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On Time-space Fractional Reaction-diffusion Equations with Nonlocal Initial Conditions*

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Abstract This paper investigates the nonlinear time-space fractional reaction-diffusion equations with nonlocal initial conditions. Based on the operator semigroup theory, we transform the time-space fractional reaction-diffusion equation into an abstract evolution equation. The existence and uniqueness of mild solution to the reaction-diffusion equation are obtained by solving the abstract evolution equation. Finally, we verify the Mittag-Leffler-Ulam stabilities of the nonlinear time-space fractional reaction-diffusion equations with nonlocal initial conditions. The results in this paper improve and extend some related conclusions to this topic.

Keywords Time-space fractional reaction-diffusion equation, Nonlocal initial condition, Mild solution, Existence and uniqueness, Mittag-Leffler-Ulam stability.

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1. Introduction

In this paper, we study the nonlocal initial-boundary value problem for the time-space fractional reaction-diffusion equations (FRDE for short) with fractional Laplacian

$$\begin{cases} {}^C D_t^{\alpha} u(x,t) + (-\Delta)^{\beta} u(x,t) = f(x,t,u(x,t)), \qquad (x,t) \in \Omega \times [0,T], \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) + \sum\limits_{k=1}^{p} c_k u(x,t_k), \qquad (x,t_k) \in \Omega \times [0,T], \end{cases} \tag{1.1}$$

where ${}^C D_t^{\alpha}$ is the Caputo fractional derivative of the order $\alpha \in (0,1)$, $(-\Delta)^{\beta}$ is a fractional Laplacian with $\beta \in (0,1)$, Ω is an open bounded domain in $\mathbb{R}^n$ with the smooth boundary $\partial\Omega$, $0 < t_1 < t_2 < \cdots < t_p < T$, $T > 0$ is a constant, $p \in \mathbb{N}$, $c_k \neq 0 (k = 1,2,...,p)$ are real numbers, $u_0 : \Omega \to \mathbb{R}$, and the nonlinear term $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ is Carathéodory continuous.

FRDE is a subject of extensive research on fractional calculus, which has a wide range of applications in modeling such as mechanics of materials, fluid mechanics,

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signal processing and control as well as biology (see [4, 10, 15, 22, 28, 32–34] for details). In recent years, many scholars have been committed to the research of time-fractional or space-fractional partial differential equation (see [11–13, 38–40, 43]). On the other hand, there are numerous works that have been devoted to fractional diffusion equations. We only list a number of the numerous papers on the analysis for fractional diffusion equation. Eidelman and Kochubei [16] considered an evolution equation with the regularized fractional derivative of an order $\alpha \in (0,1)$. Jia and Li [19] obtained the Maximum principles of time-space fractional diffusion equation with Riemann-Liouville time-fractional derivative. Kemppainen, Siljander and Zacher [21] studied the Cauchy problem for a nonlocal heat equation, which is of fractional order both in space and time. In [23], Li, Liu and Wang established $L^r - L^q$ estimates and weighted estimates of the fundamental solutions, and obtained the existence and uniqueness of mild solutions to Keller-Segel type time-space fractional diffusion equation. The paper [24] constructed the iterative sequence of mild solution for time-space fractional diffusion equation with delay. A class of time fractional diffusion equation representation of solutions was derived by using Laplace transform in [35].

In 1990, Byszewski and Lakshmikantham [8] first investigated the existence of mild solution for nonlocal differential equations. Since the nonlocal initial conditions had been applied physics with better effect than the classical initial conditions, researchers began to study differential equations with nonlocal conditions and obtained some fundamental results (see [5, 6, 8, 14, 25, 41] for more comments and citations). As some results are in relation to the time-space FRDE (1.1), when $c_k = 0$, $k = 1, 2, ..., p$, one can see [24] and [30]. Whereas, there are few articles studying time-space FRDEs with nonlocal initial conditions. The aim of this paper is to extend the current results of the classical initial conditions considered into time-space fractional diffusion equations with nonlocal initial conditions. This extension is not just a mathematical problem, but is caused by numerous physical applications. As there are multitudinous works describing the significance of fractional and nonlocal models in the anomalous diffusion, one can refer to [30] and its references for details.

In the theory of functional equations, there are some special kinds of data dependence such as Ulam-Hyers, Ulam-Hyers-Rassias, Ulam-Hyers-Bourgin and Aoki-Rassias [9, 18, 20]. Motivated by the results of [36], we further study the Mittag-leffler-Ulam stability of the time-space FRDE (1.1), and obtain some new and interesting stability results.

The main results with respect to the time-space FRDE (1.1) involving nonlocal initial conditions of this paper are as follows.

Theorem 1.1. *Assuming that the nonlinear function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory type condition and the following hypotheses,*
(H1) there exist a constant $q \in [0,\alpha)$ and a function $m(x,t)$ satisfying $\|m(t)\|_{\mathbb{H}^\beta(\Omega)} \in L^{1/q}([0,T],\mathbb{R}^+)$ with $M = \left(\int_0^T \|m(t)\|_{\mathbb{H}^\beta(\Omega)}^{1/q}\, dt\right)^q$ such that $|f(x,t,u)| \leqslant m(x,t)$ for all $x \in \Omega$, $t \in [0,T]$, $u \in \mathbb{R}$, and the norm $\|\cdot\|_{\mathbb{H}^\beta(\Omega)}$ of Sobolev space $\mathbb{H}^\beta(\Omega)$ is introduced in the following section;
(H2) $c_k > 0$, $k = 1, 2, ..., p$ and $\sum_{k=1}^{p} c_k < 1$ are satisfied, then FRDE (1.1) has at least one mild solution $u \in C(\Omega \times [0,T],\mathbb{R})$.

Theorem 1.2. *Assuming that the nonlinear function $f : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfies*