le p<\infty$ *and* $0<q\le\infty$. *Let* $\{t_k\}$ *be a p-admissible weight sequence satisfying* (2.3) *with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$. $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ *are quasi-Banach spaces. They are Banach spaces if* $1\le p<\infty$ *and* $1\le q\le\infty$.

Proof. Let $\{\lambda^{(j)}\}_{j\in\mathbb{N}_0}$ be a Cauchy sequence in $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. We use the equivalent quasi-norm given in Proposition 3.1. Then for any $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such

$$\big\|\lambda^{(i)}-\lambda^{(j)}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|^*<\varepsilon,\quad i,j\ge n_0.$$

This yields that

$$\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\Big(\sum_{m\in\mathbb{Z}^n}t_{k,m}^p|\lambda_{k,m}^{(i)}-\lambda_{k,m}^{(j)}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}<\varepsilon,\quad i,j\ge n_0.$$

We put

$$A^{(i)}=\{A_k^{(i)}\}_{k\in\mathbb{Z}},\quad A_k^{(i)}=\{2^{\frac{kn}{2}}t_{k,m}\lambda_{k,m}^{(i)}\}_{m\in\mathbb{Z}^n},\quad k\in\mathbb{Z},i\in\mathbb{N}_0,$$

it follows that $\{A^{(i)}\}_{i\in\mathbb{N}_0}$ is a Cauchy sequence in $\ell^q(\ell^p(\mathbb{Z}^n))$, so it is converges to some $A\in\ell^q(\ell^p(\mathbb{Z}^n))$, where

$$A=\{A_k\}_{k\in\mathbb{Z}},\quad A_k=\{A_{k,m}\}_{m\in\mathbb{Z}^n},\quad k\in\mathbb{Z}.$$

Define

$$\varrho_{k,m}=2^{-\frac{kn}{2}}t_{k,m}^{-1}A_{k,m},\quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n\quad\text{and}\quad\varrho=\{\varrho_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}.$$

Thus

$$\begin{aligned}&\big\|\lambda^{(j)}-\varrho|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|^*\\&=\Big(\sum_{k=-\infty}^{\infty}2^{\frac{knq}{2}}\Big(\sum_{m\in\mathbb{Z}^n}t_{k,m}^p|\lambda_{k,m}^{(j)}-\varrho_{k,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\\&=\Big(\sum_{k=-\infty}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n}|2^{\frac{kn}{2}}t_{k,m}\lambda_{k,m}^{(j)}-A_{k,m}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}\\&=\big\|A^{(j)}-A|\ell^q(\ell^p(\mathbb{Z}^n))\big\|<\varepsilon,\qquad j\ge n_1,\end{aligned}$$

where n_1 is large enough. Consequently $\{\lambda^{(j)}\}_{j\in\mathbb{N}_0}$ converges to ϱ, which belongs to the space $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. This completes the proof. □

Applying Lemma 3.5, Theorem 3.1 and by a similar argument before, see [29], we obtain the following useful properties of these function spaces.

Theorem 3.2. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\le p<\infty$ *and* $0<q\le\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a* p-*admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)', \sigma_2\ge p)$. *Then* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *are quasi-Banach spaces. They are Banach spaces if* $1\le p<\infty$ *and* $1\le q<\infty$.

Let $0<p<\infty$ and $0<q<\infty$. We know that

$$\mathcal{S}_\infty(\mathbb{R}^n)\hookrightarrow\dot{B}_{p,q}(\mathbb{R}^n,\{2^{ks}\})\hookrightarrow\mathcal{S}'_\infty(\mathbb{R}^n),\quad s\in\mathbb{R},$$

see [58, Theorem 5.1.5]. We mention that $\dot{B}_{p,q}(\mathbb{R}^n,\{2^{ks}\})$ is just the classical Besov space $\dot{B}_{p,q}(\mathbb{R}^n)$. Our aim is to extend this result to the above function spaces, but to this end we need the weighted version of Plancherel-Polya-Nikolskij inequality. This inequality (cf. [58, 1.3.2/5, Rem. 1.4.1/4]), plays an important role in theory of function spaces and PDE's, and says that $\|f|L_q(\mathbb{R}^n)\|$ can be estimated by

$$c\,R^{n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f|L_p(\mathbb{R}^n)\|$$

for any $0<p\le q\le\infty$, $R>0$ and any $f\in L_p(\mathbb{R}^n)\cap\mathcal{S}'(\mathbb{R}^n)$ with $\operatorname{supp}\mathcal{F}f\subset\{\xi\in\mathbb{R}^n:|\xi|\le R\}$. The constant $c>0$ is independent of R.

Lemma 3.6. *Let* $1<\theta\le p<\infty$ *and* $\{t_k\}$ *be a* p-*admissible weight sequence satisfying* (2.3) *with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $j=k$. *Let* $\varepsilon_{t_k^p}$ *be as in Theorem* 2.1 *and* $f_k\in L_p(\mathbb{R}^n,t_k)$ *such that* $\operatorname{supp}\mathcal{F}f_k\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2^k\}, k\in\mathbb{Z}$. *Then*

$$\|f_k|L_r(\mathbb{R}^n,t_k)\|\le c2^{nk(\frac{1}{p}-\frac{1}{r})}\|f_k|L_p(\mathbb{R}^n,t_k)\| \tag{3.14}$$

holds for any $k\in\mathbb{Z}$ *and* $p\le r\le p(1+\varepsilon_{t_k^p})$, *where* $c>0$ *is independent of* k.

Proof. Let $\omega\in\mathcal{S}(\mathbb{R}^n)$ and $\beta>1$ be such that $\mathcal{F}\omega(x)=1$ if $x\in\operatorname{supp}\mathcal{F}f_k$,

$$\begin{aligned}&\frac{r}{p}<\beta<1+\varepsilon_{t_k^p}<\infty\quad\text{if}\quad r<p(1+\varepsilon_{t_k^p}),\\&\beta=1+\varepsilon_{t_k^p}\quad\text{if}\quad r=p(1+\varepsilon_{t_k^p}).\end{aligned}$$

We put $\omega_k=2^{kn}\omega(2^k\cdot), k\in\mathbb{Z}$. By duality $\|f_k|L_r(\mathbb{R}^n,t_k)\|$ can be rewritten as

$$\sup\int_{\mathbb{R}^n}t_k(x)|\omega_k*f_k(x)||g_k(x)|dx=\sup I(g_k),$$

where the supremum is taken over all functions $g_k\in L_{r'}(\mathbb{R}^n)$ with $\|g_k|L_{r'}(\mathbb{R}^n)\|\le1$. Splitting the integration with respect to the cubes $Q_{k,m}$ we obtain

$$I(g_k)\le\sum_{m\in\mathbb{Z}^n}\int_{Q_{k,m}}t_k(x)|\omega_k*f_k(x)||g_k(x)|dx. \tag{3.15}$$

Let $x\in Q_{k,m}$. We have

$$
\begin{aligned}
|\omega_k * f_k(x)| &\lesssim \int_{\mathbb{R}^n} \eta_{k,\varrho}(x-y)|f_k(y)|dy \\
&= c\sum_{h\in\mathbb{Z}^n}\int_{Q_{k,m+h}} \eta_{k,\varrho}(x-y)|f_k(y)|dy \\
&\lesssim \sum_{h\in\mathbb{Z}^n}(1+|h|)^{-\varrho} M_{Q_{k,m+h}}(f_k) \\
&= c\sum_{h\in\mathbb{Z}^n}(1+|h|)^{-\varrho}\vartheta_{k,h,m}, \qquad \varrho>0
\end{aligned}
$$

where the positive constant c is independent of x and k, and

$$\vartheta_{k,h,m}=M_{Q_{k,m+h}}(f_k), \quad k\in\mathbb{Z}, h,\, m\in\mathbb{Z}^n.$$

Applying Hölder's inequality,

$$
\begin{aligned}
\int_{Q_{k,m}} t_k(x)|g_k(x)|dx &\le \Big(\int_{Q_{k,m}} t_k^{\tau}(x)dx\Big)^{\frac{1}{\tau}}\Big(\int_{Q_{k,m}} |g_k(x)|^{\tau'}dx\Big)^{\frac{1}{\tau'}} \\
&= |Q_{k,m}| M_{Q_{k,m},\tau}(t_k) M_{Q_{k,m},\tau'}(g_k)
\end{aligned}
$$

with $\tau>1$. Put $\tau=\frac{r}{\beta}(1+\varepsilon_{t_k^p})$. Observe that $\tau\ge r$, $\frac{r}{\beta}\le p$. Then by Theorem 2.1, we obtain

$$M_{Q_{k,m},\tau}(t_k)=M_{Q_{k,m},\frac{r}{\beta}(1+\varepsilon_{t_k^p})}(t_k)\le c\, M_{Q_{k,m},p}(t_k), \quad k\in\mathbb{Z}, m\in\mathbb{Z}^n,$$

where $c>0$ is independent of k and m. Therefore

$$\vartheta_{k,h,m}\int_{Q_{k,m}} t_k(x)|g_k(x)|dx\lesssim |Q_{k,m}| M_{Q_{k,m},p}(t_k) M_{Q_{k,m},\tau'}(g_k)\vartheta_{k,h,m}.$$

We derive from Hölder's inequality that

$$
\begin{aligned}
&\vartheta_{k,h,m}\le M_{Q_{k,m+h},\frac{\sigma_1}{\theta}}(t_k^{-1}) M_{Q_{k,m+h},\frac{p}{\theta}}(t_k f_k), \quad \sigma_1=\theta(\frac{p}{\theta})', \\
&M_{Q_{k,m+h},\frac{\sigma_1}{\theta}}(t_k^{-1})\le M_{Q_{k,m+h},\sigma_1}(t_k^{-1}).
\end{aligned}
$$

Observe that

$$Q_{k,m}\subset B(x_{k,m},\sqrt{n}2^{-k}) \quad \text{and} \quad Q_{k,m+h}\subset B(x_{k,m},\sqrt{n}(1+|h|)2^{-k}).$$

Hence

$$M_{Q_{k,m},p}(t_k) M_{Q_{k,m+h},\sigma_1}(t_k^{-1})\le c(1+|h|)^{\frac{n}{\theta}},$$

where $c>0$ is independent of k,h and m, and

$$
\begin{aligned}
\vartheta_{k,h,m}\int_{Q_{k,m}} t_k(x)|g_k(x)|dx &\lesssim (1+|h|)^{\frac{n}{\theta}}|Q_{k,m}| M_{Q_{k,m+h},\frac{p}{\theta}}(t_k f_k) M_{Q_{k,m},\tau'}(g_k) \\
&= c(1+|h|)^{\frac{n}{\theta}}\int_{Q_{k,m}} M_{Q_{k,m+h},\frac{p}{\theta}}(t_k f_k) M_{Q_{k,m},\tau'}(g)dx.
\end{aligned}
$$

Now if $x\in Q_{k,m}$, then we find that

$$M_{Q_{k,m+h},\frac{p}{\theta}}(t_kf_k)\lesssim(1+|h|)^{\delta\frac{\theta}{p}}\big(\eta_{k,\delta}*(t_k|f_k|)^{\frac{p}{\theta}}(x)\big)^{\frac{1}{\frac{p}{\theta}}},\quad \delta>n,$$
$$M_{Q_{k,m},\tau'}(g)\lesssim\big(\eta_{k,\delta}*|g_k|^{\tau'}(x)\big)^{\frac{1}{\tau'}},\quad x\in Q_{k,m}.$$

Again by Hölder's inequality we estimate

$$\vartheta_{k,h,m}\int_{\mathbb{R}^n}t_k(x)|g_k(x)|dx$$

by

$$c(1+|h|)^{\delta\frac{\theta}{p}+\frac{n}{\theta}}\big\|\eta_{k,\delta}*(t_k|f_k|)^{\frac{p}{\theta}}|L_{\frac{r\theta}{p}}(\mathbb{R}^n)\big\|^{\frac{\theta}{p}}\big\|\eta_{k,\delta}*|g_k|^{\tau'}|L_{\frac{r'}{\tau'}}(\mathbb{R}^n)\big\|^{\frac{1}{\tau'}}.$$

Put $\frac{1}{\frac{r\theta}{p}}+1=\frac{1}{\theta}+\frac{1}{\mu}$. Since $\frac{p}{r}-1\leq 0$ and $\theta>1$, we have $1\leq\mu<\infty$. Young's inequality gives

$$\begin{aligned}\big\|\eta_{k,\delta}*(t_k|f_k|)^{\frac{p}{\theta}}\big\|_{\frac{r\theta}{p}}^{\frac{\theta}{p}}&\leq\big\|\eta_{k,\delta}|L_\mu(\mathbb{R}^n)\big\|^{\frac{\theta}{p}}\big\|t_k^{\frac{p}{\theta}}|f_k|^{\frac{p}{\theta}}|L_\theta(\mathbb{R}^n)\big\|^{\frac{\theta}{p}}\\&\lesssim 2^{nk(\frac{1}{p}-\frac{1}{r})}\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\|,\end{aligned}$$

where the implicit constant is independent of k but depends on p and r. Consequently (3.15) is bounded by

$$c2^{nk(\frac{1}{p}-\frac{1}{r})}\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\|\sum_{h\in\mathbb{Z}^n}(1+|h|)^{\delta\frac{\theta}{p}-\varrho+\frac{n}{\theta}}\lesssim 2^{nk(\frac{1}{p}-\frac{1}{r})}\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\|$$

for any ϱ large enough. □

Remark 3.2. (i) This lemma can be generalized to the case of $0<\theta\leq p\leq r\leq p(1+\varepsilon_{t_k^p})$ in view of the fact that

$$|f_k|\leq c\big(\eta_{k,m}*|f_k|^{\delta}\big)^{\frac{1}{\delta}},\quad \delta>0,\ m>n$$

for any $f_k\in L_p(\mathbb{R}^n,t_k)\cap\mathcal{S}'(\mathbb{R}^n)$ such that $\operatorname{supp}\mathcal{F}f_k\subset\{\xi\in\mathbb{R}^n:|\xi|\leq 2^k\},k\in\mathbb{Z}$, see [17, Lemma A.7].

(ii) The property (3.14) can be generalized in the following way. Let $0<\theta\leq p<\infty$ and $\{t_k\}$ be a p-admissible weight sequence such that $t_k^p\in A_{\frac{p}{\theta}}(\mathbb{R}^n)$, $k\in\mathbb{Z}$. Let $\varepsilon_{t_k^p}$ be as in Theorem 2.1. Let $f_k\in L_p(\mathbb{R}^n,t_k)\cap\mathcal{S}'(\mathbb{R}^n)$ be such that $\operatorname{supp}\mathcal{F}f_k\subset\{\xi\in\mathbb{R}^n:|\xi|\leq 2^k\},k\in\mathbb{Z}$.

- If t_k^p, $k\in\mathbb{Z}$ satisfies (2.3), then

  $$\big\|f_k|L_r(\mathbb{R}^n,t_j)\big\|\leq c\,2^{\alpha_1(j-k)+nk(\frac{1}{p}-\frac{1}{r})}\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\|$$

  holds for all $p\leq r\leq p(1+\varepsilon_{t_k}),k\in\mathbb{Z}$ and $k\geq j$, where the positive constant c is independent of k and j.

- If t_k^p, $k\in\mathbb{Z}$ satisfies (2.4) with $\sigma_2\geq p$, then

$$\big\|f_k|L_r(\mathbb{R}^n,t_j)\big\|\leq c\,2^{\alpha_2(j-k)+nk(\frac{1}{p}-\frac{1}{r})}\big\|f_k|L_p(\mathbb{R}^n,t_k)\big\|$$

holds for all $p\leq r\leq p(1+\varepsilon_{t_k^p}),k\in\mathbb{Z}$ and $j\geq k$, where the positive constant c is independent of k and j.

Theorem 3.3. *Let* $0<\theta\leq p<\infty$ *and* $0<q\leq\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$ *and* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$.

(i) *We have the embedding*

$$\mathcal{S}_\infty(\mathbb{R}^n)\hookrightarrow\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\}).$$

In addition $\mathcal{S}_\infty(\mathbb{R}^n)$ *is dense in* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *with* $1<\theta\leq p<\infty$ *and* $0<q<\infty$.

(ii) *Let* $1<\theta\leq p<\infty$ *and* $0<q\leq\infty$. *Then*

$$\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\hookrightarrow\mathcal{S}'_\infty(\mathbb{R}^n).$$

Proof. **Proof of** (i)**.** Let $f\in\mathcal{S}_\infty(\mathbb{R}^n)$. Let us recall the estimate of [66, Lemma 2.2]. For any $M\in N$, there exists a positive constant $C=C(M,n)$ such that for all $\varphi,\psi\in\mathcal{S}_\infty(\mathbb{R}^n),i,j\in\mathbb{Z}$ and $x\in\mathbb{R}^n$,

$$|\varphi_j*\psi_i(x)|\leq C\big\|\varphi|S_{M+1}\big\|\big\|\psi|S_{M+1}\big\|2^{-|i-j|M}\frac{2^{-(i\wedge j)M}}{(2^{-i\wedge j}+|x|)^{n+M}},$$

where $i\wedge j=\min(i,j)$. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy (3.1) and (3.2). Then we easily obtain

$$|\varphi_k*f(x)|\lesssim\big\|f|S_{M+1}\big\|2^{kM}\eta_{k,n+M}(x),\qquad k\leq 0,$$
$$|\varphi_k*f(x)|\lesssim\big\|f|S_{M+1}\big\|2^{-kM}\eta_{0,n+M}(x),\quad k>0.$$

Using Lemma 2.2, we obtain that

$$\big\|t_k\eta_{0,n+M}|L_p(\mathbb{R}^n)\big\|\lesssim M_{B(0,1),p}(t_k),\quad k>0,\ M>\frac{n}{\theta}.$$

Similarly we obtain

$$\big\|t_k\eta_{k,n+M}|L_p(\mathbb{R}^n)\big\|\lesssim 2^{kn(1-\frac{1}{\theta})}M_{B(0,1),p}(t_k),\quad k\leq 0,\ M>\frac{n}{\theta}.$$

