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Existence of Solutions for Fractional Integro-differential Equations with Impulsive and Integral Conditions*

Yanpeng Hao[1] and Qiaoluan Li[1,†]

Abstract This paper presents the existence of solutions for a class of Cauchy problems with integral condition for impulsive fractional integro-differential equations. Based on definition of solution for impulsive fractional integro-differential equations, the existence theorems of solutions of fractional differential equation are obtained by applying fixed point methods. Finally, three examples are given to demonstrate the feasibility of the obtained results.

Keywords Existence of solutions, impulsive fractional differential equations, fixed point theorems.

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1. Introduction

During the past decades, impulsive differential equations have been attracting increasing attention due to their applications in various sciences such as Physics, Chemistry, Mechanics, Engineering, Biomedical sciences, etc. Moreover, fractional differential equations have been proved to be valuable tools to model of phenomena in both physical and social sciences.

Fractional impulsive differential equations have been extensively studied by many researchers, in which, fractional calculus, an important branch of mathematics, has been attached great importance to. For details, see [1–10] and references therein. For example, Anguraj et al. [2] considered the following initial value problems for impulsive fractional differential equation given by

$$\begin{cases} {}^cD^{\alpha}(y)(t) = f(t, y(t), \int_0^t k(t,s,y(s))ds), \ t \in J^{'} := J \backslash \{t_1, ..., t_m\}, \\ y(t_k^+) = y(t_k^-) + y_k, \qquad y_k \in R, \\ y(0) = \int_0^1 g(s)y(s)ds, \end{cases} \tag{1.1}$$

where $0 < \alpha \le 1$ and $J = [0, 1]$. They proved the existence results for the above equation by means of the contraction mapping principle and the Krasnoselskii fixed

[†]the corresponding author.
Email address: 2804123831@qq.com(Y. Hao), qll71125@163.com(Q. Li)
[1]College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang, Hebei 050024, China
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point theorem.

Liu et al. [7] studied the following problem

$$\begin{cases} {}^cD_t^\alpha u(t) = f(t, u(t), u'(t), u''(t)), \qquad t \in J' := J\backslash\{t_1, ..., t_m\}, \\ \Delta u(t_k) = A_k(u(t_k^-)), \quad \Delta u'(t_k) = B_k(u(t_k^-)), \quad \Delta u''(t_k) = C_k(u(t_k^-)), \\ u(0) = \lambda_1 u(T) + \xi_1 \int_0^T q_1(s, u(s), u'(s), u''(s))ds, \\ u'(0) = \lambda_2 u(T) + \xi_2 \int_0^T q_2(s, u(s), u'(s), u''(s))ds, \\ u''(0) = \lambda_3 u(T) + \xi_3 \int_0^T q_3(s, u(s), u'(s), u''(s))ds, \quad \lambda_i \neq 1 (i = 1, 2, 3), \end{cases} \tag{1.2}$$

where $2 < \alpha \leq 3$. By the use of the well-known fixed point theorems, they obtained the uniqueness and existence of the solutions for the above equation.

Motivated by the above mentioned works, we investigated sufficient conditions for the existence of solutions to the following impulsive fractional differential equations with integral initial condition :

$$\begin{cases} {}^cD^\alpha(x)(t) = f(t, x(t), \int_0^t k(t, s, x(s))ds), \qquad t \in J' := J\backslash\{t_1, ..., t_m\}, \\ \Delta x(t_k) = A_k(x(t_k^-)), \quad \Delta x'(t_k) = B_k(x(t_k^-)) \\ x(0) = \int_0^T g(s)x(s)ds, \quad x'(0) = \int_0^T h(s)x'(s)ds, \end{cases} \tag{1.3}$$

where $k = 1, \cdots, m$, $J = [0, T]$, $1 < \alpha \leq 2$, ${}^cD^\alpha$ is the Caputo fractional derivative, X denote a Banach space, $f : J \times X \times X \to X$ is a given function, the functions $A_k, B_k : X \to X$ are continuous, $\Omega = \{(t, s) : 0 \leq s \leq t \leq T\}$, $k : \Omega \times X \to X$, $g, h \in C[0, T]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $\Delta x'(t_k) = x'(t_k^+) - x'(t_k^-)$. For brevity, let us take $Ax(t) = \int_0^t k(t, s, x(s))ds$.

This thesis is composed of four sections. In section 2, we will introduce some definitions, lemmas and preliminary results. In section 3, we will apply some standard fixed point principles to yield existence result of problem (1.3). In section 4, three examples are given to illustrate our main results.

2. Preliminaries

In this section, we introduce definitions and preliminary results which are needed in this paper. Let X be a Banach space. Let $C(J, X)$ be the Banach space of continuous functions $x(t)$ with $x(t) \in X$ for $t \in J = [0, T]$ and $\|x\|_{C(J,X)} = \max_{t\in J}|x(t)|$. Also consider the Banach space $PC(J, X) = \{x : J \to X : x \in C((t_k, t_{k+1}], X),\ k = 0, \cdots, m$ and there exist $x(t_k^+)$ and $x(t_k^-)$, $k = 1, \cdots, m$ with $x(t_k^-) = x(t_k)\}$, with the norm $\|x\|_{PC} = \sup_{t\in J}|x(t)|$. Set $J' := [0, T]\backslash\{t_1, ..., t_m\}$.

Definition 2.1 (definition 2.1, [3]). The Riemann-Liouville fractional integral of order $\alpha > 0$, of a function $f \in L_1(\mathbb{R}_+)$ is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds, \quad \text{for} \quad \alpha > 0 \quad \text{and} \quad t > 0,$$