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The Longtime Behavior of an SIR Epidemic Model with Free Boundaries*

Meng Zhao[1,†]

Abstract In [6] , the authors studied an SIR epidemic model with free boundary. They first proved the global existence and uniqueness of solution, and then gave the criteria for spreading and vanishing. They also obtained the longtime behavior for the case of vanishing. However, the longtime behavior when spreading happens remains open. In this short paper, we aim to solve this open problem.

Keywords SIR model, free boundary, longtime behavior

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1. Introduction

The classical SIR model has been received great attention because it can describe the development of an infectious disease. In such a model, the population is separated into three categories: susceptible, infectious and recovered individuals, denoted by S, I and R respectively. If we assume that the disease incubation period is negligible so that each susceptible individual becomes infectious and later recovers with a permanently acquired immunity, then the SIR model has the following form:

$$\begin{cases} S' = b - \beta SI - \mu_1 S, \\ I' = \beta SI - \gamma I - \mu_2 I, \\ R' = \gamma I - \mu_3 R, \end{cases} \tag{1.1}$$

where b is the constant recruitment rate, β stands for the constant effective contact rate, γ represents the recovery rate of the infectious, and $\mu_i (i = 1, 2, 3)$ are the death rates of S, I and R. It was shown in [5] that the number $\widetilde{R}_0 = \frac{\beta b}{(\mu_2+\gamma)\mu_1}$ is a threshold value for the long-time dynamical behaviour of (1.1): the epidemic will eventually die out if $\widetilde{R}_0 < 1$, and remain endemic if $\widetilde{R}_0 > 1$.

In (1.1), the spatial factor is ignored. Various types of the corresponding SIR model with spatial diffusion was considered by many researchers. For example, Wang and Wang [9] considered the traveling wave phenomena in a diffusive SIR Model; Yang et al. [8] studied traveling waves in a nonlocal dispersal SIR epidemic model; Enatsu et al. [4] investigated the traveling wave solution for a diffusive simple

†the corresponding author.
Email address: zhaom@nwnu.edu.cn (M. Zhao)
[1]College of Mathematics and Statistics, Northwest Normal University, Lanzhou, Gansu 730000, PR China

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epidemic model with a free boundary; Djilali et al. [2] analysed the asymptotic behavior of SIR epidemic model with nonlocal diffusion and generalized nonlinear incidence functional, and so on. However, these works are not good enough to obtain an accurate estimate of the spreading front in the spreading of an epidemic. In order to better describe the spreading of the epidemic front, Kim et al. [6] followed the approach of [3] and investigated this problem via the free boundary version. If we further assume that $\mu_1 = \mu_2 = \mu_3 = k$, then the problem in [6] for one-dimensional case becomes the following free boundary model:

$$\begin{cases} S_t = dS_{xx} + b - \beta SI - kS, & t > 0,\ x \in \mathbb{R}, \\ I_t = dI_{xx} + \beta SI - \gamma I - kI, & t > 0,\ x \in (g(t), h(t)), \\ R_t = dR_{xx} + \gamma I - kR, & t > 0,\ x \in (g(t), h(t)), \\ I(t,x) = R(t,x) = 0, & t > 0,\ x \le g(t) \text{ or } x \ge h(t), \\ g'(t) = -\mu I_x(t, g(t)),\ h'(t) = -\mu I_x(t, h(t)), & t > 0, \\ S(0,x) = S_0(x), & x \in \mathbb{R}, \\ -g(0) = h(0) = h_0, & \\ I(0,x) = I_0(x),\ R(0,x) = R_0(x), & x \in [-h_0, h_0], \end{cases} \tag{1.2}$$

where $S_0(x), I_0(x)$ and $R_0(x)$ satisfy

$$\begin{aligned} & S_0(x) \in C^2(\mathbb{R}),\ I_0(x), R_0(x) \in C^2([-h_0, h_0]), \\ & I_0(x) = R_0(x) = 0,\ x \in \mathbb{R} \backslash (-h_0, h_0), \\ & S_0(x) > 0, x \in \mathbb{R},\ I_0(x) > 0, R_0(x) > 0, x \in (-h_0, h_0). \end{aligned} \tag{1.3}$$

Denote

$$R_0 = \frac{\beta b}{(k+\gamma)k}.$$

It follows from [6] that the solution of (1.2) exists and is unique for all $t > 0$, and the following criteria hold:

(i) if $R_0 < 1$, then the disease will vanish;

(ii) if $R_0 > 1$, then the disease will vanish for sufficiently small h_0 and μ;

(iii) if $R_0 > 1$, then the disease will spread for suitably large h_0.

Furthermore, the results in [6] show that if $h_\infty - g_\infty < \infty$, then

$$\begin{aligned} & \lim_{t\to\infty} S(t,x) = b/k \text{ locally uniformly in } \mathbb{R}; \\ & \lim_{t\to\infty} \|I(t,x)\|_{C([g(t),h(t)])} = \lim_{t\to\infty} \|R(t,x)\|_{C([g(t),h(t)])} = 0. \end{aligned}$$

Just as pointed out by Kim et al. [6], the study of the asymptotic spreading speed when spreading happens is an interesting question. To answer this question, we should firstly show the longtime behavior of (1.2) when spreading happens. Motivated by [6], Ding et al. [1] considered the effect of relapse, while we [11] considered the nonlocal version of (1.2) in 2020, but both of these two works do not give the longtime behavior when spreading happens. In this short paper, we aim to solve this problem for the cases of local diffusion. We believe this method can be used to deal with the longtime behavior in [1, 11]. It is worth mentioning that Li et al. [7] proposed a new SIR epidemic model recently, and considered the dynamical properties. They obtained the longtime behavior of this model when the diseases spread.