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Trajectory Segmentation and Symbolic Representation of Dynamics of Delayed Recurrent Inhibitory Neural Loops*

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Abstract We develop a general symbolic dynamics framework to examine the dynamics of an analogue of the integrate-and-fire neuron model of recurrent inhibitory loops with delayed feedback, which incorporates the firing procedure and absolute refractoriness. We first show that the interaction of the delay, the inhibitory feedback and the absolute refractoriness can generate three basic types of oscillations, and these oscillations can be pinned together to form interesting coexisting periodic patterns in the case of short feedback duration. We then develop a natural symbolic dynamics formulation for the segmentation of a typical trajectory in terms of the basic oscillatory patterns, and use this to derive general principles that determine whether a periodic pattern can and should occur.

Keywords Multistability, periodic pattern, neural network, time delay, pattern formation, recurrent inhibitory loops.

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1. Introduction

We propose to develop a theoretical framework that allows us to reformulate the delayed feedback as an induced action on a segment of symbols, so we can develop a systematic approach to look at the co-existence of multiple stable periodic oscillations in network of neurons with delayed feedback. For this purpose, we start with a recurrent inhibitory loop that consists of an excitatory neuron E and an inhibitory neuron I, where neuron E gives off collateral branches and excites the inhibitory neuron I, which in turn inhibits the firing of neuron E, in a delay time. The incorporation of time delays is necessary, as these are intrinsic properties of both biological and artificial loops due to axonal conduction times, distances of interneurons, the finite switching speeds of amplifiers and the passive propagation of potentials down the dendrites of neurons [6, 7, 21, 22, 24, 28, 29].

The simple recurrent inhibitory loop and its represented coupled network of neuron populations have been used to study how the interaction of the excitatory

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and the inhibitory neurons, the connection strength and the time delay affects the network's computational performance [1, 2, 15, 26, 27]. In particular, the studies [9, 10, 18] focus on the dynamical behaviors of the excitatory neuron in the loops and hence their model equation takes the form of a scalar delay differential equation that can also arise in modeling a single neuron with delayed self-feedback. The model in [9, 10, 18] takes a quite general form involving membrane potential of the excitatory neuron, ionic currents, an applied current, the effects of the inhibitory feedback on the membrane potential of the excitatory neuron, and the probability that a certain channel is open. In these models, the delay can be large in comparison with the intrinsic spiking period when we consider the recurrent inhibitory loop as a simplification of a large polysynaptic loop or neuron population network. In such a network, many factors can contribute to the delay and consequently the propagation time through the network may be considerably longer than would be estimated from the conduction velocities.

Foss and Milton [9] used the well-known Hodgkin-Huxley model to study recurrent inhibitory loops and found multiple coexisting attracting periodic solutions by computer simulations. Unfortunately, the intrinsic complexity of the conductance-based neuron models such as the Hodgkin-Huxley model makes it difficult for a detailed qualitative and theoretical analysis and hence reduced neuron models such as integrate-and-fire models become desirable from a theoretical point of view. On the other hand, Chow *et. al.* [3] showed that under some assumptions, the full conductance-based dynamics can be approximated by the integrate-and-fire neuron model $Cv'(\tilde{t}) = -g_L(v - v_L) + \tilde{I}_0 - F(\tilde{t}, \tilde{\tau})$, with the reset condition: $v(\tilde{t}+) = v_r$ when $v(\tilde{t}-) = \tilde{\vartheta}$. Such a system can be further normalized and simplified as $V'(t) = -V + I_0 - F(t, \tau)$ by re-scaling $t = \tilde{t}/\tau_m$, $\tau = \tilde{\tau}/\tau_m$ and letting $V = (v - v_r)/(\tilde{\vartheta} - v_r)$ with $\tau_m = C/g_L$. Under this normalization procedure, the $V(t)$ is reset to $V_r = 0$ whenever it reaches the threshold potential $\vartheta = 1$.

The reset condition has so far been considered as an impulse: the potential is reset immediately after it reaches the threshold. In real biological neurons, however, the reset process involves a firing procedure followed by the absolute refractory period. It turns out that the firing procedure and the absolute refractoriness have very important impact on the timing of the inhibitory feedback, and this impact is particularly significant if the feedback is delayed. Indeed, numerical results [18] showed that an integrate-and-fire model incorporating the firing procedure and absolute refractoriness is capable of generating a large number of asymptotically stable periodic solutions with predictable patterns of oscillations, in agreement with some earlier studies in [9, 13, 23, 25].

In this series of papers, we hope to develop a systematical approach to rigorously analyze the mechanism for the observed multistability in recurrent inhibitory loops. In particular, we shall show how the interaction of the time lag, the inhibition, the firing procedure and the absolute refractory period can generate some basic and analytically trackable types of oscillations, and how these basic oscillations can then be pinned together to form a large class of periodic patterns. We shall also illustrate by numerical simulations that these periodic patterns can be easily observed. In subsequent work, we will link the periodic patterns exhibited in our simple integrate-and-fire model to a variety of rhythms displayed in the nervous system.

The rest of this paper is organized as follows: we first formulate the integrate-and-fire model of recurrent inhibitory loops by incorporating the firing procedure