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On the Number of Zeros of Abelian Integrals for a Class of Quadratic Reversible Centers of Genus One*

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Abstract In this paper, using the method of Picard-Fuchs equation and Riccati equation, for a class of quadratic reversible centers of genus one, we research the upper bound of the number of zeros of Abelian integrals for the system $(r10)$ under arbitrary polynomial perturbations of degree n. Our main result is that the upper bound is $21n - 24$ $(n \geq 3)$, and the upper bound depends linearly on n.

Keywords Abelian integral, Quadratic reversible center, Weakened Hilbert's 16th problem, Limit cycle.

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1. Introduction and Main Results

We research the planar polynomial system

$$\dot{x} = \frac{H_y(x,y)}{\mu(x,y)} + \varepsilon f(x,y), \quad \dot{y} = -\frac{H_x(x,y)}{\mu(x,y)} + \varepsilon g(x,y), \tag{1.1}$$

where ε $(0 < \varepsilon \ll 1)$ is a real parameter, $H_y(x,y)/\mu(x,y)$, $H_x(x,y)/\mu(x,y)$, $f(x,y)$, $g(x,y)$ are all polynomials of x and y, with $\max\{\deg(f(x,y)), \deg(g(x,y))\} = n$, $\max\{\deg(H_y(x,y)/\mu(x,y)), \deg(H_x(x,y)/\mu(x,y))\} = m$. We suppose that when $\varepsilon = 0$, the system (1.1) is an integrable system, it has at least one center. The function $H(x,y)$ is a first integral with the integrating factor $\mu(x,y)$. That is, we can define a continuous family of periodic orbits

$$\{\Gamma_h\} \subset \left\{(x,y) \in \mathbb{R}^2 : H(x,y) = h, h \in \Delta\right\},$$

which are defined on a maximal open interval $\Delta = (h_1, h_2)$. The problem which needs to be solved in this paper is: for any small number ε, how many limit cycles in the system (1.1) can be bifurcated from the periodic orbits $\{\Gamma_h\}$. It is well known that in any compact region of the periodic orbits, the number of limit cycles of the system (1.1) is no more than the number of isolated zeros of the following Abelian

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integrals $A(h)$,

$$A(h) = \oint_{\Gamma_h} \mu(x,y)\left[g(x,y)\,dx - f(x,y)\,dy\right], \quad h \in \Delta. \tag{1.2}$$

A) If the $\mu(x,y)$ is constant, i.e., the system (1.1) is a Hamiltonian system when $\varepsilon = 0$, then the function $H(x,y)$ is a polynomial of x and y, with $\deg(H(x,y)) = m+1$. Finding an upper bound $Z(m,n)$ for the number of isolated zeros of Abelian integrals $A(h)$ is a significant and important problem, where the upper bound $Z(m,n)$ only depends on m, n, and does not depend on the concrete forms of $H(x,y)$, $f(x,y)$, and $g(x,y)$. It is called the weakened Hilbert's 16th problem by Arnold in [1]. This problem has been studied extensively, such as, for some specially planar systems, researchers obtain plentiful important results [2, 10, 13, 15], further, for some special three-dimensional differential systems, researchers obtain important results too [14], more details can be found in the review article [11] and the books [3, 5].

B) If the $\mu(x,y)$ is not constant, i.e., the system (1.1) is an integrable non-Hamiltonian system when $\varepsilon = 0$, researchers consider this problem by starting from the simplest case: m is low. For the specific case of $m = 2$, people conjecture that the upper bound of the number of zeros of Abelian integrals $A(h)$ depends linearly on n. Unfortunately, this conjecture is still far from being solved.

For quadratic reversible centers of genus one, Gautier et al. [4] showed that there are essentially 22 types, namely $(r1)$–$(r22)$. The linear dependence of case $(r1)$ was studied in [16]; cases $(r3)$–$(r6)$ were studied in [12]; cases $(r9)$, $(r13)$, $(r17)$, $(r19)$ were studied in [9]; cases $(r11)$, $(r16)$, $(r18)$, $(r20)$ were studied in [8]; cases $(r12)$, $(r21)$ were studied in [7]; and case $(r22)$ was studied in [6]. All of these upper bounds depend linearly on n. In this paper, we research the case $(r10)$, and obtain that its upper bound is $21n - 24$ $(n \geq 3)$. Our result shows that the upper bound depends linearly on n.

The form of the case $(r10)$ as follows:

$$(r10) \quad \dot{x} = -xy, \quad \dot{y} = -\frac{1}{3}y^2 + \frac{1}{3^2 \cdot 2^4}x - \frac{1}{3^2 \cdot 2^4}. \tag{1.3}$$

The (1.3) is an integrable non-Hamiltonian system. It has a center $(1,0)$, an integral curve $x = 0$, a family of periodic orbits $\{\Gamma_h\}$ $(1/2^5 < h < +\infty)$(see Figure 1), and a first integral as follows:

$$H(x,y) = x^{-\frac{2}{3}}\left(\frac{1}{2}y^2 + \frac{1}{3 \cdot 2^4}x + \frac{1}{3 \cdot 2^5}\right) = h, \quad h \in \left(\frac{1}{2^5}, +\infty\right), \tag{1.4}$$

with an integrating factor $\mu(x,y) = x^{-5/3}$.

In this paper, our main results include the following theorem.

Theorem 1.1. *If $f(x,y)$ and $g(x,y)$ are any polynomials of x and y, then the upper bound of the number of zeros of Abelian integrals $A(h)$ for the system $(r10)$ depends linearly on n. More concretely, the upper bound is $21n - 24$ for $n \geq 3$; the upper bound is 3 for $n = 1, 2$; and the upper bound is 0 for $n = 0$.*