We now consider two cases. If $k>0$, then by (2.4) we obtain

$$M_{B(0,1),p}(t_k)\leq 2^{k\alpha_2}M_{B(0,1),p}(t_0).$$

If $k\leq 0$, then by (2.3) we find that

$$M_{B(0,1),p}(t_k)M_{B(0,1),\sigma_1}(t_0^{-1})\leq 2^{k\alpha_1}.$$

Taking M large enough we get

$$\sum_{k=-\infty}^{\infty}\big\|\varphi_k*f|L_p(\mathbb{R}^n,t_k)\big\|^q\lesssim\big\|f|S_{M+1}\big\|\Big(\sum_{k=-\infty}^{0}2^{k(\alpha_1+n-\frac{n}{\theta}+M)q}+\sum_{k=1}^{\infty}2^{k(\alpha_2-M)q}\Big)$$
$$\lesssim\big\|f|S_{M+1}\big\|,$$

which completes the proof of the embedding $\mathcal{S}_\infty(\mathbb{R}^n)\hookrightarrow\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Let us prove the density of $\mathcal{S}_\infty(\mathbb{R}^n)$ in $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ if $1<\theta\leq p<\infty$ and $0<q<\infty$. Let φ, $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (3.1) through (3.3) and $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$. Let

$$f_N:=\sum_{k=-N}^{N}\tilde{\psi}_k*\varphi_k*f,\quad N\in\mathbb{N}.$$

Because

$$\varphi_j*\tilde{\psi}_k=0,\quad \text{if}\quad k\notin\{j-1,j,j+1\}$$

we have, see Lemma 2.4,

$$\big\|f_N|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|=\Big(\sum_{|k|\leq N+1}\big\|t_k(\varphi_k*\tilde{\psi}_k*\bar{\varphi}_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$
$$\lesssim\Big(\sum_{|k|\leq N+1}\big\|t_k\mathcal{M}(\varphi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$
$$\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|<\infty$$

for any $N\in\mathbb{N}$ where $\bar{\varphi}_k=\varphi_{k-1}+\varphi_k+\varphi_{k+1},k\in\mathbb{Z}$. Consequently,

$$\big\|f-f_N|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\leq\Big(\sum_{|k|\geq N+1}\big\|t_k(\varphi_k*\tilde{\psi}_k*\bar{\varphi}_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$
$$\lesssim\Big(\sum_{|k|\geq N+1}\big\|t_k\mathcal{M}(\varphi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$
$$\lesssim\Big(\sum_{|k|\geq N+1}\big\|t_k(\varphi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}},$$

where we have used again Lemma 2.4. The dominated convergence theorem implies that f_N approximates f in $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$. But f_N, $N\in\mathbb{N}$ is not necessary an element of $\mathcal{S}_\infty(\mathbb{R}^n)$, so we need to approximate f_N in $\mathcal{S}_\infty(\mathbb{R}^n)$. Let $\omega\in\mathcal{S}(\mathbb{R}^n)$ with $\omega(0)=1$ and $\operatorname{supp}\mathcal{F}\omega\subset\{\xi:|\xi|\leq1\}$. Put

$$f_{N,\delta}:=f_N\omega(\delta\cdot),\quad 0<\delta<1.$$

We have $f_{N,\delta}\in\mathcal{S}_\infty(\mathbb{R}^n)$ see [66, Lemma 5.3], and

$$f_N-f_{N,\delta}=\sum_{k=-N}^{N}(\tilde{\psi}_k*\varphi_k*f)(1-\omega(\delta\cdot)).$$

After simple calculation, we obtain

$$\varphi_j*[(\tilde{\psi}_k*\varphi_k*f)(\omega(\delta\cdot))](x)=\int_{\mathbb{R}^n}\varphi_k*f(y)\varphi_j*(\tilde{\psi}_k\omega(\delta(\cdot+y))(x-y)dy,\quad x\in\mathbb{R}^n,$$

which together with the fact that

$$\mathrm{supp}\mathcal{F}(\tilde{\psi}_k\omega(\delta(\cdot+y))\subset\{\xi:2^{k-2}\le|\xi|\le 2^{k+2}\},\quad y\in\mathbb{R}^n,\ |k|\le N$$

if $0<\delta<2^{-N-3}$ yield that

$$\varphi_j*[(\tilde{\psi}_k*\varphi_k*f)(\omega(\delta\cdot))]=0\quad\text{if}\quad |j-k|\ge 2.$$

Therefore, we obtain that $\big\|f_N-f_{N,\delta}|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$ can be estimated by

$$\begin{aligned}&\Big(\sum_{|k|\le N+2}\Big\|t_k(\varphi_k*\sum_{i=-2}^{2}[(\tilde{\psi}_{k+i}*\varphi_{k+i}*f)(1-\omega(\delta\cdot))])|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}\\ \lesssim&\sum_{i=-2}^{2}\Big(\sum_{|k|\le N+2}\Big\|t_{k+i}\big((\tilde{\psi}_{k+i}*\varphi_{k+i}*f)(1-\omega(\delta\cdot))\big)|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}.\end{aligned}$$

Again, by Lebesgue's dominated convergence theorem $f_{N,\delta}$ approximates f_N in the spaces $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$. This prove that $\mathcal{S}_\infty(\mathbb{R}^n)$ is dense in $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Proof of (ii)**.** Let $\varphi,\psi\in\mathcal{S}(\mathbb{R}^n)$ such that supp $\mathcal{F}\varphi$, supp $\mathcal{F}\psi$ are compact and bounded away from the origin and (3.3) holds. If $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ and $\omega\in\mathcal{S}_\infty(\mathbb{R}^n)$, then $\langle f,\omega\rangle$ denotes the value of the functional f of $\mathcal{S}'_\infty(\mathbb{R}^n)$ for the test function ω. We obtain

$$\begin{aligned}|\langle f,\omega\rangle|&\le\sum_{k=-\infty}^{\infty}|\langle\psi_k*\widetilde{\varphi}_k*f,\omega\rangle|\\&\le\sum_{k=-\infty}^{\infty}\big\|\widetilde{\varphi}_k*f|L_p(\mathbb{R}^n,t_0)\big\|\big\|\psi_k*\omega|L_{p'}(\mathbb{R}^n,t_0^{-1})\big\|.\end{aligned}\tag{3.16}$$

The right-hand side of (3.16) can be rewritten as I_1+I_2, where

$$\begin{aligned}I_1&:=\sum_{k=-\infty}^{0}\big\|\widetilde{\varphi}_k*f|L_p(\mathbb{R}^n,t_0)\big\|\big\|\psi_k*\omega|L_{p'}(\mathbb{R}^n,t_0^{-1})\big\|,\\ I_2&:=\sum_{k=1}^{\infty}\big\|\widetilde{\varphi}_k*f|L_p(\mathbb{R}^n,t_0)\big\|\big\|\psi_k*\omega|L_{p'}(\mathbb{R}^n,t_0^{-1})\big\|.\end{aligned}$$

We estimate I_1. Similarly, as in the proof of (i),

$$|\psi_k * \omega(x)| \lesssim 2^{kL} \eta_{k,M}(x), \quad k \le 0,\ M > 0,$$

with L large enough. Observe that $t_0^p \in A_p(\mathbb{R}^n)$ and from Lemma 2.1 (ii),

$$t_0^{-p'} \in A_{p'}(\mathbb{R}^n).$$

By Lemma 2.1 (iv) there exists $1 < \kappa < p'$ such that $t_0^{-p'} \in A_{\frac{p'}{\kappa}}(\mathbb{R}^n)$. Using Lemma 2.2, we obtain that

$$\big\| t_0^{-1} \eta_{k,M} | L_{p'}(\mathbb{R}^n) \big\| \lesssim 2^{kn(1-\frac{1}{\kappa})} M_{B(0,1),p'}(t_0^{-1}), \quad k \le 0,\ M > \frac{n}{\kappa}.$$

These estimates guarantee that, I_1 is bounded by $c\big\| f | \dot{B}_{p,\infty}(\mathbb{R}^n, \{t_k\}) \big\|$.

We proceed in a similar way and by

$$|\psi_k * \omega(x)| \lesssim 2^{-kL} \eta_{0,M}(x), \quad k > 0,$$

the term I_2 can be estimated by $c\big\| f | \dot{B}_{p,\infty}(\mathbb{R}^n, \{t_k\}) \big\|$, which completes the proof. □

3.2 Embeddings

For our spaces introduced above we want to show some embedding theorems. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f|A_2\| \le c\|f|A_1\|$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 3.4. *Let $0 < p < \infty, 0 < q \le r \le \infty$, and $\{t_k\}$ be a p-admissible weight sequence satisfying* (2.3) *with $\sigma_1 = \theta\left(\frac{p}{\theta}\right)'$ and $j = k$. We have*

$$\dot{B}_{p,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \dot{B}_{p,r}(\mathbb{R}^n, \{t_k\}).$$

The proof can be obtained by using the properties of sequence Lebesgue spaces. The main result of this subsection is the following Sobolev-type embedding. In the classical setting this was done in [37] and [58]. We set

$$w_{k,Q}(p_1) := \Big(\int_Q w_k^{p_1}(x) dx \Big)^{\frac{1}{p_1}} \quad \text{and} \quad t_{k,Q}(p_0) := \Big(\int_Q t_k^{p_0}(x) dx \Big)^{\frac{1}{p_0}}, \quad Q \in \mathcal{Q}$$

with $\ell(Q) = 2^{-k}$.

Theorem 3.5. *Let $0 < \theta \le p_0 < p_1 < \infty$ and $0 < q \le \infty$. Let $\{t_k\}$ be a p_0-admissible weight sequence satisfying* (2.3) *with $p = p_0$, $\sigma_1 = \theta\left(\frac{p_0}{\theta}\right)'$ and $j = k$. Let $\{w_k\}$ be a p_1-admissible weight sequence satisfying* (2.3) *with $p = p_1$, $\sigma_1 = \theta\left(\frac{p_1}{\theta}\right)'$ and $j = k$. If $w_{k,Q}(p_1) \lesssim t_{k,Q}(p_0)$ for all $Q \in \mathcal{Q}$ with $\ell(Q) = 2^{-k}$, then we have*

$$\dot{b}_{p_0,q}(\mathbb{R}^n, \{t_k\}) \hookrightarrow \dot{b}_{p_1,q}(\mathbb{R}^n, \{w_k\}).$$

Proof. By the embedding $\ell_{p_0} \hookrightarrow \ell_{p_1}$ and the fact that $w_{k,m}(p_1) \le t_{k,m}(p_0)$, $k \in \mathbb{Z}$, $m \in \mathbb{Z}^n$, where

$$w_{k,m}(p_1) = w_{k,Q_{k,m}}(p_1) \quad \text{and} \quad t_{k,m}(p_0) = t_{k,Q_{k,m}}(p_0),$$

we obtain our embedding. Hence the theorem is proved. □

From Theorems 3.1 and 3.5, we have the following Sobolev-type embedding conclusions for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Theorem 3.6. *Let* $1 < \theta \le p_0 < p_1 < \infty$ *and* $0 < q \le \infty$. *Let* $\{t_k\} \in \dot{X}_{\alpha_0,\sigma,p_0}$ *be a* p_0*-admissible weight sequence with* $\sigma = (\sigma_1 = \theta\left(\frac{p_0}{\theta}\right)', \sigma_2 \ge p_0)$ *and* $\alpha_0 = (\alpha_{1,0},\alpha_{2,0}) \in \mathbb{R}^2$. *Let* $\{w_k\} \in \dot{X}_{\alpha_1,\sigma,p_1}$ *be a* p_1*-admissible weight sequence with* $\sigma = (\sigma_1 = \theta\left(\frac{p_1}{\theta}\right)', \sigma_2 \ge p_1)$ *and* $\alpha_1 = (\alpha_{1,1},\alpha_{2,1}) \in \mathbb{R}^2$. *Then*

$$\dot{B}_{p_0,q}(\mathbb{R}^n,\{t_k\}) \hookrightarrow \dot{B}_{p_1,q}(\mathbb{R}^n,\{w_k\}),$$

holds if

$$w_{k,Q}(p_1) \lesssim t_{k,Q}(p_0)$$

for all $Q \in \mathcal{Q}$ *and all* $k \in \mathbb{Z}$.

From Theorem 3.6 we can obtain some special Sobolev embeddings.

Theorem 3.7. *Let* $1 < \theta \le p_0 \le p_1 < \infty$, $0 < q \le \infty$. *Let* $\{t_k\} \in \dot{X}_{\alpha,\sigma^0,p_0} \cap \dot{X}_{\alpha,\sigma^1,p_1}$ *with* $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2, \sigma^0 = (\sigma_1^0 = \theta\left(\frac{p_0}{\theta}\right)', \sigma_2^0 \ge p_0)$ *and* $\sigma^1 = (\sigma_1^1 = \theta\left(\frac{p_1}{\theta}\right)', \sigma_2^1 \ge p_1)$. *Let* $\varepsilon_{t_k^{p_0}}$ *be as in Theorem* 2.1. *Then*

$$\dot{B}_{p_0,q}(\mathbb{R}^n,\{t_k\}) \hookrightarrow \dot{B}_{p_1,q}(\mathbb{R}^n,\{2^{(\frac{n}{p_1}-\frac{n}{p_0})k} t_k\})$$

holds if $p_0 \le p_1 < p_0(1+\varepsilon_{t_k^{p_0}})$.

4 Atomic, molecular and wavelet decompositions

In recent years, it turned out that atomic and molecular, as well as wavelet decompositions of some function spaces are extremely useful in many aspects. This concerns, for instance, the investigation of (compact) embeddings between function spaces. The idea of atomic and molecular decompositions leads back to M. Frazier and B. Jawerth in their series of papers [25, 26], see also [33, 60, 68].

The main goal of this section is to prove atomic, molecular and wavelet decomposition results for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

4.1 Atomic and molecular decompositions

We will use the notation of [26]. We shall say that an operator A is associated with the matrix $\{a_{Q_{k,m}P_{v,h}}\}_{k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n}$, if for all sequences $\lambda = \{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} \subset \mathbb{C}$,

$$A\lambda = \{(A\lambda)_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} := \Big\{ \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} a_{Q_{k,m}P_{v,h}} \lambda_{v,h} \Big\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}.$$

We will use the following notation

$$J:=\frac{n}{\min(1,p)},\quad 0<p<\infty.$$

We say that A, with associated matrix $\{a_{Q_{k,m}P_{v,h}}\}_{k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n}$, is almost diagonal on $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ if there exists $\varepsilon>0$ such that

$$\sup_{k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n}\frac{|a_{Q_{k,m}P_{v,h}}|}{\omega_{Q_{k,m}P_{v,h}}(\varepsilon)}<\infty,$$

where

$$\omega_{Q_{k,m}P_{v,h}}(\varepsilon):=\Big(1+\frac{|x_{Q_{k,m}}-x_{P_{v,h}}|}{\max\left(2^{-k},2^{-v}\right)}\Big)^{-J-\varepsilon}\begin{cases}2^{(v-k)(\alpha_2+\frac{n+\varepsilon}{2})}, & \text{if}\quad v\leq k,\\ 2^{(v-k)(\alpha_1-\frac{n+\varepsilon}{2}-J+n)}, & \text{if}\quad v>k.\end{cases}$$

Due to Lemma 2.6, the following theorem is a generalization of [26, Theorem 3.3], see also [68, Theorem 3.1].

Theorem 4.1. *Let $\alpha_1,\alpha_2\in\mathbb{R}$, $0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_k\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2\geq p$. An almost diagonal operator on $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ is bounded.*

Definition 4.1. *Let $\alpha_1,\alpha_2\in\mathbb{R}$, $0<p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}$ be a p-admissible weight sequence. Let $N=\max\{J-n-\alpha_1,-1\}$ and $\alpha_2^*=\alpha_2-\lfloor\alpha_2\rfloor$.*

(i) *A function $\varrho_{Q_{k,m}}$, $k\in\mathbb{Z},m\in\mathbb{Z}^n$, is called an homogeneous smooth synthesis molecule for the space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ supported near $Q_{k,m}$ if there exist a real number $\delta\in(\alpha_2^*,1]$ and a real number $M\in(J,\infty)$ such that*

$$\int_{\mathbb{R}^n}x^\beta\varrho_{Q_{k,m}}(x)dx=0\quad\text{if}\quad 0\leq|\beta|\leq N,\tag{4.1}$$

$$|\varrho_{Q_{k,m}}(x)|\leq 2^{k\frac{n}{2}}(1+2^k|x-x_{Q_{k,m}}|)^{-\max(M,M-\alpha_1)},\tag{4.2}$$

$$|\partial^\beta\varrho_{Q_{k,m}}(x)|\leq 2^{k(|\beta|+\frac{n}{2})}(1+2^k|x-x_{Q_{k,m}}|)^{-M}\quad\text{if}\quad|\beta|\leq\lfloor\alpha_2\rfloor,\tag{4.3}$$

$$\begin{aligned}&|\partial^\beta\varrho_{Q_{k,m}}(x)-\partial^\beta\varrho_{Q_{k,m}}(y)|\\ \leq\,&2^{k(|\beta|+\frac{n}{2}+\delta)}|x-y|^\delta\sup_{|z|\leq|x-y|}(1+2^k|x-z-x_{Q_{k,m}}|)^{-M}\quad\text{if}\quad|\beta|=\lfloor\alpha_2\rfloor.\end{aligned}\tag{4.4}$$

A collection $\{\varrho_{Q_{k,m}}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ is called a family of homogeneous smooth synthesis molecules for the space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, if each $\varrho_{Q_{k,m}}$ is an homogeneous smooth synthesis molecule for the space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ supported near $Q_{k,m}$.

(ii) *A function $b_{Q_{k,m}}$, $k\in\mathbb{Z},m\in\mathbb{Z}^n$, is called an homogeneous smooth analysis molecule for the space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ supported near $Q_{k,m}$ if there exist a $\kappa\in((J-\alpha_2)^*,1]$ and an $M\in(J,\infty)$ such* *that such that*

$$\int_{\mathbb{R}^n} x^\beta b_{Q_{k,m}}(x)dx = 0 \quad \textit{if} \quad 0 \leq |\beta| \leq \lfloor \alpha_2 \rfloor, \tag{4.5}$$

$$|b_{Q_{k,m}}(x)| \leq 2^{k\frac{n}{2}}(1+2^k|x-x_{Q_{k,m}}|)^{-\max(M,M+n+\alpha_2-J)}, \tag{4.6}$$

$$|\partial^\beta b_{Q_{k,m}}(x)| \leq 2^{k(|\beta|+\frac{n}{2})}(1+2^k|x-x_{Q_{k,m}}|)^{-M} \quad \textit{if} \quad |\beta| \leq N, \tag{4.7}$$

$$|\partial^\beta b_{Q_{k,m}}(x) - \partial^\beta b_{Q_{k,m}}(y)| \tag{4.8}$$

$$\leq 2^{k(|\beta|+\frac{n}{2}+\kappa)}|x-y|^\kappa \sup_{|z|\leq|x-y|}(1+2^k|x-z-x_{Q_{k,m}}|)^{-M} \quad \textit{if} \quad |\beta| = N.$$

A collection $\{b_{Q_{k,m}}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *is called a family of homogeneous smooth analysis molecules for the space* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$*, if each* $b_{Q_{k,m}}$ *is an homogeneous smooth synthesis molecule for the space* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *supported near* $Q_{k,m}$*.*

We will use the notation $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ instead of $\{b_{Q_{k,m}}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$.

Lemma 4.1. *Let* $\alpha_1,\alpha_2,J,M,N,\delta,\kappa,p$ *and* q *be as in Definition* 4.1*. Let* $\{t_k\}$ *be a* p*-admissible weight sequence. Suppose* $\{\varrho_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}$ *is a family of smooth synthesis molecules for* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *and* $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *is a family of homogeneous smooth analysis molecules for* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$*. Then there exist a positive real number* ε_1 *and a positive constant* c *such that*

$$|\langle \varrho_{v,h}, b_{k,m}\rangle| \leq c\,\omega_{Q_{k,m}P_{v,h}}(\varepsilon), \quad k,v\in\mathbb{Z}, h, m\in\mathbb{Z}^n$$

if $\varepsilon \leq \varepsilon_1$*.*

Proof. The proof is a slight modification of [26, Corollary B.3]. Possibly reducing δ, ϱ, or M, we may assume that $\delta - \alpha_2^* = \frac{M-J}{2} = \kappa - (J-\alpha_2)^* > 0$. First we suppose that $k \geq v$ and $\alpha_2 \geq 0$. We have

$$\langle \varrho_{v,h}, b_{k,m}\rangle = g_{v,h} * b_{k,m}(x_{P_{v,h}})$$

with $g_{v,h}(x) = \overline{\varrho_{v,h}(x_{P_{v,h}} - x)}$. Applying Lemma B.1 of [26] we obtain

$$\begin{aligned}|\langle \varrho_{v,h}, b_{k,m}\rangle| &\leq c\, 2^{-(k-v)(\lfloor\alpha_2\rfloor+\frac{n}{2}+\delta)}(1+2^v|x_{P_{v,h}}-x_{Q_{k,m}}|)^{-M}\\ &\leq c\, 2^{-(k-v)(\alpha_2+\frac{n+\varepsilon}{2})}(1+2^v|x_{P_{v,h}}-x_{Q_{k,m}}|)^{-M}\end{aligned}$$

if $\lfloor\alpha_2\rfloor + \delta \geq \alpha_2 + \frac{\varepsilon}{2}$ for some $\varepsilon > 0$ small enough, but this is possible since $\delta > \alpha_2^*$. In view if the fact that $\delta \leq 1$, we will take $\varepsilon < 2(\delta - \alpha_2^*)$.

