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Non-Regular Pseudo-Differential Operators on Matrix Weighted Besov-Triebel-Lizorkin Spaces

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Abstract. In this paper we obtain the boundedness of non-regular pseudo-differential operators with symbols in Besov spaces on matrix-weighted Besov-Triebel-Lizorkin spaces. These symbols include the classical Hörmander classes.

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Key words: Pseudo-differential operator, matrix weight, Besov space, Triebel-Lizorkin space.

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1 Introduction

The pseudo-differential operators have been widely used in plenty of mathematical areas; see [1, 11, 14, 16, 17, 28, 31–33, 36, 38]. The boundedness of pseudo-differential operators on Triebel-Lizorkin and Besov spaces has been considered in [6, 22–25, 27, 30]. The authors of the paper proved the boundedness of the Hörmander classes pseudo-differential operators on matrix-weighted Besov spaces and Triebel-Lizorkin spaces in [2].

In [18, 21], Marschall obtained the boundedness of non-regular pseudo-differential operators corresponding to symbols in the class $SB^m_\delta(r,\mu,\nu;N,\lambda)$ (see Definition 2.5) on Triebel-Lizorkin spaces and Besov spaces. Then Sato obtained the boundedness of non-regular pseudo-differential operators on the weighted Triebel-Lizorkin spaces in [29], and Drihem and Hebbache obtained the boundedness of non-regular pseudodifferential operators on variable Triebel-Lizorkin spaces in [7].

In the last three decades, inspired by the applications of matrix-weighted functions, many matrix-weighted function spaces have appeared, such as matrix-weighted Lebesgue

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spaces [13,35], matrix weighted Besov and Triebel-Lizorkin spaces [4,9,10,26,37], matrix-weighted Besov type spaces and Triebel-Lizorkin type spaces [3]. In [37], Wang, Yang and Zhang obtained the characterizations of matrix-weighted Triebel-Lizorkin spaces in terms of the Peetre maximal function, the Lusin area function, and the Littlewood-Paley g_λ^*-function. They also proved the boundedness of Fourier multipliers on matrix-weighted Triebel-Lizorkin spaces under the generalized Hörmander condition. In [3], Bu, Hytönen, Yang, Yuan proposed a new concept of A_p-dimension of matrix weights. Then they obtained the boundedness of φ-transform, pseudo-differential operators, trace operators, and Calderón-Zygmund operators on matrix-weighted Besov type spaces and Triebel-Lizorkin type spaces. In particular, the symbols of their pseudo-differential operators are in the classical Hörmander class $S^m_{1,1}$.

Since the class $SB^m_\delta(r,\mu,\nu;N,\lambda)$ includes some Hörmander classes as special cases, in this paper, we consider the boundedness of non-regular pseudo-differential operators with symbols in $SB^m_\delta(r,\mu,\nu;N,\lambda)$ on matrix-weighted Besov spaces and Triebel-Lizorkin spaces.

This paper is organized as follows. In Section 2, we give some convenient notations and recall several concepts about matrix weights and function spaces. Some key lemmas and basic tools are given in Section 3. The boundedness of non-regular pseudo-differential operators on matrix-weighted Besov spaces and Triebel-Lizorkin spaces are described in Section 4.

2 Preliminaries

Let χ_E be the characteristic function of the set $E\subset\mathbb{R}^n$. Let $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. The Fourier transform of f is defined by $\mathcal{F}(f):=\hat{f}:=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\mathrm{d}x$ and the inverse Fourier transform of f by $\mathcal{F}^{-1}(f):=\check{f}:=\int_{\mathbb{R}^n} f(x)e^{2\pi i x\cdot\xi}\mathrm{d}x$. Let $\mathscr{S}(\mathbb{R}^n)$ denote the Schwartz space, and let $\mathscr{S}'(\mathbb{R}^n)$ be its dual.

Definition 2.1. *Let φ_0 be a Schwartz function such that supp$(\varphi_0)\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2\}$ and $\varphi_0(\xi)=1$ for $|\xi|\le 1$. Moreover, put $\varphi_j(\xi)=\varphi_0(2^{-j}\xi)-\varphi_0(2^{-j+1}\xi)$ for $j\in\mathbb{N}$. Then supp$(\varphi_j)\subset\{\xi:2^{j-1}\le|\xi|\le 2^{j+1}\}$ for all $j\in\mathbb{N}$ and*

$$\sum_{j=0}^{\infty}\varphi_j(\xi)=1$$

for $\xi\in\mathbb{R}^n$. Hence $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a partition of unity on $\mathbb{R}^n$ subordinated to the dyadic rings $\{\xi:2^{j-1}\le|\xi|\le 2^{j+1}\}$, $j\in\mathbb{N}$, and $\overline{B(0,2)}$.

We also set $\tilde{\varphi}_0:=\varphi_0+\varphi_1$, and $\tilde{\varphi}_j:=\varphi_{j-1}+\varphi_j+\varphi_{j+1}$ for $j\in\mathbb{N}$. Note that, $\varphi_j\tilde{\varphi}_j=\varphi_j$ for $j\in\mathbb{N}_0$ and

$$\begin{aligned}
&\mathrm{supp}(\tilde{\varphi}_j)\subset\{\xi\in\mathbb{R}^n:2^{j-2}\le|\xi|\le 2^{j+2}\} &&\text{for } j\ge 2,\\
&\mathrm{supp}(\tilde{\varphi}_j)\subset\{\xi\in\mathbb{R}^n:|\xi|\le 2^{j+2}\} &&\text{for } j=0,1.
\end{aligned}$$

For a Schwartz function φ, We define $\varphi(D)\vec{f} := \mathcal{F}^{-1}[\varphi(\xi)\mathcal{F}(\vec{f})(\xi)]$.

Definition 2.2. *Let $0<p,q\le\infty$. Let $\{f_k\}_{k=0}^{\infty}$ be a sequence of measurable functions on $\mathbb{R}^n$. Then we define*

$$\|\{f_k\}_{k=0}^{\infty}\|_{\ell^q(L^p)} = \left(\sum_{k=0}^{\infty}\|f_k\|_{L^p}^q\right)^{\frac{1}{q}},$$

where there is a usual modification for $q=\infty$.

