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Chaotic Behavior and Subharmonic Bifurcations for the Duffing-van Der Pol Oscillator*

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Abstract Chaotic behavior for the Duffing-van der Pol (DVP) oscillator is investigated both analytically and numerically. The critical curves separating the chaotic and non-chaotic regions are obtained. The chaotic feature on the system parameters are discussed in detail. The conditions for subharmonic bifurcations are also obtained. Numerical results are given, which verify the analytical ones.

Keywords Chaotic behavior, subharmonic bifurcations, Duffing-van der Pol oscillator, Melnikov function.

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1. Introduction

The Duffing oscillator is a non-linear differential equation used to model certain damped and driven oscillators. The equation is given by

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t).$$

The equation describes the motion of a damped oscillator with a more complicated potential than in simple harmonic motion (which corresponds to the case $\beta = \delta = 0$); in physical terms, it models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law.

In dynamics, the van der Pol (VDP) oscillator is a non-conservative oscillator with non-linear damping. It evolves in time according to the second-order differential equation:

$$\frac{d^2x}{dt^2} - \mu_0(1-x^2)\frac{dx}{dt} + x = 0.$$

The two classical nonlinear systems, the Duffing oscillator and the VDP oscillator can describe many kinds of practical systems. They have been extensively

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investigated [1, 9, 12, 16]. As the combination of these two classical nonlinear systems, the DVP oscillator as a model of mechanics can be applied in many fields, such as physics, engineering, electronics, biology, neurology and many other disciplines.

The mathematical expression of the DVP oscillator is assumed in the form of the second-order non-autonomous differential equation

$$\frac{d^2x}{dt^2} - \mu_0(1-x^2)\frac{dx}{dt} + \frac{dV(x)}{dx} = g(f_0, \omega, t), \tag{1.1}$$

where x stands for the displacement from the equilibrium position, f_0 is the forcing strength and $\mu_0 > 0$ is a damping parameter of the system. $g(f_0, \omega, t) = f_0 \cos(\omega t)$ represents the periodic driving function of time with period $T = \frac{2\pi}{\omega}$, ω being the angular frequency of the driving force. $V(x)$ is the potential approximated by a finite Taylor series. The DVP oscillator belongs to the category of three-dimensional dynamical system with continuous time and can be expressed in the strict feed-back form. System (1.1) is a generalization of the classic DVP oscillator equation. It can be considered in at least three physically interesting situations, wherein the potential

$$V(x) = -\alpha\frac{x^2}{2} + \beta\frac{x^4}{4} \tag{1.2}$$

is a (i) single-well ($\alpha < 0, \beta > 0$), (ii) double-well ($\alpha > 0, \beta > 0$) or (iii) double-hump ($\alpha < 0, \beta < 0$). Each of the above three cases has become a classic central model describing inherently nonlinear phenomenon exhibiting rich and baffling varieties of regular and chaotic motions.

DVP oscillator as a model of mechanics can be applied in many fields and many researches on DVP oscillator have been done. Ravisankar et al. [15] investigated the occurrence of horseshoe chaos in three different asymmetric DVP oscillators driven by a narrow-band frequency modulated force. Njah and Vincent [14] presented chaos synchronization between single and double wells DVP oscillators with potential based on the active control technique. Wang and Li [18] analyzed the nonlinear dynamical characteristics of the DVP oscillator subject to both external and parametric excitations with time delayed feedback control. Leung et al. [7] investigated the damping characteristics of two DVP oscillators having damping terms described by fractional derivative and time delay respectively. By the residue harmonic method, Leung et al. [8] investigated periodic bifurcation of DVP oscillators having fractional derivatives and time delay. Chen and Jiang [2] studied the periodic solution of the DVP oscillator by homotopy perturbation method. The nonlinear dynamics of a DVP oscillator under linear-plus-nonlinear state feedback control with a time delay are investigated by means of the averaging method and Taylor expansion [13].

In this present paper, we consider only the double-well ($\alpha > 0, \beta > 0$) and the double-hump ($\alpha < 0, \beta < 0$) cases of the following the DVP oscillator

$$\frac{d^2x}{dt^2} - \mu_0(1-x^2)\frac{dx}{dt} - \alpha x + \beta x^3 = f_0\cos(\omega t). \tag{1.3}$$

Assume the damping and excitation terms μ_0, f_0 are small, denoting them as $\epsilon\mu$, ϵf, where ϵ is a small parameter, then Eq. (1.3) can be written as the following planar system

$$\begin{cases} x' = y, \\ y' = \alpha x - \beta x^3 + \epsilon\mu(1-x^2)y + \epsilon f\cos(\omega\xi) \end{cases} \tag{1.4}$$