Now if $k \geq v$ and $\alpha_2 < 0$, then by Lemma B.2 of [26], we find that

$$\begin{aligned}|\langle \varrho_{v,h}, b_{k,m}\rangle| &\leq c\, 2^{-(k-v)\frac{n}{2}}(1+2^v|x_{P_{v,h}}-x_{Q_{k,m}}|)^{-M}\\ &\leq c\, 2^{-(k-v)(\alpha_2+\frac{n+\varepsilon}{2})}(1+2^v|x_{P_{v,h}}-x_{Q_{k,m}}|)^{-M}\end{aligned}$$

if $0 < \varepsilon < -2\alpha_2$.

We suppose that $k<v$ and $N\geq 0$. We have $\langle \varrho_{v,h},b_{k,m}\rangle = g_{k,m}*\varrho_{v,h}(x_{Q_{k,m}})$, with $g_{k,m}(x)=\overline{b_{k,m}(x_{Q_{k,m}}-x)}$. Again, using Lemma B.1 of [26], we obtain

$$\begin{aligned}|\langle \varrho_{v,h},b_{k,m}\rangle| &\leq c\, 2^{-(v-k)(N+\frac{n}{2}+\kappa)}(1+2^k|x_{Q_{v,h}}-x_{Q_{k,m}}|)^{-M}\\ &\leq c\, 2^{(v-k)(\alpha_1-J-\frac{\varepsilon-n}{2})}(1+2^k|x_{Q_{v,h}}-x_{Q_{k,m}}|)^{-M},\end{aligned}$$

since

$$N+\frac{n}{2}+\kappa>\frac{\varepsilon}{2}+J-\frac{n}{2}-\alpha_1$$

for any $0<\varepsilon<2\kappa$.

Now if that $k<v$ and $N=-1$, then we apply Lemma B.2 of [26], since $N=-1$ implies $n+\alpha_1>J$ so that $n>-\alpha_1+\frac{\varepsilon}{2}+J$, and obtain

$$\begin{aligned}|\langle \varrho_{v,h},b_{k,m}\rangle| &\leq c\, 2^{-(v-k)\frac{n}{2}}(1+2^k|x_{Q_{v,h}}-x_{Q_{k,m}}|)^{-M}\\ &\leq c\, 2^{(v-k)(\alpha_1-J-\frac{\varepsilon-n}{2})}(1+2^k|x_{Q_{v,h}}-x_{Q_{k,m}}|)^{-M}\end{aligned}$$

if $0<\varepsilon<2(\alpha_1-J+n)$. The proof is complete. □

As an immediate consequence, we have the following analogues of the corresponding results on [26, Corollary B.3].

Corollary 4.1. *Let $\alpha_1,\alpha_2,J,M,N,\delta,\kappa,p$ and q be as in Definition* 4.1. *Let $\{t_k\}$ be a p-admissible weight sequence. Let Φ and φ satisfy, respectively* (3.1) *and* (3.2).

(i) *If $\{\varrho_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ is a family of homogeneous synthesis molecules for the function space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, then the operator A with matrix entry $a_{Q_{k,m}P_{v,h}}=\langle \varrho_{v,h},\varphi_{k,m}\rangle$, $k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n$, is almost diagonal.*

(ii) *If $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ is a family of homogeneous smooth analysis molecules for the function space $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, then the operator A, with matrix entry $a_{Q_{k,m}P_{v,h}}=\langle \varphi_{v,h},b_{Q_{k,m}}\rangle$, $k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n$, is almost diagonal.*

Let $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ and $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ be a family of homogeneous smooth analysis molecules. To prove that $\langle f,b_{Q_{k,m}}\rangle$, $k\in\mathbb{Z},m\in\mathbb{Z}^n$, is well defined for all homogeneous smooth analysis molecules for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, we need the following result, which was proved in [7, Lemma 5.4]. Suppose that Φ is a smooth analysis (or synthesis) molecule supported near $Q\in\mathcal{Q}$. Then there exist a sequence $\{\varphi_k\}_{k\in\mathbb{N}}\subset\mathcal{S}(\mathbb{R}^n)$ and $c>0$ such that $c\varphi_k$ is a smooth analysis (or synthesis) molecule supported near Q for every k, and $\varphi_k(x)\to\Phi(x)$ uniformly on $\mathbb{R}^n$ as $k\to\infty$.

Now we have the following smooth molecular characterization of $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Theorem 4.2. *Let α_1, $\alpha_2\in\mathbb{R}$, $0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2\geq p$. Let J,M,N,δ and κ be as in Definition* 4.1.

(i) *If* $f=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\varrho_{k,m}\lambda_{k,m}$*, where* $\{\varrho_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *is a family of homogeneous smooth synthesis molecules for* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$*, then for all* $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

(ii) *Let* $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *be a family of homogeneous smooth analysis molecules. Then for all* $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$

$$\big\|\{\langle f,b_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Proof. **Proof of (i).** By (3.4) we can write

$$\varrho_{v,h}=\sum_{k=-\infty}^{\infty}2^{-kn}\sum_{m\in\mathbb{Z}^n}\widetilde{\varphi}_k*\varrho_{v,h}(2^{-k}m)\psi_k(\cdot-2^{-k}m)$$

for any $v\in\mathbb{Z},h\in\mathbb{Z}^n$. Therefore,

$$f=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}S_{k,m}\psi_{k,m}=T_\psi S,$$

where $S=\{S_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$, with

$$S_{k,m}=2^{-\frac{kn}{2}}\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\widetilde{\varphi}_k*\varrho_{v,h}(2^{-k}m)\lambda_{v,h}.$$

From Theorem 3.1, we have

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|=\big\|T_\psi S|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|S|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

But

$$S_{k,m}=\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}a_{Q_{k,m}P_{v,h}}\lambda_{v,h}$$

with

$$a_{Q_{k,m}P_{v,h}}=\langle\varrho_{v,h},\widetilde{\varphi}_{k,m}\rangle,\quad k,v\in\mathbb{Z},m,h\in\mathbb{Z}^n.$$

Applying Lemma 4.1 and Theorem 4.1 we find that

$$\big\|S|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Proof of (ii)**.** We have

$$\begin{aligned}\langle f,b_{k,m}\rangle&=\sum_{v=-\infty}^{\infty}2^{-vn}\sum_{m\in\mathbb{Z}^n}\langle\psi_v(\cdot-2^{-v}h),b_{k,m}\rangle\widetilde{\varphi}_v*f(2^{-v}h)\\&=\sum_{v=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle\psi_{v,h},b_{k,m}\rangle\lambda_{v,h}\\&=\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}a_{Q_{k,m}P_{v,h}}\lambda_{v,h},\end{aligned}$$

where

$$a_{Q_{k,m}P_{v,h}} = \langle \psi_{v,h}, b_{k,m} \rangle, \quad \lambda_{v,h} = 2^{-\frac{vn}{2}} \widetilde{\varphi}_v * f(2^{-v}h).$$

Again by Lemma 4.1 and Theorem 4.1 we find that

$$\begin{aligned}\big\| \{\langle f, b_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\| &\lesssim \big\| \{\lambda_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\| \\ &= c \big\| \{(S_\varphi)_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\|.\end{aligned}$$

Applying Theorem 3.1 we find that

$$\big\| \{\langle f, b_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\| \lesssim \big\| f | \dot{B}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\|. \qquad \square$$

Definition 4.2. *Let $\alpha_1,\alpha_2 \in \mathbb{R}$, $0<p<\infty$, $0<q\le\infty$ and $N=\max\{J-n-\alpha_1,-1\}$. Let $\{t_k\}$ be a p-admissible weight sequence. A function $a_{Q_{v,m}}$ is called an homogeneous smooth atom for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ supported near $Q_{k,m}$, $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^n$, if*

$$\operatorname{supp} a_{Q_{k,m}} \subseteq 3Q_{k,m}, \tag{4.9}$$

$$|\partial^\beta a_{Q_{k,m}}(x)| \le 2^{kn(|\beta|+1/2)} \quad \text{if} \quad 0\le|\beta|\le \max(0,1+\lfloor\alpha_2\rfloor), \quad x\in\mathbb{R}^n \tag{4.10}$$

and if

$$\int_{\mathbb{R}^n} x^\beta a_{Q_{k,m}}(x)dx = 0 \quad \text{if} \quad 0\le|\beta|\le N \text{ and } k\in\mathbb{Z}. \tag{4.11}$$

A collection $\{a_{Q_{k,m}}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ is called a family of homogeneous smooth atoms for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, if each $a_{Q_{k,m}}$ is an homogeneous smooth atom for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ supported near $Q_{v,m}$. We point out that in the moment condition (4.11) can be strengthened into that

$$\int_{\mathbb{R}^n} x^\beta a_{Q_{k,m}}(x)dx = 0 \quad \text{if} \quad 0\le|\beta|\le \tilde{N} \text{ and } k\in\mathbb{Z}$$

and the regularity condition (4.10) can be strengthened into that

$$|\partial^\beta a_{Q_{k,m}}(x)| \le 2^{kn(|\beta|+1/2)} \quad \text{if} \quad 0\le|\beta|\le\tilde{K}, \quad x\in\mathbb{R}^n,$$

where $\tilde{K}$ and $\tilde{N}$ are arbitrary fixed integer satisfying $\tilde{K}\ge\max(0,1+\lfloor\alpha_2\rfloor)$ and $\tilde{N}\ge\max\{J-n-\alpha_1,-1\}$. If an atom a is supported near $Q_{v,m}$, then we denote it by $a_{v,m}$.

Now we come to the atomic decomposition theorem.

Theorem 4.3. *Let α_1, $\alpha_2\in\mathbb{R}$, $0<\theta\le p<\infty$, $0<q\le\infty$. Let $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2\ge p$. Then for each $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, there exist a family $\{\varrho_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ of homogeneous smooth atoms for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ and $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ such that*

$$f=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\varrho_{k,m}, \quad \text{converging in } \mathcal{S}'_\infty(\mathbb{R}^n),$$

and

$$\big\|\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Conversely, for any family of homogeneous smooth atoms for $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *and* $\lambda =\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$

$$\Big\|\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\varrho_{k,m}|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\Big\|\lesssim\big\|\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

The proof of this theorem can be obtained by repeating the arguments of [26, Theorem 4.1].

4.2 Wavelet decompositions

Based on the decomposition system for $L_2(\mathbb{R}^n)$, the main aim of this section consists in proving another characterization of the above spaces. We use the method developed by Kyriazis [43]. First we give some notation which will be used here. Let $E=\{1,\cdots,j\}$, $j\in\mathbb{N}$ be a finite set and

$$\Phi=\{\varrho_{k,m,i}:i\in E,k\in\mathbb{Z},m\in\mathbb{Z}^n\}$$

a decomposition system for $L_2(\mathbb{R}^n)$ with dual functionals

$$\widetilde{\Phi}=\{\widetilde{\varrho}_{k,m,i}:i\in E,k\in\mathbb{Z},m\in\mathbb{Z}^n\}$$

and in addition

$$f=\sum_{i=1}^{j}\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varrho}_{k,m,i}\rangle\varrho_{k,m,i}, \tag{4.12}$$

in the sense of $L_2(\mathbb{R}^n)$. For every $\kappa\in\mathbb{N}$ we define

$$\mathcal{S}_\kappa(\mathbb{R}^n):=\Big\{\varphi\in\mathcal{S}(\mathbb{R}^n):\int_{\mathbb{R}^n}x^\gamma\varphi(x)dx=0 \text{ for all multi-indices } |\gamma|\leq\kappa\Big\}$$

and we identify the dual space of $\mathcal{S}_\kappa(\mathbb{R}^n)$ with $\mathcal{S}'_\kappa(\mathbb{R}^n)$ the set of all continuous linear functionals on $\mathcal{S}_\kappa(\mathbb{R}^n)$. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfies (3.1)–(3.3) with $\psi=\varphi$. Let $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $0<\theta\leq p\leq\infty$, $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$ and $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$. From [43, Section 2]

$$f=\sum_{k=-\infty}^{\infty}\varphi_k*\widetilde{\varphi}_k*f,\quad \text{in}\quad \mathcal{S}'_\kappa(\mathbb{R}^n),$$

with $\kappa=\max(\lfloor\alpha_2-\frac{n\delta}{p}\rfloor,-1)$ and $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$, where δ is the positive constant obtained by Lemma 2.1 (v). Indeed, again from [43, Lemma 6.2] we need only to prove that

$$\sum_{k=-\infty}^{-1}\partial^\beta(\varphi_k*\widetilde{\varphi}_k*f), \tag{4.13}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$ for every $|\beta|>\kappa$. To do this we need the following lemma.

Lemma 4.2. *Let* $0<\theta\leq p<\infty, 0<q\leq\infty$ *and* δ *be as in Lemma* 2.1 *(v). Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a* p*-admissible weight sequence with* $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$ *and* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$*. Let* $\varphi\in\mathcal{S}(\mathbb{R}^n)$ *satisfies* (3.1) *and* (3.2)*. Then*

$$|\langle\varphi_k*f,\omega_{\min(0,k)}\rangle|\lesssim\varpi_k\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$$

holds for any $k\in\mathbb{Z}$, $\omega\in\mathcal{S}(\mathbb{R}^n)$ *and any* $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$*, where*

$$\varpi_k=\begin{cases}2^{(\frac{n\delta}{p}-\alpha_2)k} & \text{if} \quad k\leq 0,\\ 2^{(\frac{n\delta}{p}-\alpha_1)k} & \text{if} \quad k\in\mathbb{N}.\end{cases}$$

Proof. We have

$$|\langle\varphi_k*f,\omega_{\min(0,k)}\rangle|\leq\int_{\mathbb{R}^n}|\varphi_k*f(y)||\omega_{\min(0,k)}(y)|dy.$$

Splitting the integration with respect to the cubes $Q_{k,m}$ we derive

$$|\langle\varphi_k*f,\omega_{\min(0,k)}\rangle|\leq\sum_{m\in\mathbb{Z}^n}\int_{Q_{k,m}}\sup_{z\in Q_{k,m}}|\varphi_k*f(z)||\omega_{\min(0,k)}(y)|dy.$$

We recall the following estimate see (2.11) in [25],

$$\sup_{z\in Q_{k,m}}|\varphi_k*f(z)|\lesssim 2^{k\frac{n}{\tau}}\Big(\sum_{h\in\mathbb{Z}^n}(1+|h|)^{-M}\int_{Q_{k,m+h}}|\varphi_k*f(x)|^\tau dx\Big)^{\frac{1}{\tau}},\tag{4.14}$$

with $M>n,\tau>0$. Taking $\frac{1}{\tau}=\frac{1}{p}+\frac{1}{\sigma_1}$, we get by Hölder's inequality that

$$\begin{aligned}\Big(\int_{Q_{k,m+h}}|\varphi_k*f(x)|^\tau dx\Big)^{\frac{1}{\tau}}&\leq|Q_{k,m+h}|^{\frac{1}{\tau}}M_{Q_{k,m+h},p}(t_k(\varphi_k*f))M_{Q_{k,m+h},\sigma_1}(t_k^{-1})\\&\leq|Q_{k,m+h}|^{\frac{1}{\sigma_1}}\big\|\varphi_k*f|L_p(\mathbb{R}^n,t_k)\big\|M_{Q_{k,m+h},\sigma_1}(t_k^{-1}).\end{aligned}$$

From (2.3),

$$M_{Q_{k,m+h},\sigma_1}(t_k^{-1})\leq C\big(M_{Q_{k,m+h},p}(t_k)\big)^{-1},$$

where the positive constant C is independent of k, m and h. First assume that $k\leq 0$. In (2.4) putting $Q=Q_{k,\widetilde{m+h}}$ and $j=0$, where

$$Q_{k,\widetilde{m+h}}=\prod_{i=1}^n[2^{-k}(m_i+h_i),2^{-k}(m_i+h_i)+2^{1-k}),$$

we get

$$M_{Q_{k,\widetilde{m+h}}}(t_0^p)\leq\big(M_{Q_{k,\widetilde{m+h}}}(t_0^{\sigma_2})\big)^{\frac{p}{\sigma_2}}\lesssim 2^{-\alpha_2pk}M_{Q_{k,\widetilde{m+h}}}(t_k^p),\quad k\leq 0,\ m\in\mathbb{Z}^n.$$

Obviously

$$Q_{k,\widetilde{m+h}} \subset E_{k,m+h}, \quad k \le 0,\, m,h \in \mathbb{Z}^n,$$

where

$$E_{k,m+h} = \prod_{i=1}^{n} (a_i, b_i)$$

with

$$\begin{aligned} a_i &= \min(-2^{1-k}, -2^{-k}|m+h|), \qquad & i = 1,\cdots,n, \\ b_i &= \max(2^{1-k}|m+h|, 2^{-k}|m+h| + 2^{1-k}), \qquad & i = 1,\cdots,n \end{aligned}$$

if $m+h \neq 0$ and

$$E_{k,m+h} = \prod_{i=1}^{n} \left[-2^{1-k}, 2^{1-k}\right),$$

if $m+h=0$. Since $t_k^p \in A_{\frac{p}{\theta}}$, $k \in \mathbb{Z}$, from Lemma 2.1 (iii) there exists constant $C>0$ such that

$$\begin{aligned} M_{Q_{k,m+h}}(t_k^p) &\ge C\Big(\frac{|Q_{k,m+h}|}{|Q_{k,\widetilde{m+h}}|}\Big)^{\frac{p}{\theta}-1} M_{Q_{k,\widetilde{m+h}}}(t_k^p), \quad k \le 0,\, m,h \in \mathbb{Z}^n, \\ M_{Q_{k,\widetilde{m+h}}}(t_0^p) &\ge C\Big(\frac{|Q_{k,\widetilde{m+h}}|}{|E_{k,m+h}|}\Big)^{\frac{p}{\theta}-1} M_{E_{k,m+h}}(t_0^p) \\ &\ge C 2^{kn(1-\frac{p}{\theta})}\Big(2^{-k} + 2^{-k}|m+h|\Big)^{\frac{-np}{\theta}+n} M_{E_{k,m+h}}(t_0^p) \\ &\ge C 2^{kn}(1+|m+h|)^{\frac{-np}{\theta}} t_0^p(E_{k,m+h}), \end{aligned}$$

where the constant C is independent of k,m and h. Using the fact that

$$A = \prod_{i=1}^{n} [-1,1) \subset E_{k,m+h},$$

and Lemma 2.1 (v) we get

$$\big(M_{Q_{k,m+h},p}(t_k)\big)^{-1} \lesssim 2^{(\frac{n\delta}{p}-\alpha_2-\frac{n}{p})k}(1+|m+h|)^{\frac{n}{\theta}-\frac{\delta n}{p}} M_{A,p}^{-1}(t_0), \qquad k \le 0,\, m,h \in \mathbb{Z}^n,$$

where the implicit constant is independent of k,m and h. Therefore

$$\Big(\int_{Q_{k,m+h}} |\varphi_k * f(y)|^\tau dy\Big)^{\frac{1}{\tau}}$$

does not exceed

$$c 2^{(\frac{n\delta}{p}-\alpha_2)k}(1+|m+h|)^{\frac{n}{\theta}-\frac{\delta n}{p}} |Q_{k,m+h}|^{\frac{1}{\tau}} \big\| f | \dot{B}_{p,\infty}(\mathbb{R}^n,\{t_k\}) \big\|.$$

