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Global Phase Portraits of Symmetrical Cubic Hamiltonian Systems with a Nilpotent Singular Point*

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Abstract Han et al. [Han et al., *Polynomial Hamiltonian systems with a nilpotent critical point*, J. Adv. Space Res. 2010, 46, 521–525] successfully studied local behavior of an isolated nilpotent critical point for polynomial Hamiltonian systems. In this paper, we extend the previous result by analyzing the global phase portraits of polynomial Hamiltonian systems. We provide 12 non-topological equivalent classes of global phase portraits in the Poincaré disk of cubic polynomial Hamiltonian systems with a nilpotent center or saddle at origin under some conditions of symmetry.

Keywords Hamiltonian systems, nilpotent singular point, global phase portraits, Poincaré transformation.

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1. Introduction

Hamiltonian systems are relevant to a variety of space and astrophysical studies such as celestial mechanics, cosmology and nonlinear plasma waves. In this paper, we study polynomial Hamiltonian systems with a nilpotent critical point. Let $H(x, y)$ be a real polynomial in (x, y). Then, as we know, a system of the form

$$\dot{x} = H_y, \quad \dot{y} = -H_x. \tag{1.1}$$

is called a polynomial Hamiltonian system. There have been many studies on the number of limit cycles of various perturbed systems to the form of system (1.1) by using the method of Melnikov functions. For the unperturbed system (1.1), one usually supposes that it has a period annulus consisting of a family of periodic orbits with its boundary containing an elementary center point or a hyperbolic saddle point. Han et al. [9] give a complete classification to nilpotent critical points for the polynomial Hamiltonian system (1.1) with exact three different types of

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a center, a cusp or a saddle. Then for quadratic and cubic Hamiltonian systems they obtain necessary and sufficient conditions for a nilpotent critical point to be a center, a cusp or a saddle. They also give local phase portraits of this kind of these systems under some conditions of symmetry.

Recently Colak, Llibre and Valls [1–4] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams. The complete classification of the phase portrait of the nilpotent centers in the last case was given in [5]. The quasi-homogeneous but non-homogeneous polynomial differential systems have also been investigated from different aspects, for example, the structural stability, the integrability, the polynomial and rational integrability, the centers and limit cycles, the normal forms. Recently, García et al. [8] provide an algorithm for obtaining all the quasi-homogeneous but not homogeneous polynomial differential systems with a given degree. Using this algorithm they obtain all the quadratic and cubic quasi-homogeneous but not homogeneous vector fields. Liang et al. [10] give a complete classification of the global phase portraits of planar quasi-homogeneous but not homogeneous polynomial differential systems of degree 4. More recently, Dias, Llibre, Valls [6] classify the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three symmetric with respect to the x-axis having a nilpotent center at the origin.

In this paper, we extend the previous result in [9] by analyzing the global phase portraits of polynomial Hamiltonian systems. We provide 12 non-topological equivalent classes of global phase portraits in the Poincaré disk of cubic polynomial Hamiltonian systems with a nilpotent center or saddle at origin under some conditions of symmetry.

2. Main results

For the cubic polynomial Hamiltonian system

$$H(x,y) = \frac{y^2}{2} + \sum_{3 \le i+j \le 4} h_{ij} x^i y^j, \tag{2.1}$$

where i, j are natural number, Han et al. give a complete classification for the nilpotent singular point (see Theorem 2 in [9]). In the following suppose (2.1) holds with

$$H(\pm x, \pm y) = H(x,y).$$

By (1.1) and (2.1) we can obtain

$$H(x,y) = \frac{y^2}{2} + cx^4 + ax^2y^2 + by^4,$$

with $a = h_{22}$, b=h_{04}, c=h_{40}. We can get the system

$$\begin{cases} \dot{x} = y(1 + 2ax^2 + 4by^2), \\ \dot{y} = -2x(ay^2 + 2cx^2). \end{cases} \tag{2.2}$$

Han et al. [9] show that the origin is a nilpotent saddle if $c < 0$, and when $c > 0$,