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Iterating a System of Variational-like Inclusion Problems in Banach Spaces

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Abstract In this manuscript, by using $(H, \varphi) - \eta$-monotone operators we study the existence of solution of a system of variational-like inclusion problems in Banach spaces. Further, we suggest an iterative algorithm for finding the approximate solution of this system and discuss the convergence criteria of the sequences generated by the iterative algorithm. The method used in this paper can be considered as an extension of methods for studying the existence of solution for various classes of variational inclusions considered and studied by many authors in Banach spaces.

Keywords System of variational-like inclusion problems, $(H, \varphi)-\eta$-monotone mappings, Resolvent operator, Cocoercive mappings, Banach spaces.

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1. Introduction

A widely studied problem known as variational inclusion problem have many applications in the fields of optimization and control, economics and transportation equilibrium, engineering sciences, etc. Several researchers used different approaches to develop iterative algorithms for solving various classes of variational inequality and variational inclusion problems. In details, we refer [2-5, 9-14, 20, 22-24] and the references therein. Recently Bhat and Shafi, Fang and Huang, Kazmi and Khan, and Lan *et al.* investigated several resolvent operators for generalized operators such as H-monotone [1, 3], H-accretive [4], (P, η)-proximal point [9], (P, η)-accretive [10], (H, η)-monotone [5], (A, η)-accretive [14] and mappings.

From the above results, in this manuscript, we intend to define the resolvent operator associated with $(H, \varphi) - \eta$-monotone mappings in Banach spaces. Using resolvent operator technique, we develop an iterative algorithm for solving the system of variational-like inclusion problems and prove that the sequences generated by the iterative algorithm converge strongly to a solution of the system. The results presented in this paper improve and extend many known results in the literature, see for example [6-8, 15-19, 23].

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2. Resolvent operator and formulation of problem

We need the following definitions and results from the literature.

Let X be a real Banach space equipped with norm $\|.\|$ and $X^\star$ be the topological dual space of X. Let $< .,. >$ be the dual pair between X and $X^\star$ and 2^X be the power set of X.

Definition 2.1 [21]. For $q > 1$, a mapping $J_q : X \to 2^{X^\star}$ is said to be generalized duality mapping, if it is defined by

$$J_q(x) = \{f \in X^\star : \langle x, f\rangle = \|x\|^q, \|x\|^{q-1} = \|f\|\}, \quad \forall x \in X.$$

In particular, J_2 is the usual normalized duality mapping on X, given as

$$J_q(x) = \|x\|^{q-2} J_2(x), \quad \forall x (\neq 0) \in X.$$

Note that if $X \equiv H$, a real Hilbert space, then J_2 becomes the identity mapping on X.

Definition 2.2 [21]. A Banach space X is said to be smooth if, for every $x \in X$ with $\|x\| = 1$, there exists a unique $f \in X^\star$ such that $\|f\| = f(x) = 1$.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$, defined by

$$\rho_X(\sigma) = \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = \sigma\right\}.$$

Definition 2.3 [21]. A Banach space X is said to be

(i) uniformly smooth if $\lim\limits_{\sigma \to 0} \dfrac{\rho_X(\sigma)}{\sigma} = 0$,

(ii) q-uniformly smooth, for $q > 1$, if there exists a constant $c > 0$ such that $\rho_X(\sigma) \leq c\sigma^q, \ \sigma \in [0, \infty)$.

Note that if X is uniformly smooth, J_q becomes single-valued.

Lemma 2.1. *[21] Let $q > 1$ be a real number and let X be a smooth Banach space. Then, the following statements are equivalent:*

(i) X is q-uniformly smooth.

(ii) There is a constant $c_q > 0$ such that for every $x, y \in X$, the following inequality holds

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 2.4. Let X be a real Banach space. Let $A : X \to X^\star$, $T : X \times X \to X^\star, \eta : X \times X \to X$ be single-valued mappings and $M : X \times X \to 2^{X^\star}$ be multi-valued mapping. Then

(i) A is said to be monotone, if

$$\Big\langle Ax - Ay, (x-y)\Big\rangle \geq 0, \quad \forall x, y \in X.$$

(ii) A is said to be η-monotone, if

$$\Big\langle Ax - Ay, \eta(x-y)\Big\rangle \geq 0, \quad \forall x, y \in X.$$