It is not difficult to see that the term $(1+|m+h|)^{\frac{n}{\theta}-\frac{\delta n}{p}}$ does not exceed

$$(1+|h|)^{\frac{n}{\theta}-\frac{\delta n}{p}}\left(1+2^k|y|\right)^{\frac{n}{\theta}-\frac{\delta n}{p}}, \qquad y\in Q_{k,m}.$$

We observe that we can ensure that the following bounds holds

$$\left\|\left(1+2^k|\cdot|\right)^{\frac{n}{\theta}-\frac{\delta n}{p}}\omega_k|L_1(\mathbb{R}^n)\right\|\lesssim 1, \quad k\le 0$$

and we finally get that

$$|\langle\varphi_k*f,\omega_k\rangle|\lesssim 2^{(\frac{n\delta}{p}-\alpha_2)k}\left\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\right\|, \quad k\le 0,$$

by taking M in (4.14) large enough. Now assume that $k\in\mathbb{N}$. We have

$$M_{Q_{k,m+h},\sigma_1}(t_k^{-1})\lesssim 2^{-k\alpha_1}\left(M_{Q_{k,m+h},p}(t_0)\right)^{-1}, \quad k\in\mathbb{N},$$

by (2.3). We have

$$Q_{k,m+h}\subset E_{k,m+h},$$

but $E_{k,m+h}=\prod_{i=1}^n(a_i,b_i)$, with

$$a_i=\min(-1,-2^{-k}|m+h|) \quad \text{and} \quad b_i=2+2^{-k}|m+h|, \quad i=1,\cdots,n$$

if $m+h\neq 0$ and $E_{k,m+h}=\prod_{i=1}^n[-1,1)$, if $m+h=0$. Again from Lemma 2.1 (iii) there exists constant $C>0$ such that

$$\begin{aligned}M_{Q_{k,m+h}}(t_0^p)&\ge C\left(\frac{|Q_{k,m+h}|}{|E_{k,m+h}|}\right)^{\frac{p}{\theta}-1}M_{E_{k,m+h}}(t_0^p)\\&\ge C2^{kn(1-\frac{p}{\theta})}\left(1+2^{-k}|m+h|\right)^{-\frac{np}{\theta}}t_0^p(E_{k,m+h}).\end{aligned}$$

Since $A\subset E_{k,m+h}$, we get

$$M_{Q_{k,m+h},\sigma_1}(t_k^{-1})\lesssim 2^{(\frac{n}{\theta}-\frac{n}{p}-\alpha_1)k}\left(1+2^{-k}|m+h|\right)^{\frac{n}{\theta}-\frac{n\delta}{p}}M_{A,p}^{-1}(t_0), \quad k\in\mathbb{N},\, m,h\in\mathbb{Z}^n.$$

By the argument same as in case $k\le 0$, we also obtain

$$|\langle\varphi_k*f,\omega\rangle|\lesssim 2^{(\frac{n\delta}{p}-\alpha_1)k}\left\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\right\|, \quad k\in\mathbb{N},$$

which completes the proof. □

From Lemma 4.2, we get

$$\begin{aligned}
&|\langle \partial^{\beta}(\varphi_k * \widetilde{\varphi}_k * f),\omega\rangle| \\
&= 2^{|\beta|k+kn} |\langle \partial^{\beta}\widetilde{\varphi}(2^k\cdot) * \varphi_k * f,\omega\rangle| \\
&= 2^{|\beta|k+kn} |\langle \varphi_k * f, \partial^{\beta}\widetilde{\varphi}(2^k\cdot) * \omega\rangle| \\
&\lesssim 2^{(|\beta|+\frac{n\delta}{p}-\alpha_2)k} \big\| f|\dot{B}_{p,s}(\mathbb{R}^n,\{t_k\})\big\|, \qquad k\le 0,
\end{aligned}$$

which ensures that the series (4.13) converges in $\mathcal{S}'(\mathbb{R}^n)$ for every $|\beta|>\kappa$. Therefore the elements of $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ can be regarded as equivalence classes of distributions modulo polynomials in $\mathcal{P}_\kappa$ with $\kappa=\max(\lfloor \alpha_2-\frac{n\delta}{p}\rfloor,-1)$ and

$$f=\sum_{k=-\infty}^{\infty} 2^{-kn} \sum_{m\in\mathbb{Z}^n} \widetilde{\varphi}_k * f(2^{-k}m)\varphi_k(\cdot-2^{-k}m) \quad \text{in} \quad \mathcal{S}'_\kappa(\mathbb{R}^n).$$

The main result of this section is the following.

Theorem 4.4. *Let* $E=\{1,\cdots,j\}, j\in\mathbb{N}$ *be a finite,* $\alpha_1,\alpha_2\in\mathbb{R}, 0<\theta\le p<\infty, 0<q\le\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\ge p$. *Let* δ *be as in Lemma* 2.1*(v). Let*

$$\begin{aligned}
\Phi&=\{\varrho_{k,m,i}: i\in E, k\in\mathbb{Z}, m\in\mathbb{Z}^n\},\\
\widetilde{\Phi}&=\{\widetilde{\varrho}_{k,m,i}: i\in E, k\in\mathbb{Z}, m\in\mathbb{Z}^n\}
\end{aligned}$$

be a decomposition system for $L_2(\mathbb{R}^n)$ *satisfying* (4.12) *in the sense of* $L_2(\mathbb{R}^n)$,

$$\begin{aligned}
&|\partial^{\beta}\varrho_{k,m,i}(x)|\le c2^{\frac{n}{2}+n|\beta|}\Big(1+2^k|x-x_{k,m}|\Big)^{-M_\Phi}, \qquad |\beta|\le r_\Phi,\\
&|\partial^{\beta}\widetilde{\varrho}_{k,m,i}(x)|\le c2^{\frac{n}{2}+n|\beta|}\Big(1+2^k|x-x_{k,m}|\Big)^{-M_{\tilde{\Phi}}}, \qquad |\beta|\le r_{\tilde{\Phi}},\\
&\int_{\mathbb{R}^n} x^{\beta}\varrho_{k,m,i}(x)dx=0 \quad \text{if} \quad 0\le|\beta|\le r_{\tilde{\Phi}}-1,\\
&\int_{\mathbb{R}^n} x^{\beta}\widetilde{\varrho}_{k,m,i}(x)dx=0 \quad \text{if} \quad 0\le|\beta|\le r_{\Phi}-1
\end{aligned}$$

for every $i\in E, k\in\mathbb{Z}$ *and* $m\in\mathbb{Z}^n$ *where* $r_\Phi, r_{\tilde{\Phi}}\in\mathbb{N}_0$ *with*

$$r_\Phi\ge\alpha_2, \quad r_{\tilde{\Phi}}\ge J-n-\alpha_1+\frac{n}{\sigma_1}$$

and

$$M_\Phi, M_{\tilde{\Phi}}>\max(J+\frac{n}{\theta}-\frac{n\delta}{p}, n+r_\Phi, n+r_{\tilde{\Phi}}).$$

Then for each $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$,

$$f=\sum_{i=1}^{j}\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varrho}_{k,m,i}\rangle\varrho_{k,m,i}, \quad \text{in} \quad \mathcal{S}'_\kappa(\mathbb{R}^n)$$

and in $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *for* $0<q<\infty$. *Moreover*

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\sum_{i=1}^{j}\big\|\{\langle f,\widetilde{\varrho}_{k,m,i}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

The proof of this theorem is based on some technical results established in [43] for the classical Besov and Triebel-Lizorkin spaces.

Lemma 4.3. *Let* $r\in\mathbb{N}_0$ *and* $M>n+r$. *Let* $\{\varrho_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}$ *and* $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *are families of functions on* $\mathbb{R}^n$ *satisfying*

$$\begin{aligned}&|\varrho_{v,h}(x)|\le c2^{v\frac{n}{2}}(1+2^v|x-x_{v,h}|)^{-M},\\&|\partial^\beta b_{k,m}(x)|\le c2^{k(\frac{n}{2}+n|\beta|)}\Big(1+2^k|x-x_{k,m}|\Big)^{-M},\quad |\beta|\le r,\\&\int_{\mathbb{R}^n}x^\beta\varrho_{v,h}(x)dx=0\quad\text{if}\quad 0\le|\beta|\le r-1\end{aligned}$$

for every $k,v\in\mathbb{Z}$ *and* $h,m\in\mathbb{Z}^n$. *Then*

$$|\langle b_{k,m},\varrho_{v,h}\rangle|\le c2^{(k-v)(r+\frac{n}{2})}\big(1+2^k|x_{v,h}-x_{k,m}|\big)^{-M},\quad k\le v.$$

For the proof, we refer the readers to [26], see also [43, Lemma 3.1]. Notice that for $r=0$ there are no moment conditions on $\varrho_{v,h}$. Let r_1, $r_2\in\mathbb{N}_0$ and $M>n+\max\{r_1,r_2\}$, $\{\varrho_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}$ and $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ are families of functions on $\mathbb{R}^n$ satisfying

$$|\partial^\beta\varrho_{v,h}(x)|\le c2^{v(\frac{n}{2}+\frac{n|\beta|}{2})}(1+2^v|x-x_{v,h}|)^{-M},\qquad |\beta|\le r_1,\tag{4.15}$$

$$|\partial^\beta b_{k,m}(x)|\le c2^{k(\frac{n}{2}+\frac{n|\beta|}{2})}\Big(1+2^k|x-x_{k,m}|\Big)^{-M},\qquad |\beta|\le r_2,\tag{4.16}$$

$$\int_{\mathbb{R}^n}x^\beta\varrho_{v,h}(x)dx=0\quad\text{if}\quad 0\le|\beta|\le r_2-1,\tag{4.17}$$

$$\int_{\mathbb{R}^n}x^\beta b_{k,m}(x)dx=0\quad\text{if}\quad 0\le|\beta|\le r_1-1\tag{4.18}$$

for every $k,v\in\mathbb{Z}$ and $h,m\in\mathbb{Z}^n$. We have the following:

Proposition 4.1. *Let* α_1, $\alpha_2\in\mathbb{R}$, $0<\theta\le p<\infty$ *and* $0<q\le\infty$. *Let* $\{t_k\}\in\dot{X}_{\alpha,\sigma,p}$ *be a* p*-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\ge p$. *Let* $\{\varrho_{v,h}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}$ *and* $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *be families of functions satisfying* (4.15)–(4.18) *for some* r_1, $r_2\in\mathbb{N}_0$ *and* $M>n+\max\{r_1,r_2\}$. *Assume that*

$$r_2>\alpha_2,\quad r_1>J-n-\alpha_1,\ M>J.$$

Then the matrix $A=\{\langle\varrho_{v,h},b_{k,m}\rangle\}_{v,k\in\mathbb{Z},h,m\in\mathbb{Z}^n}$ *is bounded on* $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. *Moreover, there exists a positive constant* c *such that*

$$\big\|A\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\le c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$$

holds for all $\lambda\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$.

Proof. Let $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. We write $A\equiv A_0+A_1$ with

$$(A_0\lambda)_{v,h}=\sum_{k=-\infty}^{v}\sum_{m\in\mathbb{Z}^n}\langle\varrho_{v,h},b_{k,m}\rangle\lambda_{k,m},$$
$$(A_1\lambda)_{v,h}=\sum_{k=v+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle\varrho_{v,h},b_{k,m}\rangle\lambda_{k,m}.$$

From Lemma 4.3, we obtain

$$\begin{aligned}\big|(A_0\lambda)_{v,h}\big|&\le\sum_{k=-\infty}^{v}\sum_{m\in\mathbb{Z}^n}2^{(k-v)(r_2+\frac{n}{2})}\frac{|\lambda_{k,m}|}{\big(1+2^k|x_{k,m}-x_{v,h}|\big)^M}\\&=\sum_{k=-\infty}^{v}2^{(k-v)(r_2+\frac{n}{2})}S_{k,v,h}.\end{aligned}$$

For each $j\in\mathbb{N}$ we define

$$\Omega_{j,k,v,h}:=\{m\in\mathbb{Z}^n:2^{j-1}<2^k|x_{k,m}-x_{v,h}|\le 2^j\},$$
$$\Omega_{0,k,v,h}:=\{m\in\mathbb{Z}^n:2^k|x_{k,m}-x_{v,h}|\le 1\}.$$

Let $\frac{n}{M}<\tau<\min(1,p)$ be such that $r_1>\frac{n}{\tau}-n-\alpha_1$. We rewrite $S_{k,v,h}$ as follows

$$S_{k,v,h}=\sum_{j=0}^{\infty}\sum_{m\in\Omega_{j,k,v,h}}\frac{|\lambda_{k,m}|}{\big(1+2^k|x_{k,m}-x_{v,h}|\big)^M}\le\sum_{j=0}^{\infty}2^{-Mj}\sum_{m\in\Omega_{j,k,v,h}}|\lambda_{k,m}|$$

and by the embedding $\ell_\tau\hookrightarrow\ell_1$ we deduce that

$$\begin{aligned}S_{k,v,h}&\le\sum_{j=0}^{\infty}2^{-Mj}\Big(\sum_{m\in\Omega_{j,k,v,h}}|\lambda_{k,m}|^\tau\Big)^{\frac{1}{\tau}}\\&=\sum_{j=0}^{\infty}2^{(\frac{n}{\tau}-M)j}\Big(2^{(k-j)n}\int_{\cup_{z\in\Omega_{j,k,v,h}}Q_{k,z}}\sum_{m\in\Omega_{j,k,v,h}}|\lambda_{k,m}|^\tau\chi_{k,m}(y)dy\Big)^{\frac{1}{\tau}}.\end{aligned}\tag{4.19}$$

We proceed as in Lemma 3.3. We easily obtain that (4.19) does not exceed

$$c\mathcal{M}_\tau\Big(\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\chi_{k,m}\Big)(x)$$

for any $x\in Q_{v,h}$ and any $k\le v$. Applying Lemma 2.6, because of $r_2>\alpha_2$, we obtain $\big\|A_0\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$ is bounded by $c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$.

Again from Lemma 4.3, we see that

$$\big|(A_1\lambda)_{v,h}\big|\le\sum_{k=v+1}^{\infty}\sum_{m\in\mathbb{Z}^n}2^{(v-k)(r_1+\frac{n}{2})}\frac{|\lambda_{k,m}|}{\big(1+2^v|x_{k,m}-x_{v,h}|\big)^M}=\sum_{k=v+1}^{\infty}2^{(v-k)(r_1+\frac{n}{2})}T_{k,v,h}.$$

As before,

$$T_{k,v,h} \leq c2^{(k-v)\frac{n}{\tau}} \mathcal{M}_\tau \Big(\sum_{m\in\mathbb{Z}^n} \lambda_{k,m}\chi_{k,m} \Big)(x), \quad k>v, x\in Q_{v,h},$$

where the positive constant c is independent of k,v,x and h. Again applying Lemma 2.6, because of $r_1 > \frac{n}{\tau} - n - \alpha_1$, we obtain $\big\| A_1\lambda | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\|$ is bounded by $c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$. Hence the lemma is proved. □

Lemma 4.4. *Let* α_1, $\alpha_2 \in \mathbb{R}$, $0<\theta\leq p<\infty, 0<q\leq\infty$. *Let* $\{t_k\}\in \dot{X}_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1 = \theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2 \geq p$. *Let* δ *be as in Lemma* 2.1 *(v) and* $\kappa = \max(\lfloor \alpha_2 - \frac{n\delta}{p} \rfloor, -1)$. *Let* $\{b_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}$ *be families of functions satisfying* (4.16) *and* (4.18) *for some* r_1, $r_2 \in \mathbb{N}_0$, *and* $M > J$. *Assume that*

$$r_2 > \alpha_2, \quad r_1 > J - n - \alpha_1 + \frac{n}{\sigma_1}$$

and

$$M > \max\left\{ J + \frac{n}{\theta} - \frac{n\delta}{p}, n+r_1, n+r_2 \right\}.$$

If $\lambda \in \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$ *then the series*

$$f = \sum_{k=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} b_{k,m}\lambda_{k,m}, \tag{4.20}$$

converges in $\mathcal{S}'_\kappa(\mathbb{R}^n)$ *and in* $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ *for* $0<q<\infty$, *and*

$$\big\| f | \dot{B}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\| \lesssim \big\| \lambda | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\|. \tag{4.21}$$

Proof. We have to subdivide the proof into three steps.

