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Explicit Traveling Wave Solutions and Their Dynamical Behaviors for the Coupled Higgs Field Equation*

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Abstract In this paper, we focus on the traveling wave solutions of the coupled Higgs field equation from the perspective of dynamical systems. Through the phase portraits, in addition to periodic wave solutions and solitary wave solutions, we also obtain explicit periodic singular wave solutions, singular wave solutions and kink wave solutions, which were not found in the previous works. The dynamical behavior of these solutions and their internal relations are revealed through asymptotic analysis. The results will help supplement the study of field equation.

Keywords Coupled Higgs field equation, Traveling wave solutions, Kink wave solutions.

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1. Introduction

The classical Higgs equation [16]

$$u_{tt} - u_{xx} - \beta u + \gamma |u|^2 u = 0 \tag{1.1}$$

has important applications in particle physics, field theory and electromagnetic waves [4]. Equation (1.1) is attributed to the classical ϕ_4 field theory in physics of elementary particles and fields. As a generalized form of equation (1.1), the coupled Higgs field equation

$$\begin{cases} u_{tt} - u_{xx} - \beta u + \gamma |u|^2 u - 2uv = 0, \\ v_{tt} + v_{xx} - \gamma (|u|^2)_{xx} = 0 \end{cases} \tag{1.2}$$

has attracted considerate attention [1,4–8,13–15,18,24,25]. Equation (1.2) describes a system of conserved scalar nucleons interacting with neutral scalar mesons in particle physics. Here, the function $v = v(x,t)$ indicates a real scalar meson field,

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and $u = u(x,t)$ stands for a complex scalar nucleon field. The subscripts t, x of u and v denote appropriate partial derivatives with respect to the time and space variables.

There have been many works on the solutions of equation (1.2) from various aspects. Tajiri [13] derived the N-soliton solution of equation (1.2) exploiting Hirota bilinear method. Later, by Hirota bilinear method, Hu et al., [6] looked for homoclinic solution of equation (1.2). The authors in [4,14,24] obtained the bright soliton, periodic wave and doubly periodic wave solutions of equation (1.2). Zha et al., [8,25] studied the first-order rogue wave solution of equation (1.2). Hon and Fan [5] used an algebraic method to construct solitary wave solutions, Jacobi periodic wave solutions and a range of other solutions of physical interest. Wazwaz and his co-author [15,18] obtained a variety of exact periodic waves and solitary wave solutions of equation (1.2). Jabbri et al., [7] combined He's semi-inverse and (G'/G)-expansion methods to construct the exact solutions of equation (1.2). Ali et al., [1] found a variety of solitary wave solutions by using rational $\exp(-\varphi(\eta))$-expansion method.

Despite the success of these attempts in understanding solutions of equation (1.2), we note that the above works did not find kink waves of equation (1.2), and the dynamical behavior of the solutions and their internal relations are not so clear. Therefore, we intend to study equation (1.2) from the viewpoint of geometry. More precisely, we exploit qualitative theory of differential equations and bifurcation method of dynamical systems [2,3,9–12,17,19–23,26] to study the traveling wave solutions of equation (1.2) and to reveal their dynamical behavior and inside relations. Through analyzing the phase portrait, in addition to periodic wave solutions and solitary wave solutions, we also obtain explicit periodic singular wave solutions, singular wave solutions and kink wave solutions, which were not found in the above works. The dynamical behavior of these solutions and their internal relations are uncovered through asymptotic analysis.

2. Qualitative analysis and phase portraits

To study the traveling wave solutions of equation (1.2), assume

$$u(x,t) = e^{i\eta}\varphi(\xi), v(x,t) = \phi(\xi), \eta = px + rt, \xi = kx + dt, \tag{2.1}$$

where $\varphi(\xi)$ and $\phi(\xi)$ are real functions, and p, r, k and d are real constants.

Substituting (2.1) into equation (1.2), we have

$$\begin{cases} (d^2 - k^2)\varphi'' - (r^2 - p^2 + \beta)\varphi + \gamma\varphi^3 - 2\varphi\phi = 0, \\ (d^2 + k^2)\phi'' - \gamma k^2(\varphi^2)'' = 0, \\ rd = kp. \end{cases} \tag{2.2}$$

Integrating the second equation of (2.2) twice and letting the first integral constant be zero, we have

$$\phi = \frac{\gamma k^2 \varphi^2}{d^2 + k^2} + g, \tag{2.3}$$

where g is an integral constant.