Definition 2.3. *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a partition of unity on $\mathbb{R}^n$ as in Definition 2.1. Let $s\in\mathbb{R}, 0<p,q\le\infty$. The classical Besov spaces $B_{p,q}^s$ is set of all functions with finite quasi-norm*

$$\|f\|_{B_{p,q}^s} = \left(\sum_{j=0}^{\infty}2^{sq}\|\mathcal{F}^{-1}(\varphi_j\hat{f})\|_{L^p}^q\right)^{\frac{1}{q}}.$$

Definition 2.4. *Let $m\in\mathbb{R}$, $n\in\mathbb{N}$, $\rho,\delta\in[0,1]$. The Hörmander class $S_{\rho,\delta}^m(\mathbb{R}^n\times\mathbb{R}^n)$ is the set of all smooth functions $\sigma:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ such that*

$$|\partial_\xi^\alpha\partial_x^\beta\sigma(x,\xi)|\le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

holds for all $\alpha,\beta\in\mathbb{N}_0^n$, where $C_{\alpha,\beta}$ is independent of $x,\xi\in\mathbb{R}^n$ and $|\alpha|$ and $|\beta|$ are their index sum. The function σ is called a pseudo-differential symbol and m is called the order of σ.

The following notion was introduced by Marschall in [18, 21]. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity as in Definition 2.1. For a function $a:\mathbb{R}^{2n}\to\mathbb{C}$, let

$$a_j(x,\xi) = \mathcal{F}_{y\to x}^{-1}(\mathcal{F}\varphi_j(y)\mathcal{F}a(y,\xi)).$$

Definition 2.5. *The space $B_{\mu,\nu}^r(B_{\lambda,\infty}^N)$ consists of all distributions $a\in\mathscr{S}'(\mathbb{R}^{2n})$ such that*

$$\|a\|_{B_{\mu,\nu}^r(B_{\lambda,\infty}^N)} := \left\|\{2^{jr}\|a_j(x,\cdot)\|_{B_{\lambda,\infty}^N}\}_{j\in\mathbb{N}_0}\right\|_{\ell^\nu(L^\mu)}<\infty,$$

where $\mu,\nu\in(0,\infty]$, $r\in[n/\mu,\infty)$, $\lambda\in[1,\infty]$, $N\in(n/\lambda,\infty)$.

Let $m\in\mathbb{R}$, $\delta\in[0,1]$, $\mu,\nu\in(0,\infty]$, $r\in(n/\mu,\infty)$, and $N\in(n/\lambda,\infty]$. Then a symbol $a\in SB_\delta^m(r,\mu,\nu;N,\lambda)$ if

$$\sup_k 2^{-km}\left\|\|a(x,2^k\cdot)\mathcal{F}\varphi_k(2^k\cdot)\|_{B_{\lambda,\infty}^N}\right\|_{L^\infty(\mathrm{d}x)}<\infty,$$
$$\sup_k 2^{-km}2^{-k\delta r}\left\|a(x,2^k\cdot)\mathcal{F}\varphi_k(2^k\cdot)\right\|_{B_{\mu,\nu}^r(B_{\lambda,\infty}^N)}<\infty.$$

Choosing $\mu=\nu=N=\lambda=\infty$, we see that the classical Hörmander classes $S_{1,\delta}^m\subset SB_\delta^m(r,\mu,\nu;N,\lambda)$. If $\sigma\in SB_\delta^m(r,\mu,\nu;N,\lambda)$ is a symbol, then

$$\sigma(x,D)f(x) := \int_{\mathbb{R}^n}e^{2\pi ix\cdot\xi}\sigma(x,\xi)\hat{f}(\xi)\mathrm{d}\xi$$

is called the pseudo-differential with the symbol σ, which is initial defined for suitable functions f, for example the Schwartz functions.

For a fixed positive integer d, if vector-valued functions $\vec{f}=(f_1,\dots,f_d)^T$ on $\mathbb{R}^n$ satisfy that each component f_i belongs to $\mathscr{S}'(\mathbb{R}^n)$ where T denotes the transpose of the row vector, then we denote $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$. The Fourier transform on vector-valued spaces is denoted by $\mathcal{F}(\vec{f}):=(\hat{f}_1,\dots,\hat{f}_d)^T$. If σ is a symbol, then

$$\sigma(x,D)\vec{f}(x):=\int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi}\sigma(x,\xi)\mathcal{F}(\vec{f})(\xi)\mathrm{d}\xi$$

defines the associated pseudo-differential operator where $\sigma(x,\xi)\mathcal{F}(\vec{f})(\xi)=(\sigma\hat{f}_1,\dots,\sigma\hat{f}_d)^T$.

2.1 Matrix weights

Let $d\in\mathbb{N}$. A matrix weight W is a map on $\mathbb{R}^n$ such that $W(x)$ is a non-negative definite $d\times d$ matrix for each $x\in\mathbb{R}^n$, where W is almost everywhere invertible and the entries of W are measurable functions on $\mathbb{R}^n$. The operator norm of a matrix A is defined by

$$\|A\|:=\sup_{|\vec{z}|=1}\frac{|A\vec{z}|}{|\vec{z}|},$$

where $\vec{z}\in\mathbb{C}^d$ and $|\vec{z}|=\left(\sum_{i=1}^d|z_i|^2\right)^{1/2}$.

For $p\in(1,\infty)$, a matrix weight $W\in A_p(\mathbb{R}^n)$ if and only if

$$\sup_Q\frac{1}{|Q|}\int_Q\left(\frac{1}{|Q|}\int_Q\|W^{1/p}(x)W^{-1/p}(y)\|^{p'}\mathrm{d}y\right)^{p/p'}\mathrm{d}x<\infty,$$

where $p'=p/(p-1)$ is the conjugate index of p, and the supremum is taken over all cubes $Q\subset\mathbb{R}^n$.

For $p\in(0,1]$, a matrix weight $W\in A_p(\mathbb{R}^n)$ if and only if

$$\sup_Q\operatorname*{ess\,sup}_{y\in Q}\frac{1}{|Q|}\int_Q\|W^{1/p}(x)W^{-1/p}(y)\|^p\mathrm{d}x<\infty.$$

We write $A_p:=A_p(\mathbb{R}^n)$ for brevity.

Let $\ell(Q)$ denote the side length of any cube $Q\subset\mathbb{R}^n$. For $j\in\mathbb{Z}$ and $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$, let $Q_{j,k}=\prod_{i=1}^n[2^{-j}k_i,2^{-j}(k_i+1)]$ be the dyadic cube of side length $\ell(Q_{j,k})=2^{-j}$ with the lower left corner $x_Q=2^{-j}k$. Let $\mathcal{D}=\{Q_{j,k}\}_{j\in\mathbb{Z},k\in\mathbb{Z}^n}$ denote the family of all dyadic cubes in $\mathbb{R}^n$, and let $\mathcal{D}_j=\{Q\in\mathcal{D}:\ell(Q)=2^{-j}\}$.

