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On the Analytical Approach of Codimension-Three Degenerate Bogdanov-Takens (B-T) Bifurcation in Satellite Dynamical System

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Abstract In this paper, we have conducted parametric analysis on the dynamics of satellite complex system using bifurcation theory. At first, five equilibrium points $\mathcal{E}_{0,1,2,3,4}$ are symbolically computed in which $\mathcal{E}_{1,3}$ and $\mathcal{E}_{2,4}$ are symmetric. Then, several theorems are stated and proved for the existence of B-T bifurcation on all equilibrium points with the aid of generalized eigenvectors and practical formulae instead of linearizations. Moreover, a special case $\alpha_2 = 0$ is observed, which confirms all the discussed cases belong to a codimension-three bifurcation along with degeneracy conditions.

Keywords Satellite dynamical system, Bogdanov-Takens bifurcation, normal form, generalized eigenvector

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1. Introduction

A system of ordinary differential equations obeying the changes in its topology with the variation in involved parameters can lead to the concept of bifurcation. This term is further categorized into local and global bifurcations such as in the study of Alle effect in predator prey models by Deeptajyoti et al., [39, 40] and Prahlad et al., [34]. However, if all parameters except one are set to be fixed, it is considered as a codimension (codim) one bifurcation including Hopf [16, 17, 32, 36, 38], zero-Hopf [18, 31, 37], saddle [3, 42] and Homoclinic/Hetroclinic [14, 15, 19, 28]. The variation in more than one parameter at a time can lead to bifurcations with a higher codimension. It is difficult to determine such bifurcations by using hit and trial methods, and that is why several analytical techniques have been discovered for achieving such a type of bifurcations, in which critical normal form on the center manifold has gained much attention. Bogdanov-Takens is one of the higher codimension bifurcations initiated with the work of Takens [11] and Bogdanov [12], but Arnold [6] and Guckenheimer [21] in 1983 derived the following normal form

$$\begin{aligned}\dot{\eta}_0 &= \eta_1,\\ \dot{\eta}_1 &= \sum_{k\geq 2}\left(\alpha_k \eta_0^k + \beta_k \eta_0^{k-1}\eta_1\right),\end{aligned} \tag{1.1}$$

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on the center manifold and discussed the exact qualitative attitude of trajectories near B-T critical point. This technique created a new way and attracted researchers by presenting the obtained normal form in a more simplified way [2,4,20].

In equation (1.1), if $\alpha_2 \times \beta_2 \neq 0$, then codim-2 bifurcation occurs and its bifurcation diagram can be seen in [29]. Cusp [33], Bogdanov-Takens [25,26] , Double Hopf [23] and Bautin [22] are the main types in codim-2 bifurcation, which include tedious computer based calculations. But when α_2 or β_2 equals to zero in equatiion (1.1), a complex situation termed as codim-3 bifurcation can occur. In 2011, Kuznetsov [30] introduced a new way of getting the normal form by using generalized eigenvectors and practical formulae instead of linearization. This methodology is not limited to numerical computation, but also useful in the analytic of symbolic computation. In 2021, Darabsah [24] used B-T bifurcation from medical point of view in the excitably class of neurons. Mondal et al., [9] used it in ecology for predator-prey system, and the same analytical formulae for B-T bifurcation are used by Mondal et al., [10] in the food chain model with two species. In 2022, Xiang, Lu and Huang [41] explored the B-T bifurcation in host-parasitoid model for the special case that the carrying capacity K equals to $\frac{r_1}{\eta}$. Due to the advancement in technology and craze of exploring cosmic dynamics, systems in mechanical engineering can never lose its importance. In 2000, Sui et al., [1] considered six-dimensional satellite system and reported its failure analysis in controlling chaos. A negligible torque term was added into satellite system as a perturbed parameter and analyzed its transversal line between stable and unstable manifolds by Kuang [27]. Similarly, Aslanov and Yudintsev [7] discussed dynamics of gyrostat in satellite by bringing modification into Melnikov's function. Whereas, for more information relating dynamics and control in spacecraft, one can follow the results given by Liu in his book [43]. In 2018, Chegini, Sadati and Salarieh [13] worked analytically and numerically as well on tri-axial rigid body moving in elliptical orbit. Whereas, Khan et al., [8] modified satellite double wing model in the following form

$$\begin{cases} \dot{x} = \frac{yz}{3} - ax + \frac{z}{\sqrt{6}}, \\ \dot{y} = -xz + by, \\ \dot{z} = xy - cz - \sqrt{6}x. \end{cases} \tag{1.2}$$

Dynamics of system (1.2) shows chaos for parameter values $a = 4$, $b = 0.17$ and $c = 4$ with initial conditions $(0.1, 0.1, 0.1)$.
In our case, while analyzing dynamics of the satellite system, we have observed that

1. Khan et al., [8] fixed all involved parameters for finding its equilibrium points and dynamical analysis;
2. recently, Anam et al., [5] have obtained their multi-scrolls;
3. the above cited literature shows that satellite system is enriched with a qualitative aspect and some research can be found on its stability.

System (1.2) has complex dynamics due to the nonlinearity in each equation and involved parameters. However, relevant literature on its bifurcation analysis has not been identified. This gap has motivated us to work on the occurrence of higher codimension bifurcation in satellite system (1.2) for all equilibrium points. This parametric study leads us to the existence of degeneracy in B-T bifurcation as well, where Hopf, Saddle and Homoclinic orbits meet. Moreover, these analytical