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Well-Posedness of MHD Equations in Sobolev-Gevery Space

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Abstract This paper is devoted to the study of the 3D incompressible magnetohydrodynamic system. We prove the local in time well-posedness for any large initial data in $\dot{H}^1_{a,1}(\mathbb{R}^3)$ or $H^1_{a,1}(\mathbb{R}^3)$. Furthermore, the global well-posedness of a strong solution in $\tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3) \cap \dot{H}^2_{a,1}(\mathbb{R}^3))$ with initial data satisfying a smallness condition is established.

Keywords MHD equation, Sobolev-Gevery space, well-posedness

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1. Introduction

The magnetohydrodynamic equations reflect the basic physics laws governing the dynamics of electrically conducting fluids. The velocity field obeys the Navier-Stokes equations, and the magnetic field satisfies the Maxwell's equations of electromagnetism. The magnetohydrodynamic equations play important roles in the study of many phenomena in geophysics, astrophysics, and cosmology(see, [1–3]). In this paper, we consider the 3D incompressible magnetohydrodynamic (short written MHD) equations, which can be written as:

$$\begin{cases} \partial_t u - \mu\triangle u + u\cdot\nabla u + \nabla p = b\cdot\nabla b, \\ \partial_t b - \nu\triangle b + u\cdot\nabla b = b\cdot\nabla u, \\ \nabla\cdot u = 0, \nabla\cdot b = 0, \\ u(x,0) = u_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

for $t > 0$, $x \in \mathbb{R}^3$. We denote $u = u(x,t)$, $b = b(x,t)$ and $p = p(x,t)$ the velocity field, magnetic field and scalar pressure respectively. The constants μ and ν are the viscosity and resistivity coefficient, u_0 and b_0 are the initial velocity field and initial magnetic field satisfying $\nabla\cdot u_0 = 0$, $\nabla\cdot b_0 = 0$. When $b = 0$, equation (1.1) reduces to the classical Navier-Stokes equation, which has been investigated in many exciting results. Leray [4] and Hopf [5] established the global existence of weak solutions. Fujita and Kato [6] obtained the local well-posedness for large initial data and the global well-posedness for small initial data in Sobolev space. And Kato [7] established similar results in $L^n(\mathbb{R}^n)$. Lei and Lin [8] proved the existence of global mild solution with small initial data in the critical space $\chi^{-1}(\mathbb{R}^3)$. Benameur

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and Jlali [16] studied the long time decay of global solution in Sobolev-Gevery spaces $\dot{H}^1_{a,\sigma}(\mathbb{R}^3)$. Sun and Liu [20] proved that if $u \in C([0,+\infty), \dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3))$ is a global solution of the 3D fractional Navier-Stokes equation, then $\|u(t)\|_{\dot{H}^{\frac{5}{2}-2\alpha}_{a,\frac{1}{\alpha}}(\mathbb{R}^3)}$ decays to zero as time approaches infinity. More results about the solutions to the Navier-Stokes equations in Sobolev-Gevery spaces can be found in ([17–19]). For the MHD equation (1.1), there are several important results. Duvaut and Lions [9] constructed a global weak solution and local strong solution. Sermange and Temam [10] established the local well-posedness of equation (1.1) in Sobolev space for any initial data. Chaabani [21] proved the local in time well-posedness for any large initial data in $H^{\frac{1}{2}}_{a,\sigma}(\mathbb{T}^3)$ as well as global in time well-posedness when initial data satisfies a smallness condition. More researches on the well-posedness for MHD euqations can be referred to ([11–14]).

In this paper, we study not only the well-posedness of the local solution in $C([0,T];\dot{H}^1_{a,1}(\mathbb{R}^3))$ and $C([0,T];H^1_{a,1}(\mathbb{R}^3))$ for large initial data but also the well-posedness of the global solution in $\tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3))\cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3)\cap \dot{H}^2_{a,1}(\mathbb{R}^3))$ for small initial data. We use the Banach contraction mapping principle to prove it. Although it is considered to be valid, the construction of the work space in the process of proof is delicate and not easy. We present it in this paper. Our results are stated in the following theorems.

Theorem 1.1. *Consider the MHD equation (1.1) with* $\mu > 0$*, and* $\nu > 0$*. Assume* $(u_0,b_0) \in \dot{H}^1_{a,1}(\mathbb{R}^3)$ *with* $divu_0 = divb_0 = 0$*. There exists a time* $T = T(\|u_0\|_{\dot{H}^1_{a,1}}, \|b_0\|_{\dot{H}^1_{a,1}}) > 0$*, such that (1.1) has a unique solution* $(u,b) \in C([0,T];\dot{H}^1_{a,1}(\mathbb{R}^3))$*.*

Theorem 1.2. *Consider the MHD equation (1.1) with* $\mu > 0$*, and* $\nu > 0$*. Assume* $(u_0,b_0) \in H^1_{a,1}(\mathbb{R}^3)$ *with* $divu_0 = divb_0 = 0$*. There exists a time* $T = T(\|u_0\|_{H^1_{a,1}}, \|b_0\|_{H^1_{a,1}}) > 0$*, such that (1.1) has a unique solution* $(u,b) \in C([0,T];H^1_{a,1}(\mathbb{R}^3))$*.*

Theorem 1.3. *Consider the MHD equation (1.1) with* $\mu > 0$*, and* $\nu > 0$*. Assume* $(u_0,b_0) \in H^1_{a,1}(\mathbb{R}^3)$ *with* $divu_0 = divb_0 = 0$*. There exists a small enough constant* $\varepsilon > 0$ *such that if* $\|(u_0,b_0)\|_{H^1_{a,1}} \le \varepsilon$*, then system (1.1) has a unique global solution* $(u,b) \in \tilde{L}^\infty(0,T;H^1_{a,1}(\mathbb{R}^3)) \cap L^2(0,T;\dot{H}^1_{a,1}(\mathbb{R}^3) \cap \dot{H}^2_{a,1}(\mathbb{R}^3))$*, for any* $T > 0$*.*

2. Notations and lemmas

In this section, we first introduce some notations and definitions that will be used later, then we present several tool lemmas which serve as preparation for the proof of our main results.

- The Fourier transformation is defined as

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{R^3} e^{-ix\cdot\xi} f(x)dx.$$

- The homogeneous Sobolev space is defined as

$$\dot{H}^s = \{f \in \mathcal{S}'(\mathbb{R}^3);\ \hat{f} \in L^1_{loc} \text{ and } |\xi|^s \hat{f} \in L^2(\mathbb{R}^3)\}.$$