Given any matrix weight W and $p\in(0,\infty)$, there exists (see, e.g., [13, Proposition 1.2]) for $p>1$ and [10, p.1237] for $0<p\le 1$) a sequence $\{A_Q\}_{Q\in\mathcal{D}}$ of positive definite $d\times d$ matrices such that

$$c_1|A_Q\vec{y}|\le\left(\frac{1}{|Q|}\int_Q|W^{1/p}(x)\vec{y}|^p\mathrm{d}x\right)^{\frac{1}{p}}\le c_2|A_Q\vec{y}|,$$

with positive constants c_1, c_2 independent of $\vec{y}\in\mathbb{C}^d$ and $Q\in\mathcal{D}$. In this case, we call $\{A_Q\}_{Q\in\mathcal{D}}$ a sequence of reducing operators of order p for W.

Definition 2.6. *A matrix weight W is called a doubling matrix weight of order $p>0$ if the scalar measures $w_{\vec{y}}(x)=|W^{1/p}(x)\vec{y}|^p$, for $\vec{y}\in\mathbb{C}^d$, are uniformly doubling: there exists $c>0$ such that for all cubes $Q\subset\mathbb{R}^n$ and all $\vec{y}\in\mathbb{C}^d$,*

$$\int_{2Q} w_{\vec{y}}(x)dx\le c\int_Q w_{\vec{y}}(x)\mathrm{d}x.$$

If $c=2^\beta$ is the smallest constant for which this inequality holds, we say that β is the doubling exponent of W. From [15, Proposition 2.10], we know that β is always not less than n.

2.2 Function spaces

For $s\in\mathbb{R}$, $p\in(0,\infty)$, $q\in(0,\infty]$, and W a matrix weight, let $F_p^{s,q}(W)$ be the set of all $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$ such that

$$\|\vec{f}\|_{F_p^{s,q}(W)}:=\left\|\left(\sum_{j=0}^{\infty}2^{jsq}|W^{1/p}\varphi_j(D)\vec{f}|^q\right)^{\frac{1}{q}}\right\|_{L^p}<\infty,$$

and let $B_p^{s,q}(W)$ be the set of all $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$ such that

$$\|\vec{f}\|_{B_p^{s,q}(W)}:=\left(\sum_{j=0}^{\infty}2^{jsq}\|W^{1/p}\varphi_j(D)\vec{f}\|_{L^p}^q\right)^{\frac{1}{q}}<\infty.$$

In the sequel, let $\ell(Q)$ denote side length of a cube Q. We will use the symbol $A\lesssim B$ to denote that there exists a positive constant c such that $A\le cB$. If $A\lesssim B$ and $B\lesssim A$, then we denote $A\approx B$. The letter c will denote various positive constants and may change in different lines.

3 Key Lemmas and basic tools

In this section, for $p\in(0,\infty)$, we always suppose that $W\in A_p$ with the doubling exponent β and that $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W.

Definition 3.1. *Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of positive definite matrices and let $\beta,p\in(0,\infty)$. We say that $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order (β,p) if there exists $c>0$ such that*

$$\|A_QA_P^{-1}\|\le c\max\left\{\Big(\frac{\ell(P)}{\ell(Q)}\Big)^n,\Big(\frac{\ell(Q)}{\ell(P)}\Big)^{\beta-n}\right\}^{\frac{1}{p}}\Big(1+\frac{|x_Q-x_P|}{\max\{\ell(P),\ell(Q)\}}\Big)^{\frac{\beta}{p}}$$

for all dyadic cubes $P,Q\in\mathcal{D}$. We say that $\{A_Q\}_{Q\in\mathcal{D}}$ is weakly doubling of order $r>0$ if there exists $c>0$ such that

$$\|A_{Q_{jk}}A_{Q_{jl}}^{-1}\|\le c(1+|k-l|)^r$$

for all $k,l\in\mathbb{Z}^n$ and all $j\in\mathbb{Z}$.

Note that a strongly doubling sequence of order (β,p) is weakly doubling of order $r=\beta/p$.

Lemma 3.1 (Lemma 2.2 of [9])**.** *For $p\in(0,\infty)$, let W be a doubling matrix weight of order $p>0$ with doubling exponent β and suppose that $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W. Then $\{A_Q\}_{Q\in\mathcal{D}}$ is strongly doubling of order (β,p).*

Lemma 3.2 (Eq. (2.8) of [9])**.** *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be as in Definition 2.1. Suppose that $\{A_Q\}_{Q\in\mathcal{D}}$ is a weakly doubling sequence of order $r>0$ of positive definite matrices. Then, for any $A\in(0,1]$ and $R\in(0,\infty)$, there exists a positive constant c, depending on $\{A_Q\}_{Q\in\mathcal{D}}$, A and R, such that, for any $j\in\mathbb{Z}$, $k\in\mathbb{Z}^n$, and $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$,*

$$\sup_{x\in Q_{jk}}|A_{Q_{jk}}\varphi_j(D)\vec{f}(x)|^A\le c\sum_{l\in\mathbb{Z}^n}(1+|k-l|)^{-A(R-r)}2^{jn}\int_{Q_{jl}}|A_{Q_{jl}}\varphi_j(D)\vec{f}(z)|^A\mathrm{d}z.$$

Lemma 3.3 (Lemma 3.7 of [37])**.** *Let $\eta>n$. Then there exists a positive constant c such that for any $j\in\mathbb{Z}$ and any complex-valued measurable function g on $\mathbb{R}^n$*

$$\sum_{k\in\mathbb{Z}^n}\sum_{l\in\mathbb{Z}^n}(1+|k-l|)^{-\eta}2^{jn}\int_{Q_{jl}}|g(s)|\mathrm{d}s\chi_{Q_{jk}}\le c\mathcal{M}(g),$$

where and what follows $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

The following lemma is the Fefferman-Stein vector-valued maximal inequality; see [8].

Lemma 3.4. *Let $p\in(1,\infty)$ and $q\in(1,\infty]$. Then there exists a positive constant c such that for any sequence $\{f_j\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^n$*

$$\left\|\left(\sum_{j\in\mathbb{Z}}(\mathcal{M}(f_j))^q\right)^{\frac{1}{q}}\right\|_{L^p}\le c\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p}.$$

Lemma 3.5 (Theorem 6.1 of [9])**.** *Suppose that $s\in\mathbb{R}$, $p\in(0,\infty)$, $q\in(0,\infty]$, $W\in A_p$, and $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W. For $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$, then*

$$\|\vec{f}\|_{F_p^{s,q}(W)}\approx\|\vec{f}\|_{F_p^{s,q}(A_Q)},$$

where

$$\|\vec{f}\|_{F_p^{s,q}(A_Q)}:=\left\|\left(\sum_{j=0}^{\infty}\sum_{Q\in\mathcal{D}_j}2^{jsq}|A_Q\varphi_j(D)\vec{f}|^q\chi_Q\right)^{\frac{1}{q}}\right\|_{L^p}.$$

We define a class of η-functions on $\mathbb{R}^n$ by

$$\eta_{v,m}(x) := \frac{2^{nv}}{(1+2^v|x|)^m}$$

with $v\in\mathbb{N}_0$ and $m>0$. Note that $\eta_{v,m}\in L^1$ when $m>n$ and $\|\eta_{v,m}\|_{L^1}=c_m$ is independent of v. Therefore $\eta_{v,m}*f\lesssim\mathcal{M}(f)$ for $v\in\mathbb{N}_0$ when $m>n$ by Theorem 2.1.10 in [12].