Step 1. We will prove the convergence of the series (4.20) in $\mathcal{S}'_\kappa(\mathbb{R}^n)$. Let $\omega \in \mathcal{S}_\kappa(\mathbb{R}^n)$. We write

$$\sum_{k=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{k,m} \langle b_{k,m}, \omega \rangle =: I_1 + I_2,$$

where

$$I_1 := \sum_{k=-\infty}^{0} \sum_{m\in\mathbb{Z}^n} \lambda_{k,m} \langle b_{k,m}, \omega \rangle \quad \text{and} \quad I_2 := \sum_{k=1}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{k,m} \langle b_{k,m}, \omega \rangle.$$

By Lemma 3.2, we find that

$$|\lambda_{k,m}| \lesssim 2^{-\frac{kn}{2}} t_{k,m}^{-1} \big\| \lambda | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\}) \big\|, \quad k\in\mathbb{Z}, m\in\mathbb{Z}^n.$$

As in Lemma 4.2, we derive that

$$t_{k,m}^{-1} \lesssim 2^{(\frac{n\delta}{p} - \alpha_2)k} (1+|m|)^{\frac{n}{\theta} - \frac{n\delta}{p}} M_{A,p}^{-1}(t_0), \quad k\leq 0, m\in\mathbb{Z}^n. \tag{4.22}$$

We see that

$$|\langle b_{k,m},\omega\rangle| \lesssim 2^{k(\kappa+1+\frac{n}{2})}\big(1+|m|\big)^{-M}, \quad k\le 0,$$

by Lemma 4.3, with the help of the fact that $r_2\ge\kappa+1$. Using this estimate, by Lemma 3.2 and (4.22), we get

$$\begin{aligned} I_1 &\lesssim \big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \sum_{k=-\infty}^{0}\sum_{m\in\mathbb{Z}^n} 2^{k(\kappa+1)} t_{k,m}^{-1}\big(1+|m|\big)^{-M} \\ &\lesssim \big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \sum_{k=-\infty}^{0} 2^{k(\kappa+1-\alpha_2+\frac{n\delta}{p})} \\ &\lesssim \big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|, \end{aligned}$$

because of $\kappa>\alpha_2-\frac{n\delta}{p}-1$. To estimate I_2, notice that

$$|\langle b_{k,m},\omega\rangle| \le c2^{-k(r_1+\frac{n}{2})}\big(1+2^{-k}|m|\big)^{-M}, \quad k\in\mathbb{N},$$

see again Lemma 4.3. For each $j\in\mathbb{N}$ we define

$$\begin{aligned} &\Omega_{j,k}:=\{m\in\mathbb{Z}^n: 2^{j+k-1}<|m|\le 2^{j+k}\},\\ &\Omega_{0,k}:=\{m\in\mathbb{Z}^n: |m|\le 2^{k}\}. \end{aligned}$$

Let $\frac{n}{M-\frac{n}{\theta}+\frac{n\delta}{p}}<\tau<\min(1,p)$ be such that $\tau>\frac{n}{r_1+n+\alpha_1-\frac{n}{\sigma_1}}$. Then we find

$$\begin{aligned} I_2 &\lesssim \sum_{k=1}^{\infty} 2^{-k(r_1+\frac{n}{2})}\sum_{m\in\mathbb{Z}^n}\frac{|\lambda_{k,m}|}{\big(1+2^{-k}|m|\big)^{M}} \\ &= c\sum_{k=1}^{\infty} 2^{-k(r_1+\frac{n}{2})}\sum_{j=0}^{\infty}\sum_{m\in\Omega_{j,k}}\frac{|\lambda_{k,m}|}{\big(1+2^{-k}|m|\big)^{M}} \\ &\lesssim \sum_{k=1}^{\infty} 2^{-k(r_1+\frac{n}{2})}\sum_{j=0}^{\infty}2^{-Mj}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|. \end{aligned}$$

Let $0<\varrho<\min(1,\theta)$ be such that $\frac{1}{\varrho}=\frac{1}{\tau}+\frac{1}{\sigma_1}$ with $0<\tau<p$. Using the embedding $\ell_\varrho\hookrightarrow\ell_1$ we find that

$$\begin{aligned} I_2 &\lesssim \sum_{k=1}^{\infty} 2^{-k(r_1+\frac{n}{2})}\sum_{j=0}^{\infty}2^{-Mj}\Big(\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\Big)^{\frac{1}{\varrho}}. \\ &= c\sum_{k=1}^{\infty} 2^{-k(r_1+\frac{n}{2})}\sum_{j=0}^{\infty}2^{(\frac{n}{\varrho}-M)j}\Big(2^{(k-j)n}\int_{\cup_{z\in\Omega_{j,k}}Q_{k,z}}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\chi_{k,m}(y)dy\Big)^{\frac{1}{\varrho}}. \end{aligned}$$

Let $y\in\cup_{z\in\Omega_{j,k}}Q_{k,z}$ and $x\in Q_{0,0}$. Then $y\in Q_{k,z}$ for some $z\in\Omega_{j,k}$ and $2^{j-1}<2^{-k}|z|\le 2^{j}$. From this it follows that

$$|y-x|\le|y-2^{-k}z|+|x-2^{-k}z|\le\sqrt{n}\,2^{-k}+|x|+2^{-k}|z|\le 2^{j+\delta_n},\quad \delta_n\in\mathbb{N},$$

which implies that y is located in the ball $B\left(x,2^{j+\delta_n}\right)$. In addition, from the fact that

$$|y|\le|y-x|+|x|\le 2^{j+\delta_n}+1\le 2^{j+c_n},\quad c_n\in\mathbb{N},$$

we have that y is located in the ball $B(0,2^{j+c_n})$. Therefore, by Hölder's inequality

$$\begin{aligned}&\Big(2^{(k-j)n}\int_{\cup_{z\in\Omega_{j,k}}Q_{k,z}}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\varrho}\chi_{k,m}(y)dy\Big)^{\frac{1}{\varrho}}\\&\le 2^{k\frac{n}{\varrho}}\Big(2^{-jn}\int_{B(x,2^{j+\delta_n})}\sum_{m\in\Omega_{j,k}}|\lambda_{k,m}|^{\tau}t_k^{\tau}(y)\chi_{k,m}(y)dy\Big)^{\frac{1}{\tau}}M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1})\\&\lesssim 2^{k\frac{n}{\varrho}}\mathcal{M}_{\tau}\Big(\sum_{m\in\mathbb{Z}^n}t_k\lambda_{k,m}\chi_{k,m}\Big)(x)M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1}),\end{aligned}$$

where the implicit constant is independent of k,j and x. By (2.3), Lemma 2.1/(v), we find that

$$M_{B(0,2^{j+c_n}),\sigma_1}(t_k^{-1})\lesssim 2^{-k\alpha_1}\left(M_{B(0,2^{j+c_n}),p}(t_0)\right)^{-1}\lesssim 2^{j(\frac{n}{p}-\frac{n\delta}{p})-k\alpha_1}\left(M_{B(0,1),p}(t_0)\right)^{-1}.$$

Therefore, since $M>\frac{n}{\tau}+\frac{n}{\theta}-\frac{n\delta}{p}$,

$$I_2\lesssim\sum_{k=1}^{\infty}2^{-k(r_1+\frac{n}{2}-\frac{n}{\varrho}+\alpha_1)}\mathcal{M}_{\tau}\big(t_k\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\chi_{k,m}\big)(x),\quad x\in Q_{0,0}.$$

The last term is bounded in the $L_p(Q_{0,0})$-quasi-norm by $c\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$, with the help of Lemma 2.4 and the fact that $r_1>J-n-\alpha_1+\frac{n}{\sigma_1}$. Hence we have proved the convergence of

$$\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}b_{k,m}\lambda_{k,m}$$

in $\mathcal{S}'_{\kappa}(\mathbb{R}^n)$.

Step 2. We will prove (4.21). From (3.13), it follows that

$$\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\approx\big\|\{\langle f,\varphi_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|. \tag{4.23}$$

Thanks to Proposition 4.1, the matrix

$$A=\{\langle\varphi_{k,m},b_{v,h}\rangle\}_{v,k\in\mathbb{Z},h,m\in\mathbb{Z}^n},$$

is bounded on $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. But

$$\langle f,\varphi_{k,m}\rangle=\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\lambda_{v,h}\langle\varphi_{k,m},b_{v,h}\rangle,\quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n.$$

Hence the right hand side of (4.23) is just

$$\big\|A\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

Step 3. We will prove the convergence of the series (4.20) in $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ for $0<q<\infty$. Again from (4.23), we obtain that

$$\Big\|f-\sum_{|v|\leq N}\sum_{h\in\mathbb{Z}^n}b_{v,h}\lambda_{v,h}|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\Big\|,$$

can be estimated by

$$\Big\|\Big\{\sum_{|v|>N}\sum_{h\in\mathbb{Z}^n}\lambda_{v,h}\langle b_{v,h},\varphi_{k,m}\rangle\Big\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\Big\|. \tag{4.24}$$

Now, we put

$$A=\{\langle\varphi_{k,m},b_{v,h}\rangle\}_{|v|>N,k\in\mathbb{Z},h,m\in\mathbb{Z}^n}\quad\text{and}\quad\lambda=\{\lambda_{k,m}\}_{|k|>N,m\in\mathbb{Z}^n}.$$

Therefore

$$A\lambda=\{(A\lambda)_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}=\Big\{\sum_{|v|>N}\sum_{h\in\mathbb{Z}^n}\lambda_{v,h}\langle b_{v,h},\varphi_{k,m}\rangle\Big\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}.$$

Hence (4.24) is just

$$\big\|A\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|=c\Big(\sum_{|k|>N}2^{\frac{knq}{2}}\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{k,m}|^p t_{k,m}^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}},$$

which tends to zero if N tends to infinity. This completes the proof of Lemma 4.4. □

Proof of Theorem 4.4. We will divide the proof into two steps.

Step 1. Let $f\in\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$. We will prove that

$$f=\sum_{i=1}^{j}\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varrho}_{k,m,i}\rangle\varrho_{k,m,i}.$$

From (3.4), we obtain

$$f=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varphi}_{k,m}\rangle\varphi_{k,m}.$$

Since $\varphi_{k,m}\in L_2(\mathbb{R}^n)$, $k\in\mathbb{Z},m\in\mathbb{Z}^n$, we find that

$$\varphi_{k,m}=\sum_{i=1}^{j}\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\varrho_{v,h,i}$$

in $L_2(\mathbb{R}^n)$ sense and consequently in the distributional sense as well as in the spaces $\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})$ for $0<q<\infty$. Indeed, by Lemma 4.4 it suffices to show that

$$\{\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_v\})$$

for any $k\in\mathbb{Z},m\in\mathbb{Z}^n$. Observe that

$$\{\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}=\Big\{\sum_{l=-\infty}^{\infty}\sum_{z\in\mathbb{Z}^n}\langle\varphi_{l,z},\widetilde{\varrho}_{v,h,i}\rangle\lambda_{l,z}\Big\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n},$$

where

$$\lambda_{l,z}=\begin{cases}1, & \text{if} \quad l=k \text{ and } z=m,\\ 0, & \text{otherwise.}\end{cases}$$

We put $\lambda=\{\lambda_{l,z}\}_{l\in\mathbb{Z},z\in\mathbb{Z}^n}$. From Proposition 4.1, we get

$$\big\|\{\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|<\infty.$$

Now

$$f=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varphi}_{k,m}\rangle\varphi_{k,m}=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\sum_{i=1}^{j}\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\langle f,\widetilde{\varphi}_{k,m}\rangle\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\varrho_{v,h,i}. \tag{4.25}$$

Define

$$d_{v,h,i}=\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}\langle f,\widetilde{\varphi}_{k,m}\rangle\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle,\quad v\in\mathbb{Z},\, h\in\mathbb{Z}^n,\, i\in\{1,\cdots,j\}$$

and $d_i=\{d_{v,h,i}\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}$, $i\in\{1,\cdots,j\}$. Thanks to Proposition 4.1, the matrix

$$A_i=\{\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle\}_{v,k\in\mathbb{Z},h,m\in\mathbb{Z}^n},\quad i\in\{1,\cdots,j\}$$

is bounded on $\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. Let $\lambda=\{\langle f,\widetilde{\varphi}_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}\in\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})$. Then

$$\big\|d_i|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|=\big\|A_i\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|\lambda|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|<\infty.$$

Therefore by Lemma 4.4, we have that

$$\sum_{v=-\infty}^{\infty}\sum_{h\in\mathbb{Z}^n}\sum_{k=-\infty}^{\infty}\sum_{m\in\mathbb{Z}^n}|\langle f,\widetilde{\varphi}_{k,m}\rangle||\langle\varphi_{k,m},\widetilde{\varrho}_{v,h,i}\rangle||\langle\varrho_{v,h,i},\omega\rangle|<\infty$$

for any $\omega \in \mathcal{S}_\kappa(\mathbb{R}^n)$. So we can change the order of the summations in (4.25), and obtain

$$\begin{aligned} f &= \sum_{i=1}^{j} \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \sum_{k=-\infty}^{\infty} \sum_{m\in\mathbb{Z}^n} \langle f, \widetilde{\varphi}_{k,m}\rangle \langle \varphi_{k,m}, \widetilde{\varrho}_{v,h,i}\rangle \varrho_{v,h,i} \\ &= \sum_{i=1}^{j} \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \langle f, \widetilde{\varrho}_{v,h,i}\rangle \varrho_{v,h,i}. \end{aligned}$$

Step 2. In this step we prove that

$$\big\| f | \dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \approx \sum_{i=1}^{j} \big\| \{\langle f, \widetilde{\varrho}_{k,m,i}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|.$$

From Step 1, we obtain

$$\langle f, \varphi_{k,m}\rangle = \sum_{i=1}^{j} \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \langle f, \widetilde{\varrho}_{v,h,i}\rangle \langle \varrho_{v,h,i}, \varphi_{k,m}\rangle, \quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n.$$

Define

$$\begin{aligned} &A_i = \{\langle \varrho_{v,h,i}, \varphi_{k,m}\rangle\}_{v,k\in\mathbb{Z},h,m\in\mathbb{Z}^n}, \quad \lambda_i = \{\lambda_{v,h,i} = \langle f, \widetilde{\varrho}_{v,h,i}\rangle\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}, \\ &A_i\lambda_i = \{(A_i\lambda_i)_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} = \Big\{ \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \langle f, \widetilde{\varrho}_{v,h,i}\rangle \langle \varrho_{v,h,i}, \varphi_{k,m}\rangle \Big\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}, \end{aligned}$$

where $i\in\{1,\cdots,j\}$. From (3.13) and Proposition 4.1 we have that

$$\begin{aligned} &\big\| f | \dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \\ \lesssim &\, \big\| \{\langle f, \varphi_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \lesssim \sum_{i=1}^{j} \big\| A_i\lambda_i | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| \\ \lesssim &\, \sum_{i=1}^{j} \big\| \lambda_i | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\| = c \sum_{i=1}^{j} \big\| \{\langle f, \widetilde{\varrho}_{k,m,i}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} | \dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|. \end{aligned}$$

Similarly,

$$\langle f, \widetilde{\varrho}_{k,m,i}\rangle = \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \langle \varphi_{v,h}, \widetilde{\varrho}_{k,m,i}\rangle \langle f, \tilde{\varphi}_{v,h}\rangle, \quad k\in\mathbb{Z},\ m\in\mathbb{Z}^n,\ i\in\{1,\cdots,j\}.$$

Define now

$$\begin{aligned} &A_i = \{\langle \varphi_{v,h}, \widetilde{\varrho}_{k,m,i}\rangle\}_{v,k\in\mathbb{Z},h,m\in\mathbb{Z}^n}, \quad \lambda = \{\lambda_{v,h} = \langle f, \tilde{\varphi}_{v,h}\rangle\}_{v\in\mathbb{Z},h\in\mathbb{Z}^n}, \\ &A_i\lambda = \{(A_i\lambda)_{k,m}\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} = \Big\{ \sum_{v=-\infty}^{\infty} \sum_{h\in\mathbb{Z}^n} \langle \varphi_{v,h}, \widetilde{\varrho}_{k,m,i}\rangle \langle f, \tilde{\varphi}_{v,h}\rangle \Big\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n} \end{aligned}$$

with $i \in \{1,\cdots,j\}$. Again by Proposition 4.1 we obtain

$$\big\|\{\langle f,\widetilde{\varrho}_{k,m,i}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|$$

does not exceed

$$\big\|\{\langle f,\tilde{\varphi}_{k,m}\rangle\}_{k\in\mathbb{Z},m\in\mathbb{Z}^n}|\dot{b}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|\lesssim\big\|f|\dot{B}_{p,q}(\mathbb{R}^n,\{t_k\})\big\|,$$

where we have used (3.13). This finishes the proof of the theorem. □

Remark 4.1. (i) Further results, concerning, for instance, characterizations via oscillations, box spline and tensor-product B-spline representations are given in [20].
(ii) We mention that the techniques of [35] are not capable of dealing with spaces of variable smoothness. Also our assumptions on the weight $\{t_k\}$ play an exceptional role in the paper.
(iii) We draw the readers attention to paper [44] where generalized Besov-type and Triebel-Lizorkin-type spaces are studied. They assumed that the weight sequence $\{t_k\}$ lies in some class different from the class $\dot{X}_{\alpha,\sigma,p}$.

5 The non-homogeneous space $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$

In this section, we present the inhomogeneous version of our results given above. Let Φ,ψ,φ and Ψ satisfy

$$\Phi,\Psi,\varphi,\psi\in\mathcal{S}(\mathbb{R}^n), \tag{5.1}$$

$$\mathrm{supp}\mathcal{F}\Phi,\ \mathrm{supp}\mathcal{F}\Psi\subset\overline{B(0,2)},\quad |\mathcal{F}\Phi(\xi)|,|\mathcal{F}\Psi(\xi)|\geq c, \tag{5.2}$$

if $|\xi|\leq\frac{5}{3}$ and

$$\mathrm{supp}\mathcal{F}\varphi,\ \mathrm{supp}\mathcal{F}\psi\subset\overline{B(0,2)}\backslash B(0,1/2),\quad |\mathcal{F}\varphi(\xi)|,|\mathcal{F}\psi(\xi)|\geq c, \tag{5.3}$$

if $\frac{3}{5}\leq|\xi|\leq\frac{5}{3}$, such that

$$\overline{\mathcal{F}\Phi(\xi)}\mathcal{F}\Psi(\xi)+\sum_{k=1}^{\infty}\overline{\mathcal{F}\varphi(2^{-k}\xi)}\mathcal{F}\psi(2^{-k}\xi)=1,\quad \xi\in\mathbb{R}^n, \tag{5.4}$$

where $c>0$. Recall that the φ-transform S_φ is defined by setting $(S_\varphi f)_{0,m}=\langle f,\Psi_m\rangle$ where $\Psi_m(x)=\Psi(x-m)$ and $(S_\varphi f)_{k,m}=\langle f,\varphi_{k,m}\rangle$ where $\varphi_{k,m}(x)=2^{\frac{kn}{2}}\varphi(2^kx-m)$, $k\in\mathbb{N}$ and $m\in\mathbb{Z}^n$. The inverse φ-transform T_ψ is defined by

$$T_\psi\lambda:=\sum_{m\in\mathbb{Z}^n}\lambda_{0,m}\Psi_m+\sum_{k=1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\psi_{k,m},$$

where $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}\subset\mathbb{C}$, see again [26]. Let $\Phi,\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy, respectively, (5.2) and (5.3). We recall that by [26, pp. 130-131] or [27, Lemma 6.9], there exist functions $\Psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (5.2) and $\psi\in\mathcal{S}(\mathbb{R}^n)$ satisfying (5.3) such that (5.4) holds.

Now we present the inhomogenous version of Definition 2.3.

Definition 5.1. *Let α_1, $\alpha_2 \in \mathbb{R}$, σ_1, $\sigma_2 \in (0,+\infty]$, $\alpha=(\alpha_1,\alpha_2)$ and let $\sigma=(\sigma_1,\sigma_2)$. We let $X_{\alpha,\sigma,p}=X_{\alpha,\sigma,p}(\mathbb{R}^n)$ denote the set of p-admissible weight sequences $\{t_k\}_{k\in\mathbb{N}_0}$ satisfying* (2.3) *and* (2.4) *for any $0\le k\le j$, with constants $C_1,C_2>0$ are independent of both the indexes k and j.*

Example 5.1. A sequence $\{\gamma_j\}_{j\in\mathbb{N}_0}$ of positive real numbers is said to be admissible if there exist two positive constants d_0 and d_1 such that

$$d_0\gamma_j \le \gamma_{j+1} \le d_1\gamma_j, \quad j\in\mathbb{N}_0. \tag{5.5}$$

For an admissible sequence $\{\gamma_j\}_{j\in\mathbb{N}_0}$, let

$$\underline{\gamma}_j := \inf_{k\ge 0}\frac{\gamma_{j+k}}{\gamma_k} \quad \text{and} \quad \overline{\gamma}_j := \sup_{k\ge 0}\frac{\gamma_{j+k}}{\gamma_k}, \quad j\in\mathbb{N}_0.$$

Let

$$\alpha_\gamma := \lim_{j\longrightarrow\infty}\frac{\log\overline{\gamma}_j}{j} \quad \text{and} \quad \beta_\gamma := \lim_{j\longrightarrow\infty}\frac{\log\underline{\gamma}_j}{j},$$

be the upper and lower Boyd index of the given sequence $\{\gamma_j\}_{j\in\mathbb{N}_0}$, respectively. Then

$$\underline{\gamma}_j\gamma_k \le \gamma_{j+k} \le \overline{\gamma}_j\gamma_k, \quad j,k\in\mathbb{N}_0$$

and for each $\varepsilon>0$,

$$c_1 2^{(\beta_\gamma-\varepsilon)j} \le \underline{\gamma}_j \le \overline{\gamma}_j \le c_2 2^{(\alpha_\gamma+\varepsilon)j}, \quad j\in\mathbb{N}_0$$

for some constants $c_1=c_1(\varepsilon)>0$ and $c_2=c_2(\varepsilon)>0$. Also, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best possible constants d_0 and d_1 in (5.5), respectively.

