Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 5, Number 1, March 2023, 123–145 DOI:10.12150/jnma.2023.123

Melnikov Functions for a Class of Piecewise Hamiltonian Systems

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Abstract This paper is concerned with the number of limit cycles for a class of piecewise Hamiltonian systems with two zones separated by two semi-straight lines. By constructing a Poincaré map, we obtain explicit expressions of the first, second and third order Melnikov functions. In addition, we apply their expressions to give upper bounds of the number of limit cycles bifurcated from a period annulus of a piecewise polynomial Hamiltonian system.

Keywords Piecewise smooth system, Melnikov function, Limit cycle, Bifurcation.

MSC(2010) 34C05, 34C07, 37G15.

1. Introduction

In recent years, the research on non-smooth systems has attracted more and more attention, especially on piecewise near-Hamiltonian systems, see [7,18,21,22] and the references therein. One of important topics in bifurcation problems for piecewise smooth systems is to study the number of limit cycles or periodic solutions of them, which is an extension of the Hilbert's 16th problem. As is known, there are two main methods to investigate limit cycle bifurcations: the Melnikov function method [1, 2, 6, 7, 12, 15, 19, 24] and the averaging method [3, 4, 9, 13, 16, 17]. It was proved in [5, 14] that the above two methods are equivalent in studying the number of limit cycles of planar C^∞ near-Hamiltonian systems or piecewise C^∞ near-integrable systems in two or higher dimensional spaces.

In 2010, Liu and Han [12] considered a piecewise near-Hamiltonian system of the form

$$\begin{cases} \dot{x} = H_y^+(x,y) + \epsilon f^+(x,y), \\ \dot{y} = -H_x^+(x,y) + \epsilon g^+(x,y), \end{cases} x > 0,$$

$$\begin{cases} \dot{x} = H_y^-(x,y) + \epsilon f^-(x,y), \\ \dot{y} = -H_x^-(x,y) + \epsilon g^-(x,y), \end{cases} x \le 0,$$

where $H_x^\pm, H_y^\pm, f^\pm, g^\pm \in C^\infty$ and $\epsilon \ge 0$ is a small real parameter, and established a formula of the first order Melnikov function which was widely used in studying the number of limit cycles bifurcated from periodic orbits, see [8, 10, 23] for example. Recently, more general results have appeared for piecewise smooth systems with

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multiple zones [11, 18, 20, 25]. For instance, Tian and Han [18] studied the number of limit cycles bifurcated from a period annulus of a class of planar piecewise near-Hamiltonian systems with three different switching curves. The authors [11] investigated limit cycle bifurcations in piecewise near-Hamiltonian systems with multiple switching curves and obtained a formula of the first order Melnikov function. Yang, Yang and Yu [24] studied a planar piecewise Hamiltonian system with two zones separated by the two semi-straight lines and presented expressions of the first and second order Melnikov functions. In [2], Chen, Li and Llibre considered piecewise smooth differential systems in $\mathbb{R}^n$ separated by a hyperplane and obtained some recursion formulas of higher order Melnikov functions.

Motivated by the works mentioned above, in this paper, we consider a piecewise Hamiltonian system of the form

$$\begin{cases} \dot{x} = H_y(x, y, \epsilon), \\ \dot{y} = -H_x(x, y, \epsilon), \end{cases} \tag{1.1}$$

where

$$H(x, y, \epsilon) = \begin{cases} H^+(x, y, \epsilon), (x, y) \in \Sigma_1, \\ H^-(x, y, \epsilon), (x, y) \in \Sigma_2, \end{cases}$$

$$H^\pm(x, y, \epsilon) = H_0^\pm(x, y) + \epsilon H_1^\pm(x, y) + \epsilon^2 H_2^\pm(x, y) + \cdots, \tag{1.2}$$

with $H_i^\pm(x, y) \in C^\infty,\ i = 0, 1, 2, \ldots$, $\epsilon \geq 0$ is a small real parameter, Σ_1 and Σ_2 are the regions with a common boundary consisting of two semi-straight lines

$$l_1 : y = k_1 x,\ \ \mu_1 x > 0$$

and

$$l_2 : y = k_2 x,\ \ \mu_2 x > 0,$$

where $\mu_1, \mu_2 = \pm 1$ with $(k_1, \mu_1) \neq (k_2, \mu_2)$, see Fig.1. By constructing a Poincaré map of system (1.1), we shall derive expressions of the first, second and third order Melnikov functions.

The rest of this paper is organized as follows. In Section 2, we establish a Ponincaré map of system (1.1) and present expressions of the first, second and third order Melnikov functions. In Section 3, we give an application to illustrate our results and estimate the number of limit cycles bifurcated from a piecewise polynomial Hamiltonian system.

2. Expressions of Melnikov functions

Consider system (1.1). We make the following basic assumptions for the unperturbed system $(1.1)|_{\epsilon=0}$ as in [11]:

(***A1***) There exist an interval $J = (\alpha, \beta)$ and two points $A_0(h) = (a_0(h), k_1 a_0(h)) \in l_1$ and $A_{10}(h) = (a_{10}(h), k_2 a_{10}(h)) \in l_2$ such that for $h \in J$

$$\begin{aligned} &H_0^+(A_0(h)) = H_0^+(A_{10}(h)) = h, \\ &H_0^-(A_0(h)) = H_0^-(A_{10}(h)). \end{aligned} \tag{2.1}$$