Lemma 3.6 (Lemma A.1 of [5])**.** *Let $v_1\geq v_0$, $m>n$, and $y\in\mathbb{R}^n$. Then*

$$\eta_{v_0,m}(y)\leq 2^m\eta_{v_1,m}(y) \qquad \text{if } |y|\leq 2^{-v_1};$$
$$\eta_{v_1,m}(y)\leq 2^m\eta_{v_0,m}(y) \qquad \text{if } |y|\geq 2^{-v_0}.$$

Lemma 3.7 (Lemma 3.1 of [7])**.** *Let $\alpha>0$ and $0<q\leq\infty$. Let $\{x_k\}_{k\in\mathbb{N}_0}$ be a sequence of positive numbers such that $\{x_k\}_k\in\ell^q$. Let*

$$\delta_k := \sum_{j=0}^{\infty}2^{-\alpha|k-j|}x_j, \qquad k\in\mathbb{N}_0.$$

Then there exists a constant c depending on α and q such that

$$\|\{\delta_k\}_k\|_{\ell^q}\leq c\|\{x_k\}_k\|_{\ell^q}.$$

Lemma 3.8 (Lemma 14 of [2])**.** *Let $A\in(0,1]$, $v\in\mathbb{N}_0$, $0<p<\infty$, $W\in A_p$, and $\{A_Q\}_{Q\in\mathcal{D}}$ is a sequence of reducing operators of order p for W. Let β be the doubling exponent of W. For any $R>0$, there exists a constant $c>0$ such that for all $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$ with* $\operatorname{supp}\mathcal{F}\vec{f}\subset\{\xi:|\xi|\leq 2^{v+1}\}$, *we have*

$$\Big(\sum_{Q\in\mathcal{D}_v}|A_Q\vec{f}(x)|\chi_Q(x)\Big)^A\leq c\eta_{v,A(R-\beta/p)}*\Big(\sum_{Q\in\mathcal{D}_v}|A_Q\vec{f}|^A\chi_Q\Big)(x), \qquad \forall x\in\mathbb{R}^n.$$

Furthermore, if $R>\beta/p+n/A$, then

$$\sum_{Q\in\mathcal{D}_v}|A_Q\vec{f}(x)|\chi_Q(x)\leq c\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_v}|A_Q\vec{f}|^A\chi_Q\Big)^{\frac{1}{A}}(x), \qquad \forall x\in\mathbb{R}^n.$$

Lemma 3.9. *Let $A,B>0$, $s\in\mathbb{R}$, $0<p<\infty$, $0<q\leq\infty$, and $W\in A_p$ with the doubling exponent β. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then there exists a constant c such that*

$$\Big\|\sum_{k=0}^{\infty}\vec{f}_k\Big\|_{F_p^{s,q}(A_Q)}\leq c\|\{2^{ks}\vec{f}_k\}_k\|_{L^p(A_Q)(\ell^q)}, \tag{3.1}$$

$$\Big\|\sum_{k=0}^{\infty}\vec{f}_k\Big\|_{B_p^{s,q}(A_Q)}\leq c\|\{2^{ks}\vec{f}_k\}_k\|_{\ell^q(L^p(A_Q))}, \tag{3.2}$$

for any sequence of functions $\{\vec{f}_k\}_{k\in\mathbb{N}_0}$ *such that*

$$\begin{aligned}&\operatorname{supp}\mathcal{F}\vec{f}_0\subset\{\xi:|\xi|\le A\},\\&\operatorname{supp}\mathcal{F}\vec{f}_k\subset\{\xi:B2^{k+1}\le|\xi|\le A2^{k+1}\},\qquad \text{for}\, k\ge 1.\end{aligned}$$

Proof. We only prove (3.1) because the proof of (3.2) is similar. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity as in Definition 2.1. In the view of the support properties of $\mathcal{F}\vec{f}_k$ and $\mathcal{F}\varphi_j$, then

$$\varphi_j*\sum_{k=0}^{\infty}\vec{f}_k=\sum_{l=-\kappa_1}^{\kappa_2}\varphi_j*\vec{f}_{j+l}$$

for some $\kappa_1,\kappa_2\in\mathbb{N}_0$. We suppose simply $s=0$. By Lemma 3.8, for $l=-\kappa_1,\dots,\kappa_2$,

$$\sum_{Q\in D_j}|A_Q\varphi_j*\vec{f}_{j+l}|^A\chi_Q\le c\eta_{j,A(R-\beta/p)}*\Bigg(\sum_{Q\in\mathcal{D}_j}|A_Q\vec{f}_{j+l}|^A\chi_Q\Bigg)$$

for any $R>0$ and any $A\in(0,1]$. Therefore, with $A\in(0,\min\{1,p,q\})$, $R>n/A+\beta/p$, and by Lemmas 3.4 and 3.8, we obtain

$$\begin{aligned}&\Big\|\Big(\sum_{j=0}^{\infty}\sum_{Q\in D_j}|A_Q\varphi_j*\sum_{k=0}^{\infty}\vec{f}_k|^q\chi_Q\Big)^{\frac{1}{q}}\Big\|_{L^p}\\
\lesssim&\sum_{l=-\kappa_1}^{\kappa_2}\Big\|\Big(\sum_{j=0}^{\infty}\sum_{Q\in D_j}|A_Q\varphi_j*\vec{f}_{j+l}|^q\chi_Q\Big)^{\frac{1}{q}}\Big\|_{L^p}\\
\lesssim&\sum_{l=-\kappa_1}^{\kappa_2}\Big\|\Big(\sum_{j=0}^{\infty}\Big(\mathcal{M}\Big(\sum_{Q\in\mathcal{D}_j}|A_Q\vec{f}_{j+l}|^A\chi_Q\Big)\Big)^{\frac{q}{A}}\Big)^{\frac{1}{q}}\Big\|_{L^p}\\
\lesssim&\sum_{l=\kappa_1}^{\kappa_2}\Big\|\Big(\sum_{j=0}^{\infty}\sum_{Q\in D_j}|A_Q\vec{f}_{j+l}|^q\chi_Q\Big)^{\frac{1}{q}}\Big\|_{L^p}\\
\lesssim&\|\{2^{ks}\vec{f}_k\}_k\|_{L^p(\{A_Q\})(\ell^q)}.\end{aligned}$$

□

Lemma 3.10. *Let* $p\in(0,\infty)$, $W\in A_p$ *with the doubling exponent* β, *and* $A\in(0,1]$. *Let* $a:\mathbb{R}^{2n}\to\mathbb{C}$ *be a bounded and measurable symbol such that* $\operatorname{supp} a(x,\cdot)\subset\{\xi:|\xi|\le c2^k\}$ *for some* $c>0$. *Suppose that* $\operatorname{supp}\mathcal{F}\vec{f}\subset\{\xi:|\xi|\le c2^k\}$. *Let* $\{A_Q\}_{Q\in\mathcal{D}}$ *be a sequence of reducing operators of order* p *for* W. *Then for* $R>0$ *we have*

$$\Big|\sum_{l\in\mathbb{Z}^n}A_{Q_{k,l}}a(x,D)\vec{f}(x)\chi_{Q_{k,l}}(x)\Big|\lesssim\|a(x,2^k\cdot)\|_{\dot{B}^R_{1,A}}\Bigg(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}\chi_Q\Big|^A(x)\Bigg)^{\frac{1}{A}}$$

for each $x\in\mathbb{R}^n$, *where the implicit constant is independent of* f *and* x.

Proof. Let $\mathcal{F}\psi\in\mathscr{S}(\mathbb{R})$ be supported in $\{\xi:1/2\le|\xi|\le 2\}$ and such that $\sum_{v\in\mathbb{Z}}\mathcal{F}\psi(2^{-v}\xi)=1$ for all $\xi\neq 0$. For $v\in\mathbb{Z}$, let $\psi_v(x)=2^{vn}\psi(2^v x)$. Then $\mathcal{F}\psi_v(\xi)=\mathcal{F}\psi(2^{-v}\xi)$.

Let

$$K(x,x-y):=\int_{\mathbb{R}^n} a(x,\xi)e^{2\pi i(x-y)\cdot\xi}\mathrm{d}\xi=\mathcal{F}_\xi(a(x,\cdot))(x-y)$$

be the kernel of $a(\cdot,D)$. Observe that

$$K(x,x-y)\mathcal{F}\psi_v(x-y)=\mathcal{F}_\xi\mathcal{F}_\sim^{-1}(\mathcal{F}\varphi_v\mathcal{F}_\xi a(x,\cdot))(x-y).$$

Therefore,

$$\begin{aligned}|K(x,x-y)\mathcal{F}\psi_v(x-y)|&\le\|\mathcal{F}_\sim^{-1}(\mathcal{F}\varphi_v\mathcal{F}_\xi a(x,\cdot))\|_{L^1}\\&=2^{kn}\|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_\xi a(x,2^k\cdot))\|_{L^1}.\end{aligned}$$

Applying the Plancherel-Pôlya-Nikol'skii inequality (see p.18 in [34]), we obtain

$$|A_{Q_{k,l}}a(x,D)\vec{f}(x)|\lesssim 2^{kn(1/A-1)}\Big(\int_{\mathbb{R}^n}|A_{Q_{k,l}}K(x,x-y)\vec{f}(y)|^A\mathrm{d}y\Big)^{1/A}$$

for $x\in Q_{k,l}$. Raising the power A, we have

$$|A_{Q_{k,l}}a(x,D)\vec{f}(x)|^A\lesssim 2^{kn(1-A)}(S_1+S_2),$$

where

$$S_1:=\sum_{v=-\infty}^{-k-1}\sup_y|K(x,x-y)\mathcal{F}\varphi_v(x-y)|^A\int_{2^{v-1}\le|x-y|\le 2^{v+1}}|A_{Q_{k,l}}\vec{f}(y)|^A\mathrm{d}y,$$

$$S_2:=\sum_{v=-k}^{\infty}\sup_y|K(x,x-y)\mathcal{F}\varphi_v(x-y)|^A\int_{2^{v-1}\le|x-y|\le 2^{v+1}}|A_{Q_{k,l}}\vec{f}(y)|^A\mathrm{d}y.$$

Then $2^{kn(1-A)}S_2$ is dominated by

$$\begin{aligned}&2^{kn}\sum_{v=-k}^{\infty}\|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_\xi a(x,2^k\cdot))\|_{L^1}^A\\&\quad\times\Big\{\sum_{\{w\in\mathbb{Z}^n:|w-l|\le 1\}}(1+|w-l|)^{A\beta/p}\int_{Q_{k,w}}|A_{Q_{k,w}}\vec{f}(y)|^A\mathrm{d}y\\&\quad+\sum_{m=1}^{k+v+1}\sum_{\{w\in\mathbb{Z}^n:2^{m-1}\le|w-l|\le 2^m\}}(1+|w-l|)^{A\beta/p}\int_{Q_{k,w}}|A_{Q_{k,w}}\vec{f}(y)|^A\mathrm{d}y\Big\}\\&\lesssim 2^{kn}\sum_{v=-k}^{\infty}\|\mathcal{F}_\xi^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_\xi a(x,2^k\cdot))\|_{L^1}^A\\&\quad\times\Big\{\sum_{\{w\in\mathbb{Z}^n:|w-l|\le 1\}}\int_{Q_{k,w}}(1+2^k|y-x|)^{A\beta/p-R}|A_{Q_{k,w}}\vec{f}(y)|^A\mathrm{d}y\end{aligned}$$

$$+\sum_{m=1}^{k+v+1} 2^{Rm-k} \int_{\bigcup_{\{w\in\mathbb{Z}^n:2^{m-1}\le|w-l|\le 2^m\}} Q_{k,w}} (1+2^k|y-x|)^{A\beta/p-R} |A_{Q_{k,w}}\vec{f}(y)|^A \mathrm{d}y \Big\}$$

$$\lesssim \sum_{v=-k}^{\infty} 2^{Rv} \|\mathcal{F}_{\xi}^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a(x,2^k\cdot))\|_{L^1}^A \eta_{k,R-A\beta/p} * \Big|\sum_{Q\in\mathcal{D}_k} A_Q\vec{f}\chi_Q\Big|^A(x)$$

$$\le \|a(x,2^k\cdot)\|_{\dot{B}^R_{1,A}}^A \eta_{k,R-A\beta/p} * \Big|\sum_{Q\in\mathcal{D}_k} A_Q\vec{f}\chi_Q\Big|^A(x).$$