Clearly the sequence $\{\gamma_j\}_{j\in\mathbb{N}_0}$ lies in $X_{\alpha,\sigma,p}$ for $\alpha_1=\beta_\gamma-\varepsilon,\alpha_2=\alpha_\gamma+\varepsilon$ and $0<p,\sigma_1,\sigma_2\le\infty$. These type of admissible sequences are used in [24] to study Besov and Lizorkin-Triebel spaces in terms of a generalized smoothness, see also [34].

Let us consider some examples of admissible sequences. The sequence $\{\gamma_j\}_{j\in\mathbb{N}_0}$,

$$\gamma_j := 2^{sj}(1+j)^b(1+\log(1+j))^c, \quad j\in\mathbb{N}_0$$

with arbitrary fixed real numbers s,b and c is a an admissible sequence with $\beta_\gamma=\alpha_\gamma=s$.

Example 5.2. Let $0<r<p<\infty$, a weight $\omega^p\in A_{\frac{p}{r}}(\mathbb{R}^n)$ and $\{s_k\}=\{2^{ks}\omega^p(2^{-k})\}_{k\in\mathbb{N}_0}$, $s\in\mathbb{R}$. Obviously, $\{s_k\}_{k\in\mathbb{N}_0}$ lies in $X_{\alpha,\sigma,p}$ for $\alpha_1=\alpha_2=s$, $\sigma=(r(\frac{p}{r})',p)$.

Now, we define the spaces under consideration.

Definition 5.2. *Let $0<p\le\infty$ and $0<q\le\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}$ be a p-admissible weight sequence. Let $\Phi,\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy* (5.2) *and* (5.3)*, respectively, and we put $\varphi_k=2^{kn}\varphi,k\in\mathbb{N}_0$. The Besov space $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ is the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\| := \Big(\sum_{k=0}^{\infty}\big\|t_k(\varphi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}} < \infty$$

with the usual modifications if $q=\infty$, where φ_0 is replaced by Φ.

Now we introduce the inhomogeneous sequence spaces $b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$. Let $0<p\leq\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}$ be a p-admissible weight sequence. Then for all complex valued sequences $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}\subset\mathbb{C}$ we define

$$b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0}):=\Big\{\lambda:\big\|\lambda|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|<\infty\Big\},$$

where

$$\big\|\lambda|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|:=\Big(\sum_{k=0}^{\infty}2^{\frac{knq}{2}}\Big\|\sum_{m\in\mathbb{Z}^n}t_k\lambda_{k,m}\chi_{k,m}|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}.$$

We have the following analogue of Theorem 3.1.

Theorem 5.1. *Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$. Let φ, ψ satisfying* (5.1) *through* (5.4)*. The operators*

$$S_\varphi: B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\to b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0}),$$
$$T_\psi: b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\to B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$$

are bounded. Furthermore, $T_\psi\circ S_\varphi$ is the identity on $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$.

As a consequence the analogues of Corollary 3.1 are now clear. We obtain the following useful properties of these function spaces.

Theorem 5.2. *Let $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$. $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ are quasi-Banach spaces. They are Banach spaces if $1\leq p<\infty$ and $1\leq q<\infty$.*

Let $0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma=(\sigma_1=\theta\left(\frac{p}{\theta}\right)',\sigma_2\geq p)$ and $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. As in Theorem 3.3 we have the embedding

$$\mathcal{S}(\mathbb{R}^n)\hookrightarrow B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0}).$$

In addition $\mathcal{S}(\mathbb{R}^n)$ is dense in $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ with $1<\theta\leq p<\infty$ and $0<q<\infty$. Also if $1<\theta\leq p<\infty$ and $0<q\leq\infty$, then

$$B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\hookrightarrow\mathcal{S}'(\mathbb{R}^n).$$

All the results in Subsection 3.2 are true for the inhomogeneous case.

We shall say that an operator A is associated with the matrix $\{a_{Q_{k,m}P_{v,h}}\}_{k,v\in\mathbb{N}_0,m,h\in\mathbb{Z}^n}$, if for all sequences $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}\subset\mathbb{C}$,

$$A\lambda=\{(A\lambda)_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}:=\Big\{\sum_{v=0}^{\infty}\sum_{h\in\mathbb{Z}^n}a_{Q_{k,m}P_{v,h}}\lambda_{v,h}\Big\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}.$$

We say that A, with associated matrix $\{a_{Q_{k,m}P_{v,h}}\}_{k,v\in\mathbb{N}_0,m,h\in\mathbb{Z}^n}$, is almost diagonal on $b_{p,q}$ $(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ if there exists $\varepsilon>0$ such that

$$\sup_{k,v\in\mathbb{N}_0,m,h\in\mathbb{Z}^n}\frac{|a_{Q_{k,m}P_{v,h}}|}{\omega_{Q_{k,m}P_{v,h}}(\varepsilon)}<\infty,$$

where $\omega_{Q_{k,m}P_{v,h}}(\varepsilon)$ as in Section 5. Let $\alpha_1,\alpha_2\in\mathbb{R},0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2\geq p$. It is obvious that an operator A on $b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ given by an almost diagonal matrix is bounded.

Let J be defined as in Section 5. we present the inhomogeneous versions of Definition 4.1.

Definition 5.3. *Let $\alpha_1,\alpha_2\in\mathbb{R}$, $0<p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}$ be a p-admissible weight sequence. Let $N=\max\{J-n-\alpha_1,-1\}$ and $\alpha_2^*=\alpha_2-\lfloor\alpha_2\rfloor$.*

(i) *We say that $\varrho_{Q_{k,m}}$, $k\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$, is an inhomogeneous smooth synthesis molecule for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ supported near $Q_{k,m}$ if it satisfies, for some real number $\delta\in(\alpha_2^*,1]$ and a real number $M\in(J,\infty)$,* (4.1)–(4.4) *if $k\in\mathbb{N}$. If $k=0$ we assume* (4.3), (4.4) *and*

$$|\varrho_{Q_{0,m}}(x)|\leq(1+|x-x_{Q_{0,m}}|)^{-M}.$$

A collection $\{\varrho_{Q_{k,m}}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ is called a family of inhomogeneous smooth synthesis molecules for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, if each $\varrho_{Q_{k,m}}$ is an inhomogeneous smooth synthesis molecule for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ supported near $Q_{k,m}$.

(ii) *We say that $b_{Q_{k,m}}$, $k\in\mathbb{N}_0$, $m\in\mathbb{Z}^n$, is an inhomogeneous smooth analysis molecule for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ supported near $Q_{k,m}$ if it satisfies, for some $\kappa\in((J-\alpha_2)^*,1]$ and an $M\in(J,\infty)$,* (4.5)–(4.8) *if $k\in\mathbb{N}$. If $k=0$ we assume* (4.7), (4.8) *and*

$$|b_{Q_{0,m}}(x)|\leq(1+|x-x_{Q_{0,m}}|)^{-M}.$$

A collection $\{b_{Q_{k,m}}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ is called a family of inhomogeneous smooth analysis molecules for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, if each $b_{Q_{k,m}}$ is an inhomogeneous smooth synthesis molecule for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ supported near $Q_{k,m}$.

As a consequence, we obtain the inhomogeneous version of Theorem 4.2.

Theorem 5.3. *Let α_1, $\alpha_2\in\mathbb{R}$, $0<\theta\leq p<\infty$ and $0<q\leq\infty$. Let $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2\geq p$. Let J,M,N,δ and κ be as in Definition* 5.3.

(i) *If $f=\sum_{k=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\varrho_{k,m}\lambda_{k,m}$, where $\{\varrho_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ is a family of inhomogeneous smooth synthesis molecules for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, then for all $\lambda\in b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$*

$$\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\lesssim\big\|\lambda|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

(ii) *Let* $\{b_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *be a family of inhomogeneous smooth analysis molecules. Then for all* $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$

$$\big\|\{\langle f,b_{k,m}\rangle\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\lesssim\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

Now we present the analogue of smooth atomic decomposition. First we need the definition of inhomogeneous smooth.

Definition 5.4. *Let* $\alpha_1,\alpha_2\in\mathbb{R},0<p<\infty,0<q\leq\infty$ *and* $N=\max\{J-n-\alpha_1,-1\}$. *Let* $\{t_k\}_{k\in\mathbb{N}_0}$ *be a p-admissible weight sequence. A function* $a_{Q_{k,m}}$ *is called an inhomogeneous smooth atom for* $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ *supported near* $Q_{k,m}$, $k\in\mathbb{N}_0$ *and* $m\in\mathbb{Z}^n$, *if it is satisfies* (4.9), (4.10) *and* (4.11) *if* $k\in\mathbb{N}$. *If* $k=0$ *we assume* (4.9) *and* (4.10).

A collection $\{a_{Q_{k,m}}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ is called a family of inhomogeneous smooth atoms for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, if each $a_{Q_{k,m}}$ is an inhomogeneous smooth atom for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ supported near $Q_{k,m}$.

Now we come to the atomic decomposition theorem.

Theorem 5.4. *Let* α_1, $\alpha_2\in\mathbb{R}$, $0<\theta\leq p<\infty$, $0<q\leq\infty$. *Let* $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\geq p$. *Then for each* $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, *there exist a family* $\{\varrho_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *of inhomogeneous smooth atoms for* $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ *and* $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}\in b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ *such that*

$$f=\sum_{k=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\varrho_{k,m},\quad \textit{converging in } \mathcal{S}'(\mathbb{R}^n)$$

and

$$\big\|\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\lesssim\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

Conversely, for any family of inhomogeneous smooth atoms for $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ *and* $\lambda=\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}\in b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$

$$\Big\|\sum_{k=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{k,m}\varrho_{k,m}|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\Big\|\lesssim\big\|\{\lambda_{k,m}\}_{k\in\mathbb{N}_0,m\in\mathbb{Z}^n}|b_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

5.1 Characterization by differences

Let f be an arbitrary function on $\mathbb{R}^n$ and $x,h\in\mathbb{R}^n$. Then

$$\Delta_h f(x):=f(x+h)-f(x),\quad \Delta_h^{M+1}f(x):=\Delta_h(\Delta_h^M f)(x),\quad M\in\mathbb{N}.$$

These are the well-known differences of functions which play an important role in the theory of function spaces. Using mathematical induction one can show the explicit formula

$$\Delta_h^M f(x):=\sum_{j=0}^{M}(-1)^j C_M^j f(x+(M-j)h),$$

where C_M^j are the binomial coefficients. By the ball means of differences we mean the quantity

$$d_t^M f(x) := t^{-n} \int_{|h|\le t} \left|\Delta_h^M f(x)\right| dh = \int_B \left|\Delta_{th}^M f(x)\right| dh,$$

where $B := \{x \in \mathbb{R}^n : |x| \le 1\}$ is the unit ball of $\mathbb{R}^n$, $t > 0$ is a real number and M is a natural number. Let $0 < p,q \le \infty$,

$$\|f|B_{p,q}^s\|^* := \|f|L_p(\mathbb{R}^n)\| + \Big(\sum_{k=0}^{\infty} 2^{ksq} \|d_{2^{-k}}^M f|L_p(\mathbb{R}^n)\|^q\Big)^{\frac{1}{q}}$$

with $\sigma_p < s < M$, where

$$\sigma_p := n \max\big(\frac{1}{p} - 1, 0\big).$$

It was proven in [58] that $\|f|B_{p,q}^s\|^*$ is an equivalent quasi-norm in $B_{p,q}^s$. Later on this type of characterization was extended by Besov in [2] and [4] to spaces $B_{p,q}^{\{t_k\}_k}$ with $p,q \in (1,\infty)$. It was assumed that the weight sequence $\{t_k\}_{k\in\mathbb{N}_0}$ lies in ${}^{\mathrm{loc}}Y_{\alpha_1,\alpha_2}^{\alpha_3}$. ${}^{\mathrm{loc}}Y_{\alpha_1,\alpha_2}^{\alpha_3}$ is the set of all weight sequences $t = \{t_k\}_{k\in\mathbb{N}_0} = \{t_k(\cdot)\}_{k\in\mathbb{N}_0}$ satisfying for α_1, $\alpha_2 \in \mathbb{R}$,

$$c_1 2^{\alpha_1(k-j)} \le \frac{t_k(x)}{t_j(x)} \le c_2 2^{\alpha_2(k-j)} \quad \text{for} \quad j \le k \in \mathbb{N}_0,\ x \in \mathbb{R}^n; \tag{5.6}$$

and

$$t_k(x) \le t_k(y) \quad \text{for} \quad x,y \in \mathbb{R}^n,\ k \in \mathbb{N}_0,\ |x-y| \le 2^{-k},$$

under the condition $0 < \alpha_1 \le \alpha_2 < M$.

Kempka and Vybíral [42] gave a characterization of Besov spaces of variable smoothness and integrability by the ball means of differences. The weight sequence satisfies (5.6) and

$$t_k(x) \le c_3 t_k(y)(1 + 2^k|x-y|)^{\alpha_3} \quad \text{for } \alpha_3 \ge 0,\ k \in \mathbb{N}_0, \quad x,y \in \mathbb{R}^n,$$

under the condition

$$0 < p,q \le \infty, \quad \sigma_p\big(1 + \frac{\alpha_3}{n} p\big) < \alpha_1 \le \alpha_2 < M.$$

They also gave the same characterization for Triebel-Lizorkin spaces of variable smoothness and integrability.

In this subsection we want to extend this type of characterization to function spaces of generalized smoothness. In the next we shall interpret $L_1^{\mathrm{loc}}(\mathbb{R}^n)$ as the set of regular distributions.

Theorem 5.5. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 1<\theta\le p<\infty, 1\le q\le\infty$ *and* $\alpha_1>0$. *Let* $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\ge p$. *Then*

$$B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n).$$

Proof. Let $\Phi,\varphi\in\mathcal{S}(\mathbb{R}^n)$ satisfy (5.2) and (5.3), respectively. From Lemma 3.6 combined with Remark 3.2, we immediately obtain

$$\big\|\varphi_k*f|L_p(\mathbb{R}^n,t_0)\big\|\le c\,2^{-\alpha_1 k}\big\|\varphi_k*f|L_p(\mathbb{R}^n,t_k)\big\| \tag{5.7}$$

holds for all $k\in\mathbb{N}_0$, with $\varphi_0=\Phi$, where $c>0$ is independent of k. This guarantees that

$$B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\hookrightarrow L_p(\mathbb{R}^n,t_0)\hookrightarrow L_1^{\mathrm{loc}}(\mathbb{R}^n),$$

because of $\alpha_1>0$. This finishes the proof. □

The following lemma plays a central role in the characterization of these spaces in terms of the difference relations.

Lemma 5.1. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 1<\theta\le p<\infty, 1\le q\le\infty$ *and* $\alpha_1>0$. *Let* $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\ge p$. *There exists a constant c such that*

$$\big\|f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|\le c\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|, \tag{5.8}$$

$$\big\|f|L_p(\mathbb{R}^n,t_0)\big\|\le c\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\| \tag{5.9}$$

hold for all $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$, *where*

$$\tilde{L}_p(\mathbb{R}^n,t_0):=\{f:\big\|f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|<\infty\},$$

where $I^n:=(-1,1)^n$ *and*

$$\big\|f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|:=\Big(\int_{\mathbb{R}^n}t_0^p(x)\big\|f|L_1(x+I^n)\big\|^p dx\Big)^{\frac{1}{p}}.$$

Proof. Let Φ,ψ,φ and Ψ satisfy (5.1) through (5.4), $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ and

$$\varrho_k=\sum_{j=0}^{k}\psi_j*\widetilde{\varphi}_j*f,\quad k\in\mathbb{N}_0,$$

where $\widetilde{\varphi}_j(\cdot):=2^{jn}\overline{\varphi(-2^j\cdot)}, j\in\mathbb{N}$, $\widetilde{\varphi_0}=\Phi$ and $\psi_0=\Psi$. From Theorem 5.5 f is a regular distribution.

Step 1. We prove (5.9). From Theorem 5.5 $\{\varrho_k\}_{k\in\mathbb{N}_0}$ converges to $h\in L_p(\mathbb{R}^n,t_0)$. Let $\varphi\in\mathcal{S}(\mathbb{R}^n)$. We write

$$\langle f-h,\varphi\rangle=\langle f-\varrho_N,\varphi\rangle+\langle h-\varrho_N,\varphi\rangle,\quad N\in\mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$. The first term tends to zero as $N\to\infty$, while by Hölder's inequality there exists a constant $c>0$ is independent of N such that

$$|\langle h-\varrho_N,\varphi\rangle|\le \big\|h-\varrho_N|L_p(\mathbb{R}^n,t_0)\big\|\big\|\varphi|L_{p'}(\mathbb{R}^n,t_0^{-1})\big\|\lesssim \big\|h-\varrho_N|L_p(\mathbb{R}^n,t_0)\big\|,$$

which tends to zero as $N\to\infty$, where the last inequality follows by Lemma 2.2. Then

$$f=\sum_{j=0}^{\infty}\psi_j*\widetilde{\varphi}_j*f,\quad \text{a.e.}$$

Step 2. We prove (5.8). We see that

$$\big\|\varrho_k|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|\le\sum_{j=0}^{k}\Big(\int_{\mathbb{R}^n}t_0^p(x)\big\|\psi_j*\widetilde{\varphi}_j*f|L_1(x+I^n)\big\|^p dx\Big)^{\frac{1}{p}}.$$

We have

$$\big\|\psi_j*\widetilde{\varphi}_j*f|L_1(x+I^n)\big\|=\int_{x+I^n}|\psi_j*\widetilde{\varphi}_j*f(y)|dy,$$

which is bounded by

$$c\mathcal{M}(\mathcal{M}(\widetilde{\varphi}_j*f))(x),\quad x\in\mathbb{R}^n, j\in\mathbb{N}_0,$$

where $c>0$ is independent of j and x. Therefore

$$\begin{aligned}\big\|\psi_j*\widetilde{\varphi}_j*f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|&\le c\big\|\mathcal{M}(\mathcal{M}(\widetilde{\varphi}_j*f))|L_p(\mathbb{R}^n,t_0)\big\|\\&\le c\big\|\widetilde{\varphi}_j*f|L_p(\mathbb{R}^n,t_0)\big\|.\end{aligned}\tag{5.10}$$

From Lemma 3.6, with the help of Remark 3.2, we obtain

$$\begin{aligned}\big\|\varrho_k|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|&\le c\sum_{j=0}^{k}\big\|\widetilde{\varphi}_j*f|L_p(\mathbb{R}^n,t_0)\big\|\\&\le c\sum_{j=0}^{k}2^{-j\alpha_1}\big\|\widetilde{\varphi}_j*f|L_p(\mathbb{R}^n,t_j)\big\|\\&\le c\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.\end{aligned}$$

Furthermore, $\{\varrho_k\}$ is a Cauchy sequence in $\tilde{L}_p(\mathbb{R}^n,t_0)$ and hence it converges to $g\in\tilde{L}_p(\mathbb{R}^n,t_0)$, and

$$\big\|g|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|\lesssim\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

Let us prove that $f=g$ a.e. We have

$$\big\|f-g|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|\le\big\|f-\varrho_k|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|+\big\|g-\varrho_k|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|,\quad k\in\mathbb{N}_0$$

and by (5.10) and (5.7) we have that

$$\begin{aligned}\big\|f-\varrho_k|\tilde{L}_p(\mathbb{R}^n,t_0)\big\| &\leq \sum_{j=k+1}^{\infty}\big\|\psi_j*\widetilde{\varphi}_j*f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\| \\ &\lesssim \sum_{j=k+1}^{\infty}\big\|\widetilde{\varphi}_j*f|L_p(\mathbb{R}^n,t_0)\big\| \\ &\lesssim \big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\sum_{j=k+1}^{\infty}2^{-j\alpha_1}.\end{aligned}$$

Taking k to infinity, we get $g=f$ a.e. This finishes the proof. □

Let $M\in\mathbb{N},k\in\mathbb{N}_0$. For $f\in L_1^{\mathrm{loc}}(\mathbb{R}^n),x\in\mathbb{R}^n$ and a cube Q, we put

$$\begin{aligned}\delta^M(Q)f&:=\frac{1}{[l(Q)]^{2n}}\int_{l(Q)I^n}\int_Q\Big|\Delta_h^M f(x)\Big|dxdh,\\ \delta^M(x+2^{-k}I^n)&:=2^{2kn}\int_{2^{-k}I^n}\int_{x+2^{-k}I^n}\Big|\Delta_h^M f(y)\Big|dydh,\end{aligned}$$

where $2^{-k}I^n=(-2^{-k},2^{-k})^n$. Now we present the definition of Besov spaces of variable smoothness $\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ as introduced recently in [62].

