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Successive Canard Explosions in a Singularly Perturbed Spruce-Budworm Model with Holling-II Functional Response*

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Abstract By combining geometric singular perturbation theory (GSPT) with qualitative method, this paper analyzes the phenomenon of successive canard explosions in a singularly perturbed Spruce-Budworm model with Holling-II functional response. We select suitable parameters such that the critical curve is S-shaped, and the full model only admits a unique equilibrium. Then, under the variation of the breaking parameter, it is found that a canard explosion followed by an inverse canard explosion successively occurs in this model. That is, a relaxation oscillation arises via the first canard explosion, which persists for a large interval of parameter until it vanishes via the so-called inverse canard explosion. All these theoretical predictions are verified by numerical simulations.

Keywords Spruce-Budworm model, geometric singular perturbation theory, canard explosion, inverse canard explosion

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1. Introduction

The spruce budworm, also referred to as the spruce curling moth, is one of the most destructive insects in mixed spruce and fir forests in the eastern United States and Canada. According to available records in the United States and Canada, large outbreaks of budworm have occurred approximately every 40 years since the early 18th century, causing billions of dollars in forest damage each time [12, 30]. When the plague of insects occurs, the density of spruce budworm will suddenly increase hundreds of times in a few months, and it wreaks havoc on spruce and fir forests. Though only a few spruce and fir trees die in a disaster, it takes about seven to ten years for damaged trees to recover to their previous health [36]. Thus, the spruce

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budworm, the spruce and fir forest evolve on different time scales. Therefore, the interaction between them would exhibit slow-fast nature. Hence, the dynamical model governing this process should be a singular perturbation problem.

In order to understand the interactions between the spruce budworm and its host, Ludwig, Jones and Holling [22] proposed the following three-dimensional spruce-budworm model

$$\begin{aligned} \frac{\mathrm{d}B}{\mathrm{d}t} &= r_B B\left(1-\frac{B}{K_B}\right)-\beta\frac{B^2}{\alpha^2+B^2}, \\ \frac{\mathrm{d}S}{\mathrm{d}t} &= r_S S\left(1-\frac{S}{K_S}\times\frac{K_E}{E}\right), \\ \frac{dE}{dt} &= r_E E\left(1-\frac{E}{K_E}\right)-P\frac{B}{S}, \end{aligned} \tag{1.1}$$

where B denotes the density of budworm, S is the total surface area of the branches in a stand, and E represents the condition of the foliage and health of the trees, which can be regarded as an "energy reserve". Additionally, K_S and K_E are the maximum values of S and E respectively, and K_B stands for the carrying capacity of B (see Ludwig for more details).

Since model (1.1) is three-dimensional, in general, its dynamical analysis is difficult. By ignoring the effect on "energy reserve" determining the condition of trees and foliage and retaining the essentials budworm and leaf area, May [24] simplified (1.1) to a two-dimensional one, namely,

$$\begin{aligned} \frac{dN}{dt} &= rN\left(1-\frac{N}{kS}\right)-\beta\frac{PN^2}{\eta^2S^2+N^2}, \\ \frac{dS}{dt} &= \rho S\left(1-\frac{S}{S_{\max}}\right)-\delta N, \end{aligned} \tag{1.2}$$

in which N is the population density of the larvae, S is the average leaf area of the spruce, r and ρ describe the intrinsic growth rates of the budworm and the leaves respectively, kS and S_{max} represent the carrying capacity of budworm population and spruce leaf area respectively, k measures the degree that leaves can accommodate the larvae, β is the coefficient of proportionality, and the term

$$\beta\frac{PN^2}{\eta^2S^2+N^2}$$

describes the predation pressure on budworm population by parasitoids, insectivorous birds, etc. For more biological motivations about (1.2), one can refer to [24]. From model (1.2), it can be noted that the carrying capacity of the budworm population and the Holling-III functional response function now depend on S, i.e. the density of the spruce, and the predation from the budworm to the spruce is assumed to be linearly dependent on N. Under these assumptions, the original three-dimensional model (1.1) is changed to the two-dimensional one (1.2). This dimension reduction is mainly biological rather than mathematical ways like the center manifold reduction, etc.

In May [24], it was assumed that $\rho \ll r$, i.e., the intrinsic growth rates of the budworm and of the leaves are of different orders. Under this condition, model (1.2) can be rewritten into the form of singular perturbation problems. Hence, it can be