And $2^{kn(1-A)}S_1$ is bounded by

$$2^{kn}\sum_{v=-\infty}^{-k-1} \|\mathcal{F}_{\xi}^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a(x,2^k\cdot))\|_{L^1}^A \int_{2^{v-1}\le|x-y|\le 2^{v+1}} |A_{Q_{k,l}}\vec{f}(y)|^A \mathrm{d}y$$

$$\lesssim 2^{kn}\sum_{v=-\infty}^{-k-1} \|\mathcal{F}_{\xi}^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a(x,2^k\cdot))\|_{L^1}^A$$

$$\times \int_{2^{v-1}\le|x-y|\le 2^{v+1}} (1+2^k|y-x|)^{A\beta/p-R} |A_{Q_{k,l}}\vec{f}(y)|^A \mathrm{d}y$$

$$\lesssim \sum_{v=-\infty}^{-k-1} \|\mathcal{F}_{\xi}^{-1}(\mathcal{F}\varphi_{v+k}\mathcal{F}_{\xi}a(x,2^k\cdot))\|_{L^1}^A \eta_{k,R-A\beta/p} * \Big|\sum_{Q\in\mathcal{D}_k} A_Q\vec{f}\chi_Q\Big|^A(x)$$

$$\lesssim \|a(x,2^k\cdot)\|_{\dot{B}^0_{1,A}}^A \eta_{k,R-A\beta/p} * \Big|\sum_{Q\in\mathcal{D}_k} A_Q\vec{f}\chi_Q\Big|^A(x)$$

$$\lesssim \|a(x,2^k\cdot)\|_{\dot{B}^R_{1,A}}^A \eta_{k,R-A\beta/p} * \Big|\sum_{Q\in\mathcal{D}_k} A_Q\vec{f}\chi_Q\Big|^A(x).$$

Thus, the proof is finished. □

Lemma 3.11. *Let $B>0$, $s>(\beta-n)/p$, $0<p<\infty$, $0<q\le\infty$, and $W\in A_p$ with the doubling exponent β. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Then there exists a constant c such that*

$$\Big\|\sum_{k=0}^{\infty}\vec{f}_k\Big\|_{F_p^{s,q}(A_Q)} \le c\|\{2^{ks}\vec{f}_k\}_k\|_{L^p(A_Q)(\ell^q)}, \tag{3.3}$$

$$\Big\|\sum_{k=0}^{\infty}\vec{f}_k\Big\|_{B_p^{s,q}(A_Q)} \le c\|\{2^{ks}\vec{f}_k\}_k\|_{\ell^q(L^p(A_Q))} \tag{3.4}$$

for any sequence of functions $\{\vec{f}_k\}_{k\in\mathbb{N}_0}$ with $\operatorname{supp}\mathcal{F}\vec{f}_k\subset\{\xi:|\xi|\le B2^{k+1}\}$.

Proof. We only prove (3.3). The proof of (3.4) is similar. Let $\{\mathcal{F}\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity as in Definition 2.1. Using the support properties of $\{\vec{f}_k\}$, we have

$$\sum_{k=0}^{\infty}\varphi_j*\vec{f}_k = \sum_{k=j+\sigma}^{\infty}\varphi_j*\vec{f}_k = \sum_{i=\sigma}^{\infty}\varphi_j*\vec{f}_{j+i},$$

for some $\sigma\in\mathbb{R}$. Then

$$\Big\|\sum_{k=0}^{\infty}\vec{f}_k\Big\|_{F_p^{s,q}(A_Q)}^{\min(1,p,q)} \le \sum_{i=\sigma}^{\infty}\Big\|\Big\{\sum_{Q\in\mathcal{D}_j}2^{js}|A_Q\varphi_j*\vec{f}_{j+i}|\chi_Q\Big\}_{j\in\mathbb{N}_0}\Big\|_{L^p(\ell^q)}^{\min(1,p,q)}$$
$$\le\Big(\sum_{i=\sigma}^{-1}+\sum_{i=0}^{\infty}\Big)\Big\|\Big\{\sum_{Q\in\mathcal{D}_j}2^{js}|A_Q\varphi_j*\vec{f}_{j+i}|\chi_Q\Big\}_{j\in\mathbb{N}_0}\Big\|_{L^p(\ell^q)}^{\min(1,p,q)}.$$

It is understood that the first sum is zero if $\sigma\ge 0$. We just consider the second sum since the first sum has finite items. For $A\in(0,1]$, $R>A\beta/p+\beta+n$, from Lemmas 3.1, 3.6 and 3.10,

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_j}|A_Q\varphi_j*\vec{f}_{j+i}(x)|\chi_Q(x)\\
&\lesssim\|\mathcal{F}\varphi_1\|_{\dot{B}^R_{1,A}}\Big\{\eta_{j,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_j}A_Q\vec{f}_{j+i}\Big|^A(x)\Big\}^{\frac{1}{A}}\\
&\le\Big\{\sum_{Q\in\mathcal{D}_j}\int_Q 2^{jn}\frac{1}{(1+2^j|x-y|)^{R-A\beta/p}}|A_Q\vec{f}_{j+i}(y)|^A\mathrm{d}y\Big\}^{\frac{1}{A}}\\
&\lesssim\Big\{\sum_{Q\in\mathcal{D}_{j+i}}\int_Q 2^{jn}\frac{1}{(1+2^j|x-y|)^{R-A\beta/p}}2^{i(\beta-n)A/p}(1+2^j|x-y|)^{\beta}|A_Q\vec{f}_{j+i}(y)|^A\mathrm{d}y\Big\}^{\frac{1}{A}}\\
&\le\Big\{\sum_{Q\in\mathcal{D}_{j+i}}\int_Q 2^{(j+i)n}2^{R-A\beta/p-\beta}\frac{1}{(1+2^{j+i}|x-y|)^{R-A\beta/p-\beta}}2^{i(\beta-n)A/p}|A_Q\vec{f}_{j+i}(y)|^A\mathrm{d}y\Big\}^{\frac{1}{A}}\\
&\le 2^{i(\beta-n)/p}2^{(R-A\beta/p-\beta)/A}\mathcal{M}\Big(\Big|\sum_{Q\in\mathcal{D}_{j+i}}A_Q\vec{f}_{j+i}\Big|^A\Big)^{\frac{1}{A}}(x)\\
&\lesssim 2^{i(\beta-n)/p}\mathcal{M}\Big(\Big|\sum_{Q\in\mathcal{D}_{j+i}}A_Q\vec{f}_{j+i}\Big|^A\Big)^{\frac{1}{A}}(x).
\end{aligned}\tag{3.5}$$