Definition 5.5. *Let $M\in\mathbb{N}$, $0<p,q\leq\infty$, and let $\{t_k\}_{k\in\mathbb{N}_0}$ be a p-admissible weight sequence. We set*

$$\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0}):=\Big\{f:f\in L_1^{\mathrm{loc}}(\mathbb{R}^n),\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|<\infty\Big\},$$

where

$$\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|:=\Big(\sum_{k=1}^{\infty}\big\|t_k\delta^M(\cdot+2^{-k}I^n)f|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}+\big\|f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|,$$

making the obvious modifications for $p=\infty$ and/or $q=\infty$.

Let $M\in\mathbb{N},0<p<\infty,0<q\leq\infty$, and let $\{t_k\}_{k\in\mathbb{N}_0}$ be a p-admissible weight sequence. We set

$$\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_k)\big\|^*:=\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_k)\big\|^{*,1}+\Big(\sum_{m\in\mathbb{Z}^n}t^p_{0,m}\big\|f|L_1(Q_{0,m})\big\|^p\Big)^{\frac{1}{p}},$$

where

$$\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_k)\big\|^{*,1}:=\Big(\sum_{k=1}^{\infty}\Big(\sum_{m\in\mathbb{Z}^n}t^p_{k,m}(\delta^M(Q_{k,m})f)^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

with

$$\delta^M(Q_{k,m})f:=\frac{1}{[l(Q_{k,m}]^{2n}}\int_{2^{-k}I^n}\int_{Q_{k,m}}\Big|\Delta_h^M f(z)\Big|dzdh.$$

Theorem 5.6. *Let $\alpha_1,\alpha_2 \in \mathbb{R}$ and $\alpha = (\alpha_1,\alpha_2)$. Let $M \in \mathbb{N}, 1 < \theta \leq p < \infty$ and $1 \leq q \leq \infty$. Let $\{t_k\}_{k\in\mathbb{N}_0} \in X_{\alpha,\sigma,p}$ be a p-admissible sequence with $\sigma = (\sigma_1,\sigma_2)$, $\sigma_1 = \theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2 \geq p$. Then*

$$\big\| \cdot |\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|^*$$

is an equivalent norm in $\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$.

This theorem is given in [62].

The following lemma plays a central role in the characterization of $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ spaces in terms of the difference relations.

Lemma 5.2. *Let $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2, 1 < \theta \leq p < \infty, 1 \leq q \leq \infty$ and $\alpha_1 > 0$. Let $\{t_k\}_{k\in\mathbb{N}_0} \in X_{\alpha,\sigma,p}$ be a p-admissible weight sequence with $\sigma_1 = \theta\left(\frac{p}{\theta}\right)'$ and $\sigma_2 \geq p$. Let $f \in \tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$. There exists a constant c such that*

$$\big\|f|L_p(\mathbb{R}^n,t_0)\big\| \leq c\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|. \tag{5.11}$$

Proof. We follow the arguments of [20]. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $0 \leq \Psi(x) \leq 1$ for all x and $\Psi(x) = 0$ for $|x| \geq \frac{1}{M}$ and $\int_{\mathbb{R}^n}\Psi(x)dx = 1$. We make use of an observation made by Nikol'skij [49, 5.2.1, p. 185] (see also [56, Lemma 10, pp. 228-229] and [64, pp. 387-388]). We put

$$\psi(x) := (-1)^{M+1}\sum_{i=0}^{M-1}(-1)^i C^i_M \frac{1}{(i-M)^n}\Psi\Big(\frac{x}{i-M}\Big).$$

The function ψ satisfies $\psi(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}^n}\psi(x)dx = \int_{\mathbb{R}^n}\Psi(x)dx = 1$. Then, taking $\varphi_0 = \psi$, $\varphi = \psi - 2^{-n}\psi(\frac{\cdot}{2})$ and $\varphi_j = 2^{jn}\varphi(2^j\cdot)$ for $j = 1,2,3,\cdots$. From [51, Theorem 1.6, p. 152] for any $N > 0$ there exist two functions $\varpi_0,\varpi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}x^\beta\varpi(x)dx = 0$ for all multi-indices β with $|\beta| \leq N$ and

$$f = \sum_{j=0}^{\infty}\varpi_j * \varphi_j * f \tag{5.12}$$

for all $f \in \mathcal{D}'(\mathbb{R}^n)$ with $\varpi_j = 2^{jn}\varpi\left(2^j\cdot\right)$, $j \in \mathbb{N}$. Let $f \in \tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$. Obviously, we need to proof the following estimate

$$\sum_{j=0}^{\infty}\big\|t_0(\varpi_j * \varphi_j * f)|L_p(\mathbb{R}^n)\big\| \lesssim \big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\| \tag{5.13}$$

and the sequence of functions $\sum_{j=0}^{N}\varpi_j * \varphi_j * f$ converges to f almost everywhere which implys (5.11). Let us prove the estimate (5.13). We have

$$|\varphi_0 * f(x)| \lesssim \int_{B(x,1)}|f(y)|dy \lesssim \int_{x+I^n}|f(y)|dy = c\big\|f|L_1(x+I^n)\big\|, \quad x \in \mathbb{R}^n,$$

which, together with Lemma 2.4, yield

$$\begin{aligned}\big\|t_0(\varpi_0*\varphi_0*f)|L_p(\mathbb{R}^n)\big\| &\lesssim \big\|t_0\mathcal{M}(\varphi_0*f)|L_p(\mathbb{R}^n)\big\| \\ &\lesssim \big\|t_0\big\|f|L_1(\cdot+I^n)\big\||L_p(\mathbb{R}^n)\big\|.\end{aligned}$$

Now, it holds for $x\in\mathbb{R}^n$ and $j=1,2,\cdots$,

$$\varphi_j*f(x)=(-1)^{M+1}2^{jn}\int_{\mathbb{R}^n}\Delta_h^M f(x)\widetilde{\Psi}(2^jh)dh,$$

where $\widetilde{\Psi}(\cdot)=\Psi(\cdot)-2^{-n}\Psi(\frac{\cdot}{2})$. The function φ_j*f is the sum of

$$\begin{aligned}&(-1)^{M+1}2^{jn}\int_{|h|\leq\frac{2^{-j}}{M}}\Delta_h^M f(x)\Psi(2^jh)dh=\omega_{1,j}(x),\\ &(-1)^{M+1}2^{(j-1)n}\int_{|h|\leq\frac{2^{1-j}}{M}}\Delta_h^M f(x)\Psi(2^{j-1}h)dh=\omega_{2,j}(x).\end{aligned}$$

Assume that $\mathrm{supp}\varpi\subset\{x\in\mathbb{R}^n:|x|\leq 2^{i-1}\},i\in\mathbb{N}$. By similarity, we only estimate the first term. Let $x\in\mathbb{R}^n$. We have

$$\begin{aligned}|\varpi_j*\omega_{1,j}(x)| &\leq \int_{x+2^{i-j}I^n}|\varpi_j(x-y)||\omega_{1,j}(y)|dy \\ &\lesssim 2^{jn}\int_{2^{-j}I^n}\int_{x+2^{i-j}I^n}|\varpi_j(x-y)||\Delta_h^M f(y)|dydh. \end{aligned}\tag{5.14}$$

Let $j\leq i+1$. Using the definition of $\Delta_h^M f$, we can estimate (5.14) by

$$c\int_{x+2^NI^n}|f(y)|dy$$

for some $N\in\mathbb{N}$, where the positive constant c is independent of x. Therefore

$$\sum_{j=1}^{i+1}\big\|t_0(\varpi_j*\omega_{1,j})|L_p(\mathbb{R}^n)\big\|\lesssim\big\|t_0\big\|f|L_1(\cdot+2^NI^n)\big\||L_p(\mathbb{R}^n)\big\|.$$

Let us prove that the last norm is bounded by

$$c\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

We estimate

$$\big\|t_0\big\|f|L_1(\cdot+2^NI^n)\big\||L_p(\mathbb{R}^n)\big\|^p \tag{5.15}$$

by

$$\sum_{m\in\mathbb{Z}^n}t_{0,m}^p\big\|f|L_1(CQ_{0,m})\big\|^p \tag{5.16}$$

for some positive constant C. Let $m'\in\mathbb{Z}^n$ be such that $Q_{0,m'}\cap CQ_{0,m}\neq\emptyset$. Then

$$|m'-m|\leq c$$

for some positive constant c independent of m and m'. Therefore

$$\big\|f|L_1(CQ_{0,m})\big\|^p\lesssim\sum_{\substack{m'\in\mathbb{Z}^n\\ Q_{0,m'}\cap CQ_{0,m}\neq\emptyset}}\big\|f|L_1(Q_{0,m'})\big\|^p. \tag{5.17}$$

Substitute (5.17) into (5.16). This gives (5.15) can be estimated by

$$\begin{aligned}&c\sum_{m\in\mathbb{Z}^n}t_{0,m}^p\sum_{\substack{m'\in\mathbb{Z}^n\\ Q_{0,m'}\cap CQ_{0,m}\neq\emptyset}}\big\|f|L_1(Q_{0,m'})\big\|^p\\ \lesssim&\sum_{m\in\mathbb{Z}^n}\sum_{\substack{m'\in\mathbb{Z}^n\\ Q_{0,m'}\cap CQ_{0,m}\neq\emptyset}}t_{0,m'}^p\big\|f|L_1(Q_{0,m'})\big\|^p\lesssim\sum_{m\in\mathbb{Z}^n}t_{0,m}^p\big\|f|L_1(Q_{0,m})\big\|^p,\end{aligned}$$

where for the first inequality we used Lemma 2.1 (iii). Now let $j>i+1$. Obviously,

$$\begin{aligned}|\varpi_j*\omega_{1,j}(x)|&\lesssim 2^{2jn}\int_{2^{-j}I^n}\int_{x+2^{i-j}I^n}|\Delta_h^M f(y)|dydh\\ &\lesssim 2^{2jn}\int_{2^{i-j}I^n}\int_{x+2^{i-j}I^n}|\Delta_h^M f(y)|dydh\\ &=c\delta^M(x+2^{i-j}I^n)f,\quad x\in\mathbb{R}^n.\end{aligned}$$

Let $x\in Q_{j-i,m}$, $m\in\mathbb{Z}^n$. We find that $x+2^{i-j}I^n\subset Q_{j-i,m}+2^{i-j}I^n$. Therefore

$$\delta^M(x+2^{i-j}I^n)f\leq\delta^M(Q_{j-i,m}+2^{i-j}I^n)f=\frac{1}{|Q_{j-i,m}|}\int_{Q_{j-i,m}}\delta^M(Q_{j-i,m}+2^{i-j}I^n)fdy.$$

Obviously

$$Q_{j-i,m}+2^{i-j}I^n=y+Q_{j-i,m}-y+2^{i-j}I^n,\quad y\in Q_{j-i,m}.$$

We have $Q_{j-i,m}-y\subset 2^{i-j}I^n$ for all $y\in Q_{j-i,m}$ and this implies that

$$Q_{j-i,m}+2^{i-j}I^n\subset y+2^{1+i-j}I^n,\quad y\in Q_{j-i,m}.$$

Therefore, for any $x\in Q_{j-i,m}$,

$$\begin{aligned}\delta^M(x+2^{i-j}I^n)f&\leq\frac{1}{|Q_{j-i,m}|}\int_{Q_{j-i,m}}\delta^M(y+2^{1+i-j}I^n)fdy\\ &\lesssim\mathcal{M}(\delta^M(\cdot+2^{1+i-j}I^n)f)(x),\end{aligned}$$

where the implicit constant is independent of j,m and x, which yields that

$$\begin{aligned}\big\|t_0(\varpi_j*\omega_{1,j})|L_p(\mathbb{R}^n)\big\|&\lesssim\big\|t_0\mathcal{M}(\delta^M(\cdot+2^{1+i-j}I^n)f)|L_p(\mathbb{R}^n)\big\|\\ &\lesssim 2^{\alpha_1(i-j)}\big\|\delta^M(\cdot+2^{1+i-j}I^n)f|L_p(\mathbb{R}^n,t_{j-i-1})\big\|,\end{aligned}$$

by Lemma 2.4 combined with Remark 2.3 (ii). Consequently

$$\sum_{j=i+2}^{\infty}\big\|t_0(\varpi_j*\omega_{1,j})|L_p(\mathbb{R}^n)\big\|\lesssim\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

Now we prove that

$$\Big\{\varrho_N=\sum_{j=0}^{N}\varpi_j*\varphi_j*f\Big\}$$

converges to f almost everywhere. From (5.13), the sequence $\{\varrho_N\}$ converges to

$$g=\sum_{j=0}^{\infty}\varpi_j*\varphi_j*f$$

almost everywhere and g belongs to $L_p(\mathbb{R}^n,t_0)$. Let $\varphi\in\mathcal{D}(\mathbb{R}^n)$. We write

$$\langle f-g,\varphi\rangle=\langle f-\varrho_N,\varphi\rangle+\langle g-\varrho_N,\varphi\rangle,\quad N\in\mathbb{N}_0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality bracket between $\mathcal{D}'(\mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n)$. By (5.12) the first term tends to zero as $N\to\infty$, while by Hölder's inequality there exists a constant $c>0$ independent of N such that

$$|\langle g-\varrho_N,\varphi\rangle|\le\big\|g-\varrho_N|L_p(\mathbb{R}^n,t_0)\big\|\big\|\varphi|L_{p'}(\mathbb{R}^n,t_0^{-1})\big\|\lesssim\big\|g-\varrho_N|L_p(\mathbb{R}^n,t_0)\big\|,$$

which tends to zero as $N\to\infty$, where the last inequality follows by Lemma 2.2. Therefore $f=g$ almost everywhere. □

Remark 5.1. Let ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying

$$0\le\psi\le1\quad\text{and}\quad\psi(x)=\begin{cases}1,&\text{if}\quad|x|\le1,\\0,&\text{if}\quad|x|\ge\frac{3}{2}.\end{cases}$$

We put $\mathcal{F}\phi_0=\psi$, $\mathcal{F}\phi_1=\psi(\frac{\cdot}{2})-\psi$ and $\mathcal{F}\phi_k=\mathcal{F}\phi_1(2^{1-k}\cdot)$ for $k=2,3,\cdots$. Then $\{\mathcal{F}\phi_k\}_{k\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity,

$$\sum_{k=0}^{\infty}\mathcal{F}\phi_k(x)=1$$

for all $x\in\mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f=\sum_{k=0}^{\infty}\phi_k*f$$

of all $f\in\mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$). We can easily prove that

$$\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\approx\Big(\sum_{k=0}^{\infty}\big\|t_k(\phi_k*f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}$$

for any $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$.

In the following theorem we will establish characterizations of $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$ in terms of the difference relations.

Theorem 5.7. *Let* $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2, 1<\theta\le p<\infty, 1\le q\le\infty$ *and* $M\in\mathbb{N}\backslash\{0\}$. *Let* $\{t_k\}_{k\in\mathbb{N}_0}\in X_{\alpha,\sigma,p}$ *be a p-admissible weight sequence with* $\sigma_1=\theta\left(\frac{p}{\theta}\right)'$ *and* $\sigma_2\ge p$. *Assume that*

$$0<\alpha_1\le\alpha_2<M.$$

Then

$$B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})=\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0}),$$

in the sense of equivalent norm.

Proof. We will divide the proof into two steps.

Step 1. We prove that

$$\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|\le c\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|$$

for all $f\in B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$. By Lemma 5.1, $\big\|f|\tilde{L}_p(\mathbb{R}^n,t_0)\big\|$ can be estimated from above by $c\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|$. Let $\{\mathcal{F}\phi_j\}_{j\in\mathbb{N}_0}$ be a smooth dyadic resolution of unity. We need to estimate

$$\Big(\sum_{k=1}^{\infty}\Big\|\sum_{j=0}^{k}t_k\delta^M(\cdot+2^{-k}I^n)(\phi_j*f)|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}, \tag{5.18}$$

$$\Big(\sum_{k=1}^{\infty}\Big\|\sum_{j=k+1}^{\infty}t_k\delta^M(\cdot+2^{-k}I^n)(\phi_j*f)|L_p(\mathbb{R}^n)\Big\|^q\Big)^{\frac{1}{q}}. \tag{5.19}$$

We have

$$\delta^M(x+2^{-k}I^n)f\lesssim 2^{kn}\int_{x+2^{-k}I^n}d^M_{\sqrt{n}2^{-k}}f(z)dz,\quad x\in\mathbb{R}^n.$$

Let

$$\phi_j^{*,a}f(x):=\sup_{y\in\mathbb{R}^n}\frac{|\phi_j*f(y)|}{(1+2^j|x-y|)^a}$$

be the Peetre maximal function. As in [58, 2.5.10] we immediately obtain that

$$d^M_{\sqrt{n}2^{-k}}(\phi_j*f)(z)\le c2^{(j-k)M}\phi_j^{*,a}f(z)\le c2^{(j-k)M}\phi_j^{*,a}f(x)$$

if $a>0, 0\le j\le k, k\in\mathbb{N}$, $x\in\mathbb{R}^n$ and $z\in x+2^{-k}I^n$, where $c>0$ is independent of j,k and x. Since $\alpha_2<M$, (5.18) does not exceed

$$\Big(\sum_{j=0}^{\infty}\big\|\phi_j^{*,a}f|L_p(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|^q\Big)^{\frac{1}{q}}\lesssim\big\|f|B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|,$$

by Lemma 2.6 combined with the fact that

$$\phi_j^{*,a} f \lesssim \mathcal{M}(\phi_j * f),$$

where a is large enough. Let $j>k$. Let $x\in Q_{k,m}$, for some $m\in\mathbb{Z}^n$. Recalling the definition of $d^M_{2^{-k}}(\varphi_j * f)$, we can estimate

$$2^{kn}\int_{x+2^{-k}I^n} d^M_{\sqrt{n}2^{-k}} f(y)dy \tag{5.20}$$

by

$$2^{2kn}\sum_{i=0}^{M} C_M^i \int_{\sqrt{n}2^{-k}I^n}\int_{x+N2^{-k}I^n} |\phi_j * f(y)|\,dydh \lesssim 2^{kn}\int_{CQ_{k,m}} |\phi_j * f(y)|\,dy,$$

where the positive constants C and N do not depend on k and m. Therefore (5.20) is bounded by $c\mathcal{M}(\phi_j * f)(x)$ for any $j>k$ and $x\in\mathbb{R}^n$. Hence

$$t_k \sum_{j=k+1}^{\infty} \delta^M(\cdot + 2^{-k}I^n)(\phi_j * f)$$

can be estimated from above by

$$c\; t_k \sum_{j=k+1}^{\infty} \mathcal{M}(\phi_j * f),$$

where the positive constant c is independent of k. We obtain that (5.19) is bounded by

$$\Big(\sum_{j=0}^{\infty} \big\| t_j \mathcal{M}(\phi_j * f) | L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}} \le c \big\| f | B_{p,q}(\mathbb{R}^n, \{t_k\}_{k\in\mathbb{N}_0})\big\|,$$

where we used Lemma 2.6.

