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Existence and Decay of Global Strong Solution to 3D Density-Dependent Boussinesq Equations with Vacuum*

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Abstract This paper is concerned with the initial boundary problem for the three-dimensional density-dependent Boussinesq equations with vacuum. We obtain the existence of the global strong solution under the initial density in the norm L^∞ is small enough without any smallness condition of u and θ. Furthermore, the exponential decay rates of the solution and their derivatives in some norm was established. In addition, we show that the solution and their derivatives are monotonically decreasing with respect to time t on $[0, T]$.

Keywords Boussinesq equation, vacuum, global strong solution, exponential decay-in-time

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1. Introduction

The density-dependent Boussinesq equations with vacuum were presented as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P - \operatorname{div}(\mu(\rho,\theta)\nabla u) = \rho\theta e_3, \\ \rho\theta_t + \rho u \cdot \nabla\theta = \operatorname{div}(\kappa(\rho,\theta)\nabla\theta), \\ \operatorname{div} u = 0 \end{cases}$$

in $\Omega \in \mathbb{R}^3$, where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ denotes the fluid velocity vector field, $P(x,t)$, $\rho(x,t)$ and $\theta(x,t)$ are the scalar pressure, density and temperature, respectively. $e_3 = (0,0,1)$. The constants μ and κ are the viscosity and the thermal diffusivity, respectively.

The Boussinesq equation [2,4,10] is an important model in mathematics physics. This system describes the influence of the convection phenomena on the dynamics of the ocean or the atmosphere. Fan and Ozawa [3] obtained the local existence

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of the strong solution to the Cauchy problem for the system (1.1)-(1.2) in $\mathbb{R}^3$, and they also established some blow-up criteria $u \in L^{2/(1-r)}(0,T;\dot{X}_r)$, $(0 < r < 1)$ or $u \in L^{2q/(q-3)}(0,T;L^q)$, $3 < q \leq \infty$. Later, Zhang [15] proved the regularity criterion in BMO space $u \in L^2(0,T;BMO)$. In [11], they established the local wellposedness for the incompressible Boussinesq system without dissipation terms under the framework of the Besov spaces in dimension $N \geq 2$. They also obtained a Beale-Kato-Majda type regularity criterion. Zhong [17] considered the Cauchy problem of the 2D density-dependent Boussinesq equations without a dissipation term in the temperature equation with vacuum as far field density. He proved that there exists a unique local strong solution provided the initial density and the initial temperature decay not too slow at infinity. Global well-posedness of two-dimensional density-dependent boussinesq equations with large initial data and vacuum was investigated by Zhong in [18]. In [12], Ye and Zhu got the zero limit of thermal diffusivity for the 2D density-dependent Boussinesq equations with vacuum.

When $\rho = C$, system (1.1) reduces to the classical homogeneous incompressible Boussinesq system which is widely studied. Chae [1](see also [8]) proved the global in time regularity for the 2D Boussinesq system with either the zero diffusivity or the zero viscosity. He [6] studied the blow-up criterion of classical solution to the Boussinesq equations with temperature-dependent viscosity and zero thermal diffusivity in $\mathbb{R}^2$ and $\mathbb{R}^3$. Larios and Pei [9] studied the local well-posedness of solutions to the 3D Boussinesq-MHD system. Some regularity criteria were also investigated in [9]. Later, Zhao [16] investigated the well-posedness of the Cauchy problem to the Boussinesq-MHD system with partial viscosity and zero magnetic diffusion.

Inspired by [5, 13, 14], we consider the following density-dependent Boussinesq equations

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \nabla P - \mathrm{div}(\mu(\rho,\theta)\nabla u) = \rho\theta e_3, \\ \rho\theta_t + \rho u \cdot \nabla\theta = \mathrm{div}(\kappa(\rho,\theta)\nabla\theta), \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

where $\mu(\rho,\theta)$ and $\kappa(\rho,\theta)$ are all function of ρ and θ, which are assumed to satisfy

$$(\mu(\rho,\theta),\kappa(\rho,\theta)) \in C^1[0,\infty), 0 < \underline{\kappa} \leq \kappa(\rho,\theta) \leq C < \infty, 0 < \underline{\mu} \leq \mu(\rho,\theta) \leq C < \infty, \tag{1.2}$$

and

$$(\mu_\rho(\rho,\theta), \mu_\theta(\rho,\theta), \kappa_\rho(\rho,\theta), \kappa_\theta(\rho,\theta)) \leq C \tag{1.3}$$

for some positive constants $\underline{\mu}$, $\underline{\kappa}$ and C.

The initial and boundary conditions satisfy that

$$(\rho,u,\theta)|_{t=0} = (\rho_0,u_0,\theta_0)(x), \quad x \in \Omega; \quad (u,\theta)|_{x\in\partial\Omega} = 0. \tag{1.4}$$

Our main purpose is to study the existence of the global strong solution to the initial boundary value problem of (1.1)-(1.3). Now, we present our results as follows:

Theorem 1.1. *Assume that the initial data* (ρ_0,u_0,θ_0) *satisfies*

$$0 \leq \rho_0 \leq \bar{\rho}, \ \nabla\rho_0 \in L^p (p > 3), \ (u_0,\theta_0) \in H_0^1 \cap H^2$$