Since $s>(\beta-n)/p$, then

$$\begin{aligned}
&\sum_{i=0}^{\infty}\Big\|\Big\{\sum_{Q\in\mathcal{D}_j}2^{js}|A_Q\varphi_j*\vec{f}_{j+i}|\chi_Q\Big\}_{j\in\mathbb{N}_0}\Big\|_{L^p(\ell^q)}^{\min(1,p,q)}\\
&\le\sum_{i=0}^{\infty}\Big\|\Big\{2^{js}2^{i(\beta-n)/p}\mathcal{M}\Big(\Big|\sum_{Q\in\mathcal{D}_{j+i}}A_Q\vec{f}_{j+i}\Big|^A\Big)^{\frac{1}{A}}\Big\}_{j\in\mathbb{N}_0}\Big\|_{L^p(\ell^q)}^{\min(1,p,q)}\\
&\le\sum_{i=0}^{\infty}2^{i((\beta-n)/p-s)\min(1,p,q)}\Big\|\Big\{2^{js}2^{is}\sum_{Q\in\mathcal{D}_{j+i}}|A_Q\vec{f}_{j+i}|\chi_Q\Big\}_{j\in\mathbb{N}_0}\Big\|_{L^p(\ell^q)}^{\min(1,p,q)}
\end{aligned}$$

$$\lesssim \|\{2^{ks}\vec{f}_k\}_k\|_{L^p(A_Q)(\ell^q)}^{\min(1,p,q)}.$$

Hence, the proof is finished. □

Remark 3.1. The most difference between Lemmas 3.9 and 3.11 is the support of the Fourier transform of f_k for $k \geq 1$. The support of the Fourier transform of f_k in Lemma 3.9 is a ring, but the support of the Fourier transform of f_k in Lemma 3.9 is a ball.

4 Boundedness of pseudo-differential operators

Theorem 4.1. *Let $0<p<\infty$, $0<q\leq\infty$, $W \in A_p$ with the doubling exponent β. Let $a \in SB^m_\delta(r,\mu,\nu;N,\lambda)$ with $\mu=\infty$, $\nu=\infty$, $r \in (s,\infty)$, $\delta \in [0,1)$, and $\lambda=1$.*

(i) *Let $N > \min(1,p,q)\beta/p+n$ and $(\beta-n)/p<s<r$. Then $a(\cdot,D)$ is a continuous linear mapping from $F_p^{s+m,q}(W)$ to $F_p^{s,q}(W)$.*

(ii) *Let $N > \min(1,p)\beta/p+n$ and $(\beta-n)/p<s<r$. Then $a(\cdot,D)$ is a continuous linear mapping from $B_p^{s+m,q}(W)$ to $B_p^{s,q}(W)$.*

Proof. We only prove (i). The proof of (ii) is similar. Let $\{\mathcal{F}\varphi_k\}_{k\in\mathbb{N}_0}$ be a resolution of unity as in Definition 2.1. We set

$$a_{j,k}(x,\xi)=\mathcal{F}^{-1}(\mathcal{F}\varphi_j(\eta)\mathcal{F}_x a(\cdot,\xi))\mathcal{F}\varphi_k(\xi).$$

We decompose the symbol into three parts:

$$a(x,\xi)=\sum_{i=1}^{3} a^{(i)}(x,\xi),$$

where

$$a^{(1)}(x,\xi):=\sum_{k=4}^{\infty}\sum_{j=0}^{k-4} a_{j,k}(x,\xi),$$
$$a^{(2)}(x,\xi):=\sum_{k=0}^{\infty}\sum_{j=k-3}^{k+3} a_{j,k}(x,\xi),$$
$$a^{(3)}(x,\xi):=\sum_{k=4}^{\infty}\sum_{j=k+4}^{\infty} a_{j,k}(x,\xi).$$

Let $\vec{f}_k=\varphi_k*\vec{f}$. Let $\{A_Q\}_{Q\in\mathcal{D}}$ be a sequence of reducing operators of order p for W. Since $\|\vec{f}\|_{F_p^{s,q}(W)} \approx \|\vec{f}\|_{F_p^{s,q}(A_Q)}$, it suffices to prove that

$$\|a(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)} \lesssim \|\vec{f}\|_{F_p^{s+m,q}(A_Q)}.$$

Firstly, we consider the symbol $a^{(1)}$. For every $\vec{f}\in\mathscr{S}'(\mathbb{R}^n)$,

$$\|a^{(1)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}\lesssim\|\vec{f}\|_{F_p^{s+m,q}(A_Q)}.$$

Indeed, $\sum_{j=0}^{k-4}a_{j,k}(\cdot,D)\vec{f}_k$ has its spectrum in $\{\xi:c_1 2^k\le|\xi|\le c_2 2^k\}$ where $c_1,c_2>0$ are independent of k. Then we can apply Lemma 3.9,

$$\begin{aligned}\|a^{(1)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}&=\Big\|\sum_{k=4}^{\infty}\sum_{j=0}^{k-4}a_{j,k}(\cdot,D)\vec{f}\Big\|_{F_p^{s,q}(A_Q)}\\&\lesssim\Big\|\Big\{2^{ks}\sum_{j=0}^{k-4}a_{j,k}(\cdot,D)\vec{f}\Big\}_k\Big\|_{L^p(A_Q)(\ell^q)}.\end{aligned}$$

Let us show the last norm is bounded by $c\|\vec{f}\|_{F_p^{s+m,q}(A_Q)}$. By Lemma 3.10, the right hand side does not exceed

$$\begin{aligned}&\Big\|\Big(\sum_{k\ge4}2^{ksq}\sum_{j=0}^{k-4}\|\mathcal{F}^{-1}(\mathcal{F}\varphi_j\mathcal{F}_x a(\cdot,\cdot))\|_{\dot{B}^R_{1,A}}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A\Big)^{\frac{q}{A}}\Big)^{\frac{1}{q}}\Big\|_{L^p}\\\le&\Big\|\Big(\sum_{k\ge4}2^{ksq}\sum_{j=0}^{k-4}\|\mathcal{F}^{-1}(\mathcal{F}\varphi_j\mathcal{F}_x a(\cdot,\cdot))\|_{\dot{B}^{R+\epsilon}_{1,\infty}}\Big(\eta_{k,R-A\beta/p}*|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q|^A\Big)^{\frac{q}{A}}\Big)^{\frac{1}{q}}\Big\|_{L^p}\\\le&\Big\|\Big(\sum_{k\ge4}2^{ksq}2^{kmq}(k-3)\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A\Big)^{\frac{q}{A}}\Big)^{\frac{1}{q}}\Big\|_{L^p}\\\lesssim&\|\vec{f}\|_{F_p^{s+m,q}(A_Q)},\end{aligned}$$

where $A<\min(1,p,q)$, $R>A\beta/p+n$, $N:=R+\epsilon>\min(1,p,q)\beta/p+n$.