Step 2. Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x)=1$ for $|x|\le 1$ and $\Psi(x)=0$ for $|x|\ge \frac{3}{2}$, and in addition radial symmetric. We make use again of an observation made by Nikol'skij [49] (see also [56]). We put

$$\psi(x) := (-1)^{M+1}\sum_{i=0}^{M-1} (-1)^i C_M^i \Psi(x(M-i)).$$

The function ψ satisfies $\psi(x)=1$ for $|x|\le 1/M$ and $\psi(x)=0$ for $|x|\ge 3/2$. Then, taking $\mathcal{F}\varphi_0(x)=\psi(x)$, $\mathcal{F}\varphi_1(x)=\psi(x)-\psi(\frac{x}{2})$ and $\mathcal{F}\varphi_k(x)=\varphi_1(2^{-k+1}x)$ for $k=2,3,\cdots$, we obtain that $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is a smooth dyadic resolution of unity. This yields that

$$\Big(\sum_{k=0}^{\infty} \big\| t_k(\varphi_k * f) | L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}} \tag{5.21}$$

is an equivalent norm in $B_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})$. Let us prove that the last term is bounded by

$$c\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|. \tag{5.22}$$

We observe that

$$\varphi_0*f(x)=(-1)^{M+1}\int_{\mathbb{R}^n}\mathcal{F}^{-1}\Psi(z)\Delta^M_{-z}f(x)dz+f(x)\int_{\mathbb{R}^n}\mathcal{F}^{-1}\Psi(z)dz,\quad x\in\mathbb{R}^n.$$

From Lemma 5.2, $\big\|f|L_p(\mathbb{R}^n,t_0)\big\|$, can be estimated by (5.22). Moreover, it holds for $x\in\mathbb{R}^n$ and $k=1,2,\cdots$

$$\varphi_k*f(x)=(-1)^{M+1}\int_{\mathbb{R}^n}\Delta^M_{-2^{-k}y}f(x)\widetilde{\Psi}(y)dy,$$

where $\widetilde{\Psi}(\cdot)=\mathcal{F}^{-1}\Psi(\cdot)-2^{-n}\mathcal{F}^{-1}\Psi(\frac{\cdot}{2})$. Obviously we need to estimate

$$\int_{\mathbb{R}^n}|\Delta^M_{2^{-k}y}f(x)g(y)|dy,$$

where $g\in\mathcal{S}(\mathbb{R}^n)$, $x\in\mathbb{R}^n$ and $k\in\mathbb{N}_0$. We split the last integral into two parts:

$$\int_{|y|\le 1}|\Delta^M_{2^{-k}y}f(x)||g(y)|dy+\int_{|y|>1}|\Delta^M_{2^{-k}y}f(x)||g(y)|dy.$$

By similarity we estimate only the second integral. We write

$$\begin{aligned}\int_{|y|>1}|\Delta^M_{2^{-k}y}f(x)|g(y)|dy&=\sum_{j=0}^{\infty}\int_{2^j<|y|\le 2^{j+1}}|\Delta^M_{2^{-k}y}f(x)||g(y)|dy\\&\le c\sum_{j=0}^{\infty}2^{kn-Vj}\int_{2^{j-k}<|h|\le 2^{j-k+1}}|\Delta^M_h f(x)|dh,\end{aligned}$$

where the positive constant c is independent of k, $V>n$ is at our disposal and we have used the properties of the function g,

$$|g(x)|\le c(1+|x|)^{-V}$$

for any $x\in\mathbb{R}^n$ and any $V>n$. Let us recall the following useful estimate which essentially from [55] and [68]. In [35], but traced back to Brudnyi [9] and Nevskii [48], is established the following inequality. Let $r\in(0,\infty]$, $M\in\mathbb{N}$, $\varrho\in(0,1]$ and $f\in L^{\text{loc}}_r(\mathbb{R}^n)$. Then there exists a positive constant $C=C(M,\varrho,r,n)$ such that for any cube Q with side length a there is a polynomial $P\in\mathcal{P}_{M-1}(\mathbb{R}^n)$ satisfying

$$\int_Q|f(y)-P(y)|^r dy\le C\,a^{-n}\int_{|h|\le\varrho a}\int_Q\Big|\Delta^M_h f(y)\Big|^r dydh.$$

From this inequality and the fact that,

$$\Delta^M_h f(x)=\Delta^M_h(f-P)(x)=\sum_{v=0}^{M}(-1)^{M-v}C^v_M(f-P)(x+vh),$$

we obtain

$$\int_{|h|\leq 2^{j-k+1}}|(f-P)(x+vh)|dh\leq \int_{x+v2^{j-k+1}I^n}|(f-P)(z)|dz$$
$$\leq c2^{(k-j)n}\int_{|h|\leq 2^{N+j-k}}\int_{x+2^{N+j-k}I^n}\left|\Delta_h^M f(y)\right|dydh$$
$$\leq c2^{(j-k)n}\delta^M(x+2^{N+j-k}I^n)f,$$

where $0\leq v\leq M, N\in\mathbb{N}$ and the positive constant c is independent of j,k and x. Therefore

$$\int_{|y|>1}|\Delta_{2^{-k}y}^M f(x)||g(y)|dy\leq c\sum_{j=0}^{\infty}2^{-(V-n)j}\delta^M(x+2^{N+j-k}I^n)f$$

and (5.21) can be estimated by

$$\left\|f|\tilde{B}_{p,q}^M(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\right\|+\sum_{j=0}^{\infty}2^{-(V-n)j}\Big(\sum_{k=0}^{\infty}\left\|t_k\delta^M(\cdot+2^{N+j-k}I^n)f|L_p(\mathbb{R}^n)\right\|^q\Big)^{\frac{1}{q}}.$$

We put

$$J_1:=\sum_{j=0}^{\infty}2^{-(V-n)j}\Big(\sum_{k=j+N+1}^{\infty}\left\|t_k\delta^M(\cdot+2^{N+j-k}I^n)f|L_p(\mathbb{R}^n)\right\|^q\Big)^{\frac{1}{q}}, \tag{5.23}$$

$$J_2:=\sum_{j=0}^{\infty}2^{-(V-n)j}\Big(\sum_{k=0}^{j+N}\left\|t_k\delta^M(\cdot+2^{N+j-k}I^n)f|L_p(\mathbb{R}^n)\right\|^q\Big)^{\frac{1}{q}}. \tag{5.24}$$

For the sake of simplicity we suppose $N=0$. We use the technique of [20].

Estimate of J_1. Let $x\in Q_{k,m}$ with $k\geq j+1$ and $m\in\mathbb{Z}^n$. We find that

$$x+2^{j-k}I^n\subset Q_{k,m}+2^{j-k}I^n.$$

Therefore

$$\delta^M(x+2^{j-k}I^n)f\leq\tilde{\delta}^M(Q_{k,m}+2^{j-k}I^n)f=\frac{1}{|Q_{k,m}|}\int_{Q_{k,m}}\tilde{\delta}^M(Q_{k,m}+2^{j-k}I^n)fdy,$$

where

$$\tilde{\delta}^M(Q_{k,m}+2^{j-k}I^n)f:=2^{2(k-j)n}\int_{2^{j-k}I^n}\int_{Q_{k,m}+2^{j-k}I^n}\left|\Delta_h^M f(v)\right|dvdh.$$

Obviously

$$Q_{k,m}+2^{j-k}I^n=y+Q_{k,m}-y+2^{j-k}I^n,\quad y\in Q_{k,m}.$$

We have $Q_{k,m}-y\subset 2^{-k}I^n\subset 2^{j-k}I^n$ for all $y\in Q_{k,m}$ and this implies that

$$Q_{k,m}+2^{j-k}I^n\subset y+2^{j-k+1}I^n,\quad y\in Q_{k,m}.$$

Therefore, for any $x\in Q_{k,m}$,

$$\delta^M(x+2^{j-k}I^n)f\lesssim\frac{1}{|Q_{k,m}|}\int_{Q_{k,m}}\delta^M(y+2^{j-k+1}I^n)fdy\lesssim\mathcal{M}(\delta^M(\cdot+2^{j-k+1}I^n)f)(x),$$

where the implicit constant is independent of k,j,m and x. Using this estimate, the quantity

$$\Big(\sum_{k=j+1}^{\infty}\big\|t_k\delta^M(\cdot+2^{j-k}I^n)f|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}},\tag{5.25}$$

can be estimated from above by

$$c\Big(\sum_{k=j+1}^{\infty}\big\|t_k\mathcal{M}(\delta^M(\cdot+2^{j-k+1}I^n)f)|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}.$$

Applying Lemma 2.4 combined with Remark 2.3 we get the estimate

$$\big\|t_k\mathcal{M}(\delta^M(\cdot+2^{j-k+1}I^n)f)|L_p(\mathbb{R}^n)\big\|\lesssim 2^{\alpha_2 j}\big\|t_{k-j-1}\delta^M(\cdot+2^{j-k+1}I^n)f|L_p(\mathbb{R}^n)\big\|,$$

which yields that (5.25) does not exceed

$$c2^{\alpha_2 j}\Big(\sum_{i=0}^{\infty}\big\|t_i\delta^M(\cdot+2^{-i}I^n)f|L_p(\mathbb{R}^n)\big\|^q\Big)^{\frac{1}{q}}\lesssim 2^{\alpha_2 j}\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|,$$

where the implicit constant is independent of j. Taking V large enough, (5.23) is bounded by

$$c\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|.$$

Estimate of J_2. Observe that

$$\delta^M(x+2^{j-k+1}I^n)f\lesssim 2^{(k-j)n}\big\|f|L_1(CQ_{k-j,m})\big\|$$

if $x\in Q_{k-j,m}$, $m\in\mathbb{Z}^n$, where the positive constant C is independent of x,j,k and m. This yields that

$$\big\|t_k\delta^M(\cdot+2^{j-k+1}I^n)f|L_p(\mathbb{R}^n)\big\|^p$$

can be rewritten as

$$\sum_{m\in\mathbb{Z}^n}\big\|t_k\delta^M(\cdot+2^{j-k+1}I^n)f|L_p(Q_{k-j,m})\big\|^p,$$

which can be estimated by

$$\begin{aligned}&c2^{(k-j)np}\sum_{m\in\mathbb{Z}^n}\big\|t_k|L_p(Q_{k-j,m})\big\|^p\big\|f|L_1(CQ_{k-j,m})\big\|^p\\&=c2^{(k-j)n(1+p)}\sum_{m\in\mathbb{Z}^n}\int_{Q_{k-j,m}}\big\|t_k|L_p(Q_{k-j,m})\big\|^p\big\|f|L_1(CQ_{k-j,m})\big\|^p dx,\end{aligned}$$

where the positive constant c is independent of j and k. From Lemma 2.1 (iii)

$$\big\|t_k|L_p(Q_{k-j,m})\big\|\lesssim 2^{(j-k)\frac{n}{\theta}}\big\|t_k|L_p(Q_{0,\bar{m}})\big\|$$

for any dyadic cube $Q_{0,\bar{m}}$ such that $Q_{0,\bar{m}}\cap CQ_{k-j,m}\neq\emptyset$. Therefore we obtain the following chain of estimates

$$\begin{aligned}
&\big\|t_k|L_p(Q_{k-j,m})\big\|\big\|f|L_1(CQ_{k-j,m})\big\|\\
&\lesssim 2^{(j-k)\frac{n}{\theta}}\sum_{\substack{\bar{m}\in\mathbb{Z}^n\\ Q_{0,\bar{m}}\cap CQ_{k-j,m}\neq\emptyset}}\big\|t_k|L_p(Q_{0,\bar{m}})\big\|\big\|f|L_1(Q_{0,\bar{m}})\big\|\\
&\lesssim 2^{(j-k)\frac{n}{\theta}+k\alpha_2}\sum_{\substack{\bar{m}\in\mathbb{Z}^n\\ Q_{0,\bar{m}}\cap CQ_{k-j,m}\neq\emptyset}}t_{0,\bar{m}}\big\|f|L_1(Q_{0,\bar{m}})\big\|,
\end{aligned}$$

where in the last estimate we used (2.4). This estimate and the fact that

$$(1+|x-\bar{m}|)^d\lesssim 2^{(j-k)d}$$

for any $x\in Q_{k-j,m}$ and any $d\in\mathbb{N}$, such that $Q_{0,\bar{m}}\cap CQ_{k-j,m}\neq\emptyset$ yield that

$$\begin{aligned}
&\big\|t_k|L_p(Q_{k-j,m})\big\|\big\|f|L_1(CQ_{k-j,m})\big\|\\
&\lesssim 2^{(j-k)(\frac{n}{\theta}+d)+k\alpha_2}\sum_{\bar{m}\in\mathbb{Z}^n}(1+|x-\bar{m}|)^{-d}t_{0,\bar{m}}\big\|f|L_1(Q_{0,\bar{m}})\big\|.
\end{aligned}$$

Our estimate uses partially some decomposition techniques already used in [26]. For any $i\in\mathbb{N}$ and any $x\in Q_{k-j,m}$, we define

$$\begin{aligned}
&\Omega_{i,x}=\{\bar{m}\in\mathbb{Z}^n: 2^{i-1}<|x-\bar{m}|\leq 2^i\},\\
&\Omega_{0,x}=\{\bar{m}\in\mathbb{Z}^n: |x-\bar{m}|\leq 1\}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\sum_{\bar{m}\in\mathbb{Z}^n}t_{0,\bar{m}}(1+|x-\bar{m}|)^{-d}\big\|f|L_1(Q_{0,\bar{m}})\big\|\\
&=\sum_{i=0}^{\infty}\sum_{\bar{m}\in\Omega_{i,x}}t_{0,\bar{m}}(1+|x-\bar{m}|)^{-d}\big\|f|L_1(Q_{0,\bar{m}})\big\|\\
&\lesssim\sum_{i=0}^{\infty}2^{-di}\sum_{\bar{m}\in\Omega_{i,x}}t_{0,\bar{m}}\big\|f|L_1(Q_{0,\bar{m}})\big\|.
\end{aligned}$$

The last expression can be rewritten as

$$c\sum_{i=0}^{\infty}2^{(n-d)i}2^{-in}\int_{\cup_{\bar{m}\in\Omega_{i,x}}\tilde{Q}_{0,\bar{m}}}\sum_{h\in\Omega_{i,x}}t_{0,h}\big\|f|L_1(Q_{0,h})\big\|\chi_{0,h}(y)dy. \tag{5.26}$$

If $y \in \cup_{\bar{m}\in\Omega_{i,x}} Q_{0,\bar{m}}$, then $y \in Q_{0,\bar{m}}$ for some $\bar{m} \in \Omega_{i,x}$ and $2^{i-1} < |x-\bar{m}| \le 2^i$. From this it follows that

$$\begin{aligned}|y-x| &\le |y-\bar{m}|+|x-\bar{m}| \\ &\le |y-\bar{m}|+2^i \\ &\lesssim \sqrt{n}+2^i \\ &< 2^{i+\varrho_n}, \qquad \varrho_n \in \mathbb{N},\end{aligned}$$

which implies that y is located in the ball $B(x,2^{i+\varrho_n})$. By taking d large enough, (5.26) does not exceed

$$c\mathcal{M}\Big(\sum_{h\in\mathbb{Z}^n} t_{0,h}\big\|f|L_1(Q_{0,h})\big\|\chi_{0,h}\Big)(x), \quad x\in Q_{k-j,m}.$$

Hence

$$\begin{aligned}&\big\|t_k(\delta^M(\cdot+2^{j-k+1}I^n)f|L_p(\mathbb{R}^n)\big\| \\ \lesssim\, &2^{(j-k)(\frac{n}{\theta}+d-\frac{n}{p}-n)+k\alpha_2}\Big\|\mathcal{M}\Big(\sum_{h\in\mathbb{Z}^n} t_{0,h}\big\|f|L_1(Q_{0,h})\big\|\chi_{0,h}\Big)|L_p(\mathbb{R}^n)\Big\|.\end{aligned}$$

Clearly the last term is bounded by

$$\begin{aligned}&c2^{(j-k)(\frac{n}{\theta}+d-\frac{n}{p}-n)+k\alpha_2}\Big(\sum_{h\in\mathbb{Z}^n} t_{0,h}^p\big\|f|L_1(Q_{0,h})\big\|^p\Big)^{\frac{1}{p}} \\ &\lesssim 2^{(j-k)(\frac{n}{\theta}+d-\frac{n}{p}-n)+k\alpha_2}\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|^*.\end{aligned}$$

Therefore (5.24) is bounded by

$$\begin{aligned}&c\sum_{j=0}^{\infty}2^{j(\frac{n}{\theta}+d-V)}\Big(\sum_{k=0}^{j+1}2^{k(\alpha_2+\frac{n}{p}+n-\frac{n}{\theta}-d)q}\Big)^{\frac{1}{q}}\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|^* \\ \lesssim\, &\big\|f|\tilde{B}^M_{p,q}(\mathbb{R}^n,\{t_k\}_{k\in\mathbb{N}_0})\big\|^*,\end{aligned}$$

by taking V large enough. The proof is complete. □

Corollary 5.1. *Let $1<p<\infty, 1<q<\infty$ and $M\in\mathbb{N}\backslash\{0\}$. Let w denote a positive, locally integrable function. Suppose that $w^p\in A_p(\mathbb{R}^n)$. Assume that $0<s<M$. Then*

$$B_{p,q}(\mathbb{R}^n,\{2^{sk}w\}_{k\in\mathbb{N}_0}) = \tilde{B}^M_{p,q}(\mathbb{R}^n,\{2^{sk}w\}_{k\in\mathbb{N}_0}),$$

in the sense of equivalent norm.

Proof. Since $w^p\in A_p(\mathbb{R}^n)$, from Lemma 2.1 there exists $1<\theta<p<\infty$ such that

$$\{2^{-sk}w\}_k \in X_{\alpha,\sigma,p}$$

with $\sigma=(\sigma_1=\theta\big(\frac{p}{\theta}\big)', \sigma_2\ge p)$ and $\alpha=(s,s)$. Hence, this corollary is a special case of Theorem 5.7. □

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