Secondly, we consider the symbol $a^{(2)}$. Since $\sum_{j=k-3}^{k+3}a_{j,k}(\cdot,D)\vec{f}_k$ has its spectrum in $\{\xi:|\xi|\le c_2 2^k\}$ where $c_2>0$ is independent of k, then by Lemma 3.11 ($s>(\beta-n)/p$),

$$\|a^{(2)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}\le\Big\|\Big\{2^{ks}\sum_{j=k-3}^{k+3}a_{j,k}(\cdot,D)\vec{f}_k\Big\}_k\Big\|_{L^p(A_Q)(\ell^q)}.$$

Since $a\in SB^m_\delta(r,\mu,\nu;N,\lambda)$, we obtain

$$\begin{aligned}&\Big|\sum_{Q\in\mathcal{D}_k}A_Q\sum_{j=k-3}^{k+3}a_{j,k}(\cdot,D)\vec{f}_k\chi_Q\Big|\\\le&\sup_k\Big\|\sum_{j=k-3}^{k+3}a_{j,k}(x,2^k\cdot)\Big\|_{\dot{B}_1^{R,A}}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}\end{aligned}$$

$$\lesssim 2^{km}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}.$$

Let $R>A\beta/p+n$. Then we obtain

$$\begin{aligned}\|a^{(2)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(W)}&\lesssim\Big\|\Big\{2^{k(s+m)}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}\Big\}_k\Big\|_{L^p(l^q)}\\&\lesssim\Big\|2^{k(s+m)}\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big\|_{L^p(\ell^q)}\\&\lesssim\|\vec{f}\|_{F_p^{s+m,q}(A_Q)}.\end{aligned}$$

Finally, we consider the symbol $a^{(3)}$. Applying Lemma 3.9, we obtain

$$\|a^{(3)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}\lesssim\Big\|\Big\{2^{js}\sum_{k=0}^{j-4}a_{j,k}(\cdot,D)\vec{f}_k\Big\}_j\Big\|_{L^p(A_Q)(\ell^q)}.$$

Let $R>A\beta/p+n$. Then by Lemma 3.10, we have

$$\begin{aligned}&\Big|\sum_{Q\in\mathcal{D}_k}A_Qa_{j,k}(\cdot,D)\vec{f}_k(x)\chi_Q\Big|\\&\lesssim\|a_{j,k}(x,2^k\cdot)\|_{\dot{B}^R_{1,A}}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}\\&\le 2^{-rj}\sup_i\|2^{ri}a_{i,k}(x,2^k\cdot)\|_{\dot{B}^{R+\epsilon}_{1,\infty}}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}\\&=2^{-rj}2^{-ks}\sup_i\|2^{ri}a_{i,k}(x,2^k\cdot)\|_{\dot{B}^{R+\epsilon}_{1,\infty}}2^{ks}\Big(\eta_{k,R-A\beta/p}*\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A(x)\Big)^{\frac{1}{A}}\\&\le 2^{-rj}2^{-ks}\sup_i\|2^{ri}a_{i,k}(x,2^k\cdot)\|_{\dot{B}^{R+\epsilon}_{1,\infty}}2^{ks}\mathcal{M}\Big(\Big|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\Big|^A\Big)^{\frac{1}{A}}(x).\end{aligned}$$

Let $g_k(x):=2^{ks}\mathcal{M}(|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q|^A)^{1/A}(x)$. Let $r>s>(\beta-n)/p$. Applying Lemmas 3.4 and 3.7, set $N=R+\epsilon$, and we conclude that

$$\begin{aligned}\|a^{(3)}(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}&\lesssim\Big\|\Big\{\sum_{j\ge0}^{\infty}\Big(2^{js}\sum_{k=0}^{j-4}2^{-rj}2^{-ks}\sup_i\|2^{ri}a_{i,k}(\cdot,2^k\cdot)\|_{\dot{B}^{R+\epsilon}_{1,\infty}}g_k\Big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^p}\\&=\Big\|\Big\{\sum_{j\ge0}^{\infty}\Big(\sum_{k=0}^{j-4}2^{js}2^{-rj}2^{kr}2^{-kr}2^{-ks}\sup_i\|2^{ri}a_{i,k}(\cdot,2^k\cdot)\|_{\dot{B}^{N}_{1,\infty}}g_k\Big)^q\Big\}^{\frac{1}{q}}\Big\|_{L^p}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\{\sup_i \|2^{ri}a_{i,k}(x,2^k\cdot)\|_{\dot{B}_1^{N,\infty}}2^{-kr}g_k\}_k\|_{L^p(\ell^q)}\\
&\lesssim \|\{2^{-k(m+\delta r)}\sup_i \|2^{ri}a_{i,k}(\cdot,2^k\cdot)\|_{\dot{B}_1^{N,\infty}}2^{-kr}2^{k(m+\delta r)}g_k\}_k\|_{L^p(\ell^q)}\\
&\lesssim \left\|\left\{2^{-kr}2^{k(m+\delta r)}2^{ks}\mathcal{M}\left(\left|\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\right|^A\right)^{\frac{1}{A}}\right\}_k\right\|_{L^p(\ell^q)}\\
&\lesssim \left\|\left\{2^{k(m+s)}\sum_{Q\in\mathcal{D}_k}A_Q\vec{f}_k\chi_Q\right\}_k\right\|_{L^p(\ell^q)}\\
&\approx \|\vec{f}\|_{F_p^{s+m,q}(A_Q)}.
\end{aligned}
$$

Thus, we have

$$
\|a(\cdot,D)\vec{f}\|_{F_p^{s,q}(A_Q)}\lesssim \|\vec{f}\|_{F_p^{s+m,q}(A_Q)}.
$$

Hence, we finish the proof. □

Remark 4.1. Marschall obtained the boundedness of non-regular pseudo-differential with double symbols on weighted L^p spaces in [19]. The symbols in [19] are different with Theorem 4.1.

Let $0<p<\infty$, $0<q\le\infty$, $a\in SB_\delta^m(r,\mu,\infty,N,\lambda)$ with $m\in\mathbb{R}$, $0<\mu<\infty$, $(1-\delta)r\ge n/\mu$, $1\le\lambda\le\infty$, $N>n/\lambda$. Suppose that $N>n\max\{1/2,1/\lambda,1/p,1/q\}$, and $n(\max\{1,1/\mu+1/p\}-1)-(1-\delta)r<s<r-n\max\{1/mu-1/p,0\}$. Then Marschall proved that $a(\cdot,D)$ is a bounded operator from $F_{p,q}^{s+m}$ to $F_{p,q}^m$ in [20].

However for $\mu=\infty$ in Theorem 4.1, we can not get the Theorem 7 in [20] if let $d=